# Accepted Manuscript

## International Journal of Geometric Methods in Modern Physics

Article Title: Non-Standard Hamiltonian Structures of the Lienard Equation and Contact Geometry

Author(s): Jose F. Carinena, Partha Guha

DOI: 10.1142/S0219887819400012

Received: 01 March 2017

Accepted: 08 September 2018

To be cited as: Jose F. Carinena, Partha Guha, Non-Standard Hamiltonian Structures of the Lienard Equation and Contact Geometry, *International Journal of Geometric Methods in Modern Physics*, doi: 10.1142/S0219887819400012

Link to final version: https://doi.org/10.1142/S0219887819400012

International Journal of Geometric Methods in Modern Physics


# NON-STANDARD HAMILTONIAN STRUCTURES OF THE LIÉNARD EQUATION AND CONTACT GEOMETRY

JOSÉ F. CARIÑENA

*Departamento de Física Teórica*
*Universidad de Zaragoza*
*50009 Zaragoza, Spain jfc@unizar.es*

PARTHA GUHA

*S.N. Bose National Centre for Basic Sciences*
*JD Block, Sector III, Salt Lake,Kolkata-700106, India*
*Instituto de Física de São Carlos; IFSC/USP*
*and*
*CEP 13560-970, São Carlos-SP, Brazil*
*partha@bose.res.in*

Received (Day Month Year)
Revised (Day Month Year)

The construction of non-standard Lagrangians and Hamiltonian structures for Liénard equations satisfying Chiellini condition is presented and their connection to time-dependent Hamiltonian formalism is shown. We also show that such non-standard Lagrangians are deformations of simpler standard Lagrangians. We also exhibit its connection with contact Hamiltonian mechanics.

*Keywords*: Reeb vector field; non-standard Hamiltonian; contact structure; deformed Lagrangian

*MSC 2010:* 70G45, 53D10, 53Z05

## 1. Introduction

Liénard type second-order nonlinear differential equations [1]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

appear very often modelling phenomena in many different areas of applied science (see e.g. [2]) and have receiving a lot of attention the last years [2,3,4,5,6] These equations are very related with first and second-kind Abel equations [7,8,9]. Some generalisations of the Levinson and Smith type [10], where the function $f(x)$ is replaced in (1) by another one $f(x, \dot{x})$, are also relevant and in particular the differential equation

$$\ddot{x} + h(x)\,\dot{x}^2 + f(x)\dot{x} + g(x) = 0, \tag{2}$$

that for $h \equiv 0$ reduces to the standard Liénard equation, while it is the quadratic Liénard equation when $f \equiv 0$. Some of these terms seem to be dissipative and it is not clear how to find first-integrals (see [6] for recent results). The problem becomes simpler when we are able to find an autonomous Lagrange function $L$ whose Euler-Lagrange equation is equivalent to the given Liénard equation, because then the associated energy function is a constant of motion. This has been the case for many of these equations, and in general the method for finding such Lagrangians is based on the approach of the Jacobi last multiplier (hereafter JLM) as developed by Jacobi [11]. Moreover, the knowledge of alternative but non gauge-equivalent Lagrangians for the same dynamics leads to constants of motion. This result, that is a direct consequence of JLM theory, is attributed in the physics literature to Currie and Saletan for one-dimensional systems [12]. See also for the $n$-dimensional systems [13], and [14] for a more geometric approach. The considered Lagrangians are usually of the standard type, i.e. they can be expressed as a difference between a 'kinetic energy term' and a 'potential energy term', but other non-standard Lagrangians can also be useful as those obtained from the JLM approach, because the method can be applied to a broader range of physical problems.

As indicated above, Liénard and Abel equations are very related and in particular for the reduced form of the first-kind first-order Abel equation

$$\frac{dy}{dx} = f(x)y^2 + g(x)y^3$$

Chiellini [15] was able to find an integrability condition: there exists a nonzero real number $k$ such that (see also [2,7,16,17,18])

$$\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right) = k\, f(x).$$

We prove in Section 2 that the same condition is valid for the existence in a certain family of functions of a JLM for the Liénard differential equation. Once this multiplier has been found, and based on the knowledge of this Jacobi last multiplier, we can derive a (regular) Lagrangian which is a non-standard Lagrangian but it yields the equations of motion of a deformation of a simpler standard Lagrangian. Section 3 develops the corresponding Hamiltonian formulation which is not that of a mechanical type system. The theory is particularised for a damped oscillator and then we also exhibit its connection to contact Hamiltonian mechanics. Finally, in section 4 we analyse the Lagrangian formulation of Liénard equation satisfying Chiellini condition from the perspective of the recently developed theory of deformed Lagrangians [19,20].

