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Nonstandard Student Conceptions About Infinitesimals

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This is a case study of an undergraduate calculus student's nonstandard conceptions of the real number line. Interviews with the student reveal robust conceptions of the real number line that include infinitesimal and infinite quantities and distances. Similarities between these conceptions and those of G. W. Leibniz are discussed and illuminated by the formalization of infinitesimals in A. Robinson's nonstandard analysis. These similarities suggest that these student conceptions are not mere misconceptions, but are nonstandard conceptions, pieces of knowledge that could be built into a system of real numbers proven to be as mathematically consistent and powerful as the standard system. This provides a new perspective on students' "struggles" with the real numbers, and adds to the discussion about the relationship between student conceptions and historical conceptions by focusing on mechanisms for maintaining cognitive and mathematical consistency.

Key words: Advanced mathematical thinking; College/university; Constructivism; Historical analysis; Learning theories

The conclusions we reach when we research student thinking are highly influenced by the way we regard a conception that differs from the standard ones held by communities of mathematicians. If we treat such a conception as a mere misconception, as a simple lack of knowledge or as an incorrect idea that should be eradicated, then we may miss some important aspects of how that conception functions within a student's understanding. This article offers an extreme example of how examining the functionality and structure of a student's conceptions, rather than dismissing these conceptions as misconceptions, reveals a meaningful structure to the student's conceptions that otherwise might have been overlooked.

THEORETICAL BACKGROUND

During the past 40 years, mathematics education researchers have accumulated a large body of research about student conceptions that differ from standard conceptions. During this time their outlook on such conceptions has undergone two broad changes, one after another. The first general transition was to stop treating such a student conception as a complete lack of knowledge and to start treating it as a piece of knowledge, albeit an incorrect one, a "misconception" (e.g., Schwarzenberger & Tall, 1978). Rather than viewing learners as empty vessels waiting to be filled, the perspective assumed that learners often approach a topic with misconceptions arising from their experiences and that learning occurs when these misconceptions are replaced by correct conceptions. This perspective therefore suggests that it is important for researchers to learn more about these student misconceptions, particularly what kinds of environmental and pedagogical factors create or reinforce them. Studies using this approach have provided useful information not only about what misconceptions often occur but also about how they can arise from overgeneralization or inappropriate transfer (e.g., Matz, 1982).

A subsequent important shift in thinking has been to stop treating learning as a process of replacing misconceptions with correct conceptions, but rather as a process of building new knowledge from prior understandings. This shift has its roots in the Piagetian tradition of genetic epistemology, which treats prior knowledge as something that is never overwritten, but is always integrated within the new knowledge structures (Piaget & Garcia, 1983/1989). This perspective is articulated in more detail by proponents of radical constructivism, who claim that a learner has cause to reconstruct her knowledge only when she experiences a perturbation, an experience or conception that does not fit with her current schema (von Glasersfeld, 1995). With this perspective, the intent of research is less to eliminate "misconceptions" than to understand the nature of conceptions, in order to give learners tasks that induce perturbations leading to the productive restructuring of knowledge. Researchers who use this perspective advocate focusing not upon replacing faulty knowledge but upon refining and reorganizing it (Smith, diSessa, & Roschelle, 1993).

This perspective opens new avenues for inquiry, such as studying the experiences that induce productive perturbation for the learner, but it also entails the acknowledgment of subjectivity in several ways and requires the careful consideration of the role of the observer. For instance, the holder of a given conception has no reason to suspect that it is a misconception unless it is perturbed. This suggests that the term "misconception" should just be replaced with conception (Confrey, 1991), because we have no access to an objective frame of reference by which a conception can be judged to be correct or incorrect. Nonetheless, it is worthwhile to be able to specify when the observer projects that a learner's conception will eventually conflict with one of the learner's other stable conceptions or experiences and will be restructured during the resulting accommodation. We say that the observer considers the learner's conception to be *perturbable*.

There are several ways that subjectivity arises when the objective notion of misconception is replaced with the idea of perturbability. One well-known way is the issue of intersubjectivity: The researcher's interpretation of a student conception is itself a conception, and thus might itself be perturbed and updated due to further experiences with the student (von Glasersfeld, 1995).

A less-explored manifestation of subjectivity is that we have no objective way to know that any conception of ours is unperturbable. The fact that a conception has not yet been perturbed does not mean that it is unperturbable. For instance, suppose I am a teacher and I encounter a student conception that seems to contradict one of mine. I assume it to be perturbable and begin looking for ways to induce a perturbation for the student. If, after a while, I cannot produce an experience that induces a perturbation for my student, I might wonder, is it perhaps my conception—not the student's—that is perturbable? After all, just because my conception differs from hers, and mine has not been perturbed (yet), does not mean that it never will be perturbed. This scenario may sound absurd, especially if, as the teacher, my conception is the standard one taken as shared by other mathematically educated persons. If I cannot readily induce a perturbation for the student, surely it is because I know too little about the student's conceptions and how they relate to one another, not because the standard conception that I hold will someday be perturbed.

But another possibility exists, revealed by this subjectivity: It could still be that neither my conception nor that of my student is perturbable. Perhaps the student's conception can be part of a stable cognitive structure and so can mine. Yet these conceptions still contradict each other, meaning that it is ultimately impossible for either my student or me to simultaneously and stably hold both conceptions.

It is important to note that the student's conception could still soon experience a perturbation, even if I as a teacher do not project it. After all, I may only be paying attention to the mathematical implications of the student's conceptions and ignoring myriad other factors that might give rise to a perturbation. It is certainly not the case that the only sources of perturbation are mathematical inconsistency, although these are the sources of perturbation primarily focused on in this article. For instance, the conception could be perturbed if it someday functions less powerfully than the standard one, or less flexibly across contexts, or less generalizably in novel situations.

But what if the stable structures into which these two competing conceptions can be built are in fact not appreciably different in power and flexibility either? In other words, suppose there is no reason, based on stability, viability, power, or flexibility, that I as the teacher can anticipate preferring one conception to the other. In this scenario, however unlikely, we are justified in calling the learner's conception a *nonstandard conception*. It is a conception that contradicts the standard conception, yet it is no objective "misconception," nor is it a conception that appears to await perturbation by the learner's other conceptions or future experiences due to inconsistency, lack of power, or viability.

There are several important things to notice about a nonstandard conception. First of all, this reasoning is entirely theoretical so far. I have not shown that nonstandard conceptions actually occur, nor have I suggested where one might look to find them; I have simply made a case that they could exist based on a radical constructivist perspective on conceptions.

Second, a nonstandard conception would be an unusual entity, one that to my knowledge does not appear in the literature. To make this point clear, it is worth describing a few things that a nonstandard conception is *not*. A nonstandard conception is not simply an alternate strategy or algorithm. Certainly research has shown that in order to solve problems students often develop strategies and algorithms, sometimes as alternatives to the standard ones they are taught (e.g., Carpenter & Moser, 1984; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). But these are not necessarily nonstandard conceptions, because a novel strategy for, say, adding two-digit whole numbers, need not conflict with any standard conceptions. A student's conception about how the decompose-and-add-like-units algorithm (Fuson, 2003) works for adding two-digit whole numbers need not conflict with her conception of how the standard