## 2. The Lagrangian formulation of a Liénard equation satisfying Chiellini condition

Consider the classical Liénard equation (1), a class of second-order differential equations (SODE) in which the damping term is proportional to the velocity $\dot{x}$:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where we assume that $f$ is a never vanishing function [2,3,4,5,6].

This second-order differential equation has associated a system of two first-order differential equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{3}$$

where $x$ is the coordinate in the configuration space and $y$ is its corresponding velocity coordinate. The solutions of such system are the integral curves of the following vector field in $T\mathbb{R}$:

$$\Gamma = y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y}. \tag{4}$$

Recall (see e.g.[11,21,22]) that if we consider the 2-form $\omega = dx \wedge dy$ in $T\mathbb{R}$, a Jacobi last multiplier for $\Gamma$ is a strictly positive function $M(x, y)$ such that $\mathcal{L}_{M\Gamma}\omega = 0$. The divergence (w.r.t. $\omega$) of a vector field $X \in \mathfrak{X}(T\mathbb{R})$ is defined by $\mathcal{L}_X\omega = \operatorname{div} X\, \omega$, and then in the particular case of a SODE vector $\Gamma = y\,\partial/\partial x + F(x, y)\,\partial/\partial y$ is $\operatorname{div}\Gamma = \partial F/\partial y$, and we can see that in this case the equation defining the JLM, $\Gamma(M) + M \operatorname{div}\Gamma = 0$, reduces to

$$\Gamma(M) + M\,\frac{\partial F}{\partial y} = 0, \tag{5}$$

or equivalently to

$$\Gamma(\log M) + \frac{\partial F}{\partial y} = 0, \tag{6}$$

and in our particular case (4) both expressions reduce, respectively, to

$$\Gamma(M) - f\,M = 0, \qquad \Gamma(\log M) - f = 0. \tag{7}$$

This is a partial differential equation for $M$ whose most general solution is not easily found. However we can look for a particular one of some specific type. For instance we can try to determine whether it is possible to find a real number $\alpha \neq 0$ and a function $W(x)$ such that

$$M = (y - W(x))^{1/\alpha} \tag{8}$$

is a JLM. Since

$$\begin{aligned}\Gamma((y - W(x))^{1/\alpha}) &= \frac{1}{\alpha}(y - W(x))^{(1/\alpha)-1}\,\Gamma(y - W(x)) \\ &= \frac{1}{\alpha}(y - W(x))^{(1/\alpha)-1}\,(-f(x)y - g(x) - y\,W'(x)),\end{aligned}$$

with $W'(x) = dW/dx$, the equation (7) for the JLM is in this case

$$f(x)y + g(x) + W'(x)y + \alpha\, f(x)(y - W(x)) = 0,$$

and consequently the function $W(x)$ must be such that

$$W'(x) = -(\alpha+1)f(x) \qquad \text{and} \qquad \alpha\, W(x) = \frac{g(x)}{f(x)}. \tag{9}$$

This shows that a necessary and sufficient condition for the function $M$ defined by (8) to be a JLM is (9). The compatibility condition between functions $f$ and $g$, the so called Chiellini condition [15], is:

$$\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right) = -\alpha\,(\alpha+1)f(x)\ , \quad -1 \neq \alpha \neq 0\,. \tag{10}$$

In this case we can find a JLM which can be used for determining a non-standard Lagrangian and the corresponding energy first-integral.

On the other hand, it is to be remarked that if the functions $f$ and $g$ satisfy (10), then we can introduce a new variable $u$ by

$$u = y - \frac{1}{\alpha}\frac{g(x)}{f(x)}, \tag{11}$$

and the given system of differential equations is equivalent to the new system

$$\begin{cases} \dot{x} = u + \dfrac{1}{\alpha}\dfrac{g(x)}{f(x)}, \\ \dot{u} = \alpha\, f(x)\, u \end{cases} \tag{12}$$

because if $x$ and $y$ satisfy (3), then using (11),

$$\dot{u} = \dot{y} - \frac{1}{\alpha}\frac{d}{dt}\left(\frac{g(x)}{f(x)}\right) = -f(x)y - g(x) + (\alpha+1)f(x)\,y = \alpha f(x)\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right) = \alpha f(x)u,$$

while if $x$ and $u$ satisfy (12), then

$$\dot{y} = \dot{u} + \frac{1}{\alpha}\frac{d}{dt}\left(\frac{g(x)}{f(x)}\right) = \alpha\, f(x)\, u + \frac{1}{\alpha}y(-\alpha\,(\alpha+1)f(x)) = -y\, f(x) - g(x).$$

Recall also that in the geometrical approach to Lagrangian mechanics [24,25,26] one identifies the configuration space with a differentiable manifold whose tangent bundle is the the velocity phase space [27,28]. The geometry of the tangent bundle is encoded in two tensor fields: the vertical endomorphism, a $(1,1)$ tensor field $S$, and the Liouville vector field $\Delta$ generating dilations along the fibres (see [27,28]). We can then define the 1-form $\theta_L = dL \circ S$, and $\omega_L = -d\theta_L$, which is a symplectic form when $L$ is regular. In this case the dynamical vector field $\Gamma_L$ is the uniquely defined solution of $i(\Gamma_L)\omega_L = dE_L$, where $E_L = \Delta(L) - L$ is the energy function. Moreover, $\Gamma_L$ turns out to be a second-order vector field, i.e. $S(\Gamma_L) = \Delta$. For such vector fields the dynamical equation is equivalent to $\mathcal{L}_{\Gamma_L}\theta_L - dL = 0$. Local coordinates

$(q^1, \ldots, q^n)$ in the configuration space induce coordinates $(q^1, \ldots, q^n, v^1, \ldots, v^n)$ on its tangent bundle and the local expression of $\theta_L$ and $\Delta$ are:

$$\theta_L(q, v) = \frac{\partial L}{\partial v^i}\, dq^i, \qquad \Delta(q, v) = v^i\, \frac{\partial}{\partial v^i}. \tag{13}$$

It is a well known fact (see e.g. [22,23] and references therein) that in one-dimensional problems there exists a uniquely defined (up to addition of a gauge term) Lagrangian for each JLM whose Euler Lagrangian equations are equivalent to the given SODE. This Lagrangian is such that

$$M = \frac{\partial^2 L}{\partial y^2}, \tag{14}$$

which in our case reduces to

$$\frac{\partial^2 L}{\partial y^2} = \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{1/\alpha},$$

so that a simple integration leads to

$$L(x, y) = \frac{1}{((1/\alpha) + 1)((1/\alpha) + 2)} \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+2} + h_1(x)y + h_2(x).$$

Here $h_1(x)$ and $h_2(x)$ are arbitrary functions of integration, and the term corresponding to $h_1$ may be eliminated because it is a gauge term.

Note that

$$\frac{\partial L}{\partial y} = \frac{1}{(1/\alpha) + 1} \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+1},$$

while

$$\frac{\partial L}{\partial x} = \frac{1}{(1/\alpha) + 1} \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+1} (\alpha + 1) f(x) + h_2'(x), \tag{15}$$

and therefore, as the equation of motion $\mathcal{L}_\Gamma \theta_L - dL = 0$, we obtain

$$\Gamma\left(\frac{\partial L}{\partial y}\right) = \frac{\partial L}{\partial x},$$

that in our case, as

$$\Gamma\left(\frac{\partial L}{\partial y}\right) = \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{1/\alpha} (y(\alpha + 1) f(x) - (f(x)y + g(x)),$$

i.e.,

$$\Gamma\left(\frac{\partial L}{\partial y}\right) = \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+1} \alpha\, f(x),$$

and when comparing with (15), it turns out to be $h_2' = 0$ and consequently $h_2$ is a constant, and the (non-sandard) Lagrangian is

$$L(x, y) = \frac{1}{((1/\alpha) + 1)((1/\alpha) + 2)} \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+2}. \tag{16}$$

The corresponding conserved energy function, $E_L = y\,\partial L/\partial y - L$, is given by

$$E_L = y\,\frac{1}{(1/\alpha)+1}\left(y-\frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+1} - \frac{1}{((1/\alpha)+2)((1/\alpha)+1)}\left(y-\frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+2},$$

and consequently,

$$E_L = \frac{1}{(1/\alpha)+2}\left(y-\frac{1}{\alpha}\frac{g}{f}\right)^{(1/\alpha)+1}\left(y+\frac{1}{1+\alpha}\frac{g}{f}\right). \tag{17}$$

which can also be rewritten as

$$E_L = \frac{1}{(1/\alpha)+2}\left(y-\frac{1}{\alpha}\frac{g}{f}\right)^{1/\alpha}\left(y^2-\frac{1}{\alpha(1+\alpha)}\frac{g}{f}y-\frac{1}{\alpha(1+\alpha)}\left(\frac{g}{f}\right)^2\right). \tag{18}$$

## 3. Non-standard Hamiltonians of Liénard equation and contact structure

The Legendre transformation for the Lagrangian (16) associates each $(x, y) \in T\mathbb{R}$ with a point $(x, p) \in T^*\mathbb{R}$ in the usual manner:

$$p = \frac{\partial L}{\partial y} = \frac{1}{(1/\alpha)+1}\left(y-\frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)+1},$$

and therefore

$$y = \frac{1}{\alpha}\frac{g(x)}{f(x)} + ((1/\alpha)+1)p)^{\alpha/(\alpha+1)},$$

and then the Hamiltonian, which corresponds to the energy function in the phase space, is given by

$$H = p\,y - L = p\frac{1}{\alpha}\frac{g(x)}{f(x)} + p\left(\frac{\alpha+1}{\alpha}\,p\right)^{\frac{\alpha}{\alpha+1}} - \frac{1}{((1/\alpha)+1)((1/\alpha)+2)}\left(\frac{\alpha+1}{\alpha}\,p\right)^{\frac{2\alpha+1}{\alpha+1}}$$

or in a more reduced way,

$$H = \frac{1}{\alpha}\,p\,\frac{g(x)}{f(x)} + \frac{\alpha}{2\alpha+1}\left(\frac{\alpha+1}{\alpha}p\right)^{\frac{2\alpha+1}{\alpha+1}}, \tag{19}$$

which is a non-standard Hamiltonian.

The corresponding canonical equations, under Chiellini condition (10), are:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \dfrac{1}{\alpha}\dfrac{g(x)}{f(x)} + \left(\dfrac{\alpha+1}{\alpha}p\right)^{\frac{\alpha}{\alpha+1}}, \\ \dot{p} = -\dfrac{\partial H}{\partial x} = (\alpha+1)\,f(x)\,p. \end{cases} \tag{20}$$

### 3.1. *Constant of motion of the damped oscillator from non-standard Hamiltonian*

Let us now consider the second-order differential equation describing the damped oscillator,

$$\ddot{x} + \gamma\,\dot{x} + x = 0, \qquad \gamma \in \mathbb{R}, \tag{21}$$

with associated vector field

$$\Gamma_{\rm do} = y\frac{\partial}{\partial x} - (\gamma\,y + x)\frac{\partial}{\partial y}, \tag{22}$$

which is a particular case of Liénard equation corresponding to $f(x) = \gamma$ and $g(x) = x$, and then in this case

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \frac{d}{dx}\left(\frac{x}{\gamma}\right) = \frac{1}{\gamma},$$

i.e. Chiellini condition is satisfied for $\gamma^{-2} = -\alpha(\alpha+1)$.

Note also that the last term in (18) becomes in this case

$$\left(y - \frac{x}{\alpha\gamma}\right)\left(y + \frac{1}{1+\alpha}\frac{x}{\gamma}\right) = y^2 - \frac{1}{\alpha\,\gamma^2}\frac{1}{\alpha+1}x^2 + \left(-\frac{1}{\alpha\,\gamma} + \frac{1}{(\alpha+1)\gamma}\right)x\,y$$

that using Chiellini condition reduces to $y^2 + \gamma\,x\,y + x^2$. This shows that the energy of the system is

$$E_L = \frac{1}{(1/\alpha)+2}\left(y - \frac{1}{\alpha}\frac{x}{\gamma}\right)^{1/\alpha}(y^2 + \gamma\,x\,y + x^2).$$

On the other hand, we can also see that

$$\Gamma_{\rm do}\left(y - \frac{1}{\alpha}\frac{x}{\gamma}\right)^{1/\alpha} = \frac{1}{\alpha}\left(y - \frac{1}{\alpha}\frac{x}{\gamma}\right)^{(1/\alpha)-1}\left(-y\frac{1}{\alpha\gamma} - (\gamma\,y + x)\right),$$

that using Chiellini condition for this case can be rewritten as

$$\Gamma_{\rm do}\left(y - \frac{1}{\alpha}\frac{x}{\gamma}\right)^{1/\alpha} = \gamma\left(y - \frac{1}{\alpha}\frac{x}{\gamma}\right)^{1/\alpha}.$$

This relation shows that along the time evolution the quantity $(y - x/(\alpha\gamma))^{1/\alpha}$ takes a value proportional to $e^{\gamma t}$, and therefore the quantity

$$I = e^{\gamma t}(\dot{x}^2 + \gamma\,x\,\dot{x} + x^2)$$

is constant for the damped harmonic oscillator (21) [29,30].

In the more general case of arbitrary functions $f$ and $g$ satisfying (10) we have that

$$\begin{aligned}\Gamma\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{1/\alpha} &= \left(y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y}\right)\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{1/\alpha}\\ &= \frac{1}{\alpha}\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{(1/\alpha)-1}\alpha f\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right) = f\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^{1/\alpha}.\end{aligned}$$

In full similarity with the previous particular case this relation shows that along the time evolution the quantity $F(x,y) = (y - g(x)/(\alpha\, f(x)))^{1/\alpha}$ is such that $\varphi(t) = F(x(t), y(t))$ takes a value proportional to

$$\varphi(t) \propto \exp\left(\int^t f(x(\zeta))\, d\zeta\right), \tag{23}$$

and as the energy is a constant of the motion, the quantity

$$I = \varphi(t)\left(y^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}y - \frac{1}{\alpha(1+\alpha)}\left(\frac{g}{f}\right)^2\right)$$

is constant for the dynamics provided by a vector field (4) satisfying Chiellini condition.

### 3.2. *From a non-standard Hamiltonian to a contact Hamiltonian*

Coming back to the damped harmonic oscillator (21), one can see it admits a description in terms of contact geometry. We introduce a new variable $s$, consider in $\mathbb{R}^3$ the surface defined by $s(x,y) = xy$, for damped oscillator, but $s$ differs from system to system. In particular we will see that for the Liénard equation $s$ is proportional to $(g(x)/f(x))y$. Define the vector field in $\mathbb{R}^3$

$$\bar{\Gamma}_{\rm do} = y\frac{\partial}{\partial x} - (\gamma\, y + x)\frac{\partial}{\partial y} + (y^2 - x^2 - \gamma\, s)\frac{\partial}{\partial s}, \tag{24}$$

whose integral curves are the solutions of the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \gamma\, y \\ \dot{s} = y^2 - x^2 - \gamma\, s \end{cases} \tag{25}$$

This set of equations of motion describes a system with a friction force that depends linearly on the velocity. In general, for arbitrary Hamilton $H$ this set can incorporate the description of a much larger class of models, like generalised dissipative systems and systems in equilibrium with a heat bath, i.e. the so-called *thermostated dynamics*, where $s$ or a function of $s$ characterises effectively the interaction with the environment.

Recall that a contact manifold is a pair $(M, \eta)$ where $\eta$ is a 1-form in the $(2n+1)$-dimensional manifold $M$ such that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form in $M$. There is a uniquely defined vector field $E$ (called Reeb or characteristic vector field) such that (see e.g. [31,32,33,34])

$$i(E)\eta = 1, \qquad i(E)d\eta = 0. \tag{26}$$

For each function $f$ in $M$ there is a contact Hamiltonian vector field $X_f$ such that

$$i(X_f)\eta = f, \qquad i(X_f)d\eta = df - E(f)\,\eta. \tag{27}$$

It is easy to check that if $c \in \mathbb{R}$, then $X_c = c\,E$ and in particular $X_1 = E$, and that $X_{f_1+f_2} = X_{f_1} + X_{f_2}$, while $X_{f_2 f_1} = f_2\,X_{f_1} + f_1\,X_{f_2} - f_2\,f_1\,E$. Furthermore, $X_f(f) = f\,E(f)$ and more generally, $X_f(f^k) = k\,f^k\,E(f)$. This contact structure defines a Jacobi structure where the bivector field $\Lambda$ is defined by $\Lambda(df_2, df_1) = d\eta(X_{f_2}, X_{f_1})$. The analogous to Darboux theorem for symplectic manifolds establishes the existence of a set of coordinates $(s, q^i, p^i)$, with $i = 1, \dots, n$, such that

$$\eta = ds - \sum_{i=1}^{n} p_i\,dq^i \tag{28}$$

In such a coordinate set the expressions of $E$, $\Lambda$, $\{\cdot,\cdot\}$ and $X_f$ are:

$$\begin{aligned}
E &= \frac{\partial}{\partial s}\\
\Lambda &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i} + p_i\,\frac{\partial}{\partial s}\wedge\frac{\partial}{\partial p_i}\right)\\
\{f_1, f_2\} &= \left(f_1 - \sum_{i=1}^{n} p_i\frac{\partial f_1}{\partial p_i}\right)\frac{\partial f_2}{\partial s} - \left(f_2 - \sum_{i=1}^{n} p_i\frac{\partial f_2}{\partial p_i}\right)\frac{\partial f_1}{\partial s} + \sum_{i=1}^{n}\left(\frac{\partial f_1}{\partial q^i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q^i}\frac{\partial f_1}{\partial p^i}\right)\\
X_f &= \left(\sum_{i=1}^{n} p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial s} + \sum_{i=1}^{n}\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial s}\right)\frac{\partial}{\partial p_i}
\end{aligned} \tag{29}$$

In the particular case of the damped harmonic oscillator the manifold $M$ is $\mathbb{R}^3$ and $\eta$ is then $\eta = ds - y\,dx$, and then the contact Hamiltonian vector field defined by the function $f$ is:

$$X_f = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \left(\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial s}\right)\frac{\partial}{\partial y} + \left(y\frac{\partial f}{\partial y} - f\right)\frac{\partial}{\partial s}$$

and choosing the function $f(x, y, s) = \frac{1}{2}(y^2 + 2\gamma\,s + x^2)$, which plays the role of Hamiltonian, we see that the corresponding contact Hamiltonian vector field is (24).

In the case of the Liénard equation things are not so simple. If we define

$$s = -\frac{1}{\alpha(1+\alpha)}\,\frac{g}{f}\,y\,, \tag{30}$$

then we obtain

$$\begin{aligned}
\dot{s} &= -\frac{1}{\alpha(1+\alpha)}\frac{d}{dx}\left(\frac{g}{f}\right)y^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\,(-f(x)\,y - g(x))\\
&= f(x)\left(y^2 + \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\,y + \frac{1}{\alpha(1+\alpha)}\left(\frac{g}{f}\right)^2\right).
\end{aligned}$$

Now once again we can choose as Hamiltonian the function

$$h(x, y, s) = y^2 - \frac{1}{\alpha(1+\alpha)}\left(\frac{g}{f}\right)^2 + s,$$

and then a straightforward computation shows

$$\dot{s} = f(x)\left(y\frac{\partial h}{\partial y} - h\right). \tag{31}$$

Unfortunately we cannot recast the Liénard equation in contact mechanics form, unless the function $f(x)$ is a constant. Similar result is obtained when the system (12) is considered and the contact 1-form $\eta = ds - u\,dx$.

## 4. Deformation of Lagrangians

We will show that the Lagrangian derived here is a *deformation* of a more elementary Lagrangian $L$. Consider a differential function $\phi : \mathbb{R} \to \mathbb{R}$, then for a Lagrangian $L$, the deformation of Lagrangian function is $\phi(L)$ (see [19], and [20] for the $t$-dependent case).

Using the geometrical approach to Lagrangian dynamics mentioned before, we can get the relation between the equations of motion for a Lagrangian $L$ and for its deformation $\phi(L)$ as given by [19]:

**Theorem 1.** *Let be $\phi : \mathbb{R} \to \mathbb{R}$ a differentiable function and $L$ a regular Lagrangian for a SODE vector field $\Gamma_L$, i.e. such that $i(\Gamma_L)\omega_L = dE_L$. Then, the equations of motion for $\phi(L)$, $\mathcal{L}_{\Gamma_{\phi(L)}}(\theta_{\phi(L)}) - d(\phi(L)) = 0$, are equivalent to*

$$\Gamma_{\phi(L)}(\phi'(L))\,\theta_L + \phi'(L)(\mathcal{L}_{\Gamma_{\phi(L)}}\theta_L - dL) = 0. \tag{32}$$

Note that equation (32) can be rewritten as

$$\phi''(L)\,\Gamma_{\phi(L)}(L)\,\theta_L + \phi'(L)(\mathcal{L}_{\Gamma_{\phi(L)}}\theta_L - dL) = 0. \tag{33}$$

or more explicitly in coordinates,

$$\Gamma_{\phi(L)}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} + \frac{\phi''}{\phi'}\,\Gamma_{\phi(L)}(L)\,\frac{\partial L}{\partial y^i} = 0. \tag{34}$$

### 4.1. *Application to Liénard system*

Let us start by remarking that the Euler Lagrange equation of the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = \left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)^2, \tag{35}$$

assuming that the functions $f$ and $g$ satisfy Chiellini condition (10), is

$$\ddot{x} + \frac{\alpha+1}{\alpha}g(x) = 0. \tag{36}$$

Therefore, the Lagrangian $L$ given in (16) for equation (1), which was derived earlier by using JLM with Chiellini condition, can also be seen as the *deformation* of the simpler elementary Lagrangian $\mathcal{L}$, i.e., up to a factor, $L = \mathcal{L}^{(1/\alpha)+2}$, and therefore $\phi(x) = x^{(2\alpha+1)/(2\alpha)}$. Actually one can check from (34), using Chiellini condition (10), that

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y} &= 2\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right) \Longrightarrow \Gamma_{\phi(\mathcal{L})}\left(\frac{\partial \mathcal{L}}{\partial y^i}\right) = 2(\ddot{x} + \alpha(\alpha+1)\, f\, \dot{x}) \\ \frac{\partial \mathcal{L}}{\partial x} &= 2\left(y - \frac{1}{\alpha}\frac{g}{f}\right)(\alpha+1)\, f \\ \Gamma_{\phi(\mathcal{L})}\mathcal{L} &= 2\left(y - \frac{1}{\alpha}\frac{g(x)}{f(x)}\right)(\ddot{x} + \alpha(\alpha+1)\, f(x)\, \dot{x}) \end{aligned}$$

that together with

$$\frac{\phi''}{\phi'} = \frac{1}{2\alpha}\frac{1}{\mathcal{L}} = \frac{1}{2\alpha}\left(y - \frac{1}{\alpha}\frac{g}{f}\right)^{-2}, \tag{37}$$

this leads to the Liénard equation (1).

## Acknowledgments

One of us (PG) wishes to thank Anindya Ghose Choudhury for many enlighting discussions, he also gratefully acknowledges the Organizing Committee of the XXV International Fall Workshop of Geometry and Physics, ICMAT Madrid, August 30–September 02, 2016, for their kind invitation and gracious hospitality. He is also grateful to Dynamics of Complex Systems 2016 programme at ICTS-TIFR (ICTS/Prog-dcs/2016/05) and Department of Theoretical Physics, University of Zaragoza for their hospitality where part of the work has been done. JFC acknowledges partial financial support by the research projects MTM2015-64166-C2-1-P (MINECO) and DGA E24/1.

## References

[1] A. Liénard, Étude des oscillations entretenues, *Revue Générale de l'électricité* **23** (1928) 901–912; Ibid. (1928) 946–954.
[2] T. Harko, F.S.N. Lobo and M.K. Mak, A class of exact solutions of the Liénard-type ordinary nonlinear differential equation, *J. Eng. Math.* **89** (2014) 193–205.
[3] B. Bagchi, A. Ghose Choudhury and P. Guha, On quantized Liénard oscillator and momentum dependent mass, *J. Math. Phys.* **56** (2015) 012105.
[4] A.G. Choudhury and P. Guha, Chiellini integrability condition, planar isochronous systems and Hamiltonian structures of Liénard equation, *Discrete Contin. Dyn. Syst. Ser. B* (DCDS-B) **22** (2017) 2465–2478.
[5] N.A. Kudryashov and D.I. Sinelshchikov, On the Connection of the Quadratic Liénard Equation with an Equation for the Elliptic Functions, *Regul. Chaot. Dyn.* **20** (2015) 486–496.
[6] N.A. Kudryashov and D.I. Sinelshchikov, On the criteria for integrability of the Liénard equation, *Appl. Math. Lett.* **57** (2016) 114–120.
[7] L. Bougoffa, New exact general solutions of Abel equation of the second kind, *Appl. Math. Comput.* **216** (2010) 689–691.
[8] J.F. Cariñena, J. de Lucas and M.F. Rañada, A geometric approach to integrability of Abel differential equations, *Int. J. Theor. Phys.* **50** (2011) 2114–2124.

[9] J.F. Cariñena and J. de Lucas, Quasi-Lie families, schemes, invariants and their applications to Abel equations, *J. Math. Anal. Appl.* **430** (2015) 648–671.

[10] N. Levinson and O. Smith, A general equation for relaxation oscillations, *Duke Math J.* **9** (1942) 382–403.

[11] C.G.J. Jacobi, Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi, *J. Reine Angew. Math. (Crelle J.)* **27** (1844) 199–268; *J. Reine Angew. Math. (Crelle J.)* **29** (1845) 213–279, 333–376.

[12] D.G. Currie, E.J. Saletan, $q$-Equivalent Particle Hamiltonians I. The Classical one-dimensional case, *J. Math. Phys.* **7** (1966) 967–974.

[13] S. Hojman and H. Harleston, Equivalent Lagrangians: Multidimensional case, *J. Math. Phys.* **22** (1981) 1414–1419.

[14] J.F. Cariñena and L.A. Ibort, Non-Noether constants of motion, *J. Phys. A: Math. Gen.* **16** (1983) 1–7.

[15] A. Chiellini, Sull'integrazione dell'equazione differenziale $y' + Py^2 + Qy^3 = 0$. *Boll. Unione Mat. Italiana* **10** (1931) 301–307.

[16] T. Harko, F.S.N. Lobo and M.K. Mak, A Chiellini type Integrability condition for the generalized first kind Abel differential equation, *Univ. J. Appl. Math.* **1** (2013) 101–104.

[17] S.C. Mancas and H.C. Rosu, Integrable dissipative nonlinear second order differential equations via factorizations and Abel equations, *Phys. Lett.* A **377** (2013) 1434–1438.

[18] A.G. Choudhury and P. Guha, An analytic technique for the solutions of nonlinear oscillators with damping using the Abel Equation, *Discontinuity, Nonlinearity, and Complexity* **6** (2017) 65–74.

[19] J.F. Cariñena and J. Fernández–Núñez, Geometric approach to dynamics obtained by deformation of Lagrangians, *Nonlinear Dynam.* **83** (2016) 457–461.

[20] J.F. Cariñena and J. Fernández–Núñez, Geometric approach to dynamics obtained by deformation of time-dependent Lagrangians, *Nonlinear Dynam.* **86** (2016) 1285–1291.

[21] C.G.J. Jacobi, *Jacobi's Lectures on Dynamics: Second Edition*, Edited by: A. Clebsch, Hindustan Book Agency, New Delhi, 2009.

[22] J.F. Cariñena, J. de Lucas and M.F. Rañada, Jacobi multipliers, non-local symmetries and nonlinear oscillators, *J. Math. Phys.* **56** (2015) 063505.

[23] M.C. Nucci and K.M. Tamizhmani, Lagrangians for dissipative nonlinear oscillators: the method of Jacobi last multiplier, *J. Nonl. Math. Phys.* **17** (2010) 167–178.

[24] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1989

[25] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Addison-Wesley, Redwood City, 1987.

[26] J.F. Cariñena, A. Ibort, G. Marmo and G. Morandi, *Geometry from Dynamics, Classical and Quantum*, Springer-Heidelberg, 2015.

[27] M. Crampin, On the differential geometry of the Euler-Lagrange equations and the inverse problem in Lagrangian dynamics, *J. Phys. A: Math. Gen.* **14** (1981) 2567–2575.

[28] M. Crampin, Tangent bundle geometry for Lagrangian dynamics, *J. Phys. A: Math. Gen.* **16** (1983) 3755–3772.

[29] P. Caldirola, Forze non conservative nella meccanica quantistica Il, *Nuovo Cimento* **18** (1941) 393–400.

[30] E. Kanai, On the quantization of the dissipative systems, *Progr. Theor. Phys.* **3** (1948) 440–444.

[31] P. Liberman and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Translated from the French by Bertram Eugene Schwarzbach, Mathematics and its

Applications **35**, D. Reidel Publ. Comp., 1987.

[32] A. Banyaga and D.F. Houenou, *An Introduction to Symplectic and Contact Manifolds*, Nankai Tracts in Mathematics **15**, World Sci, 2016.

[33] A. Bravetti, H. Cruz and D. Tapias, Contact Hamiltonian mechanics, *Ann. Phys.* **376** (2017) 17– 39.

[34] M. Visinescu, Generalized action-angle coordinates in toric contact spaces, XVII International Conference on Symmetry Methods in Physics (SYMPHYS-XVII) Yerevan, 2017, to be published in: Physics of Atomic Nuclei. See also: arXiv:1704.04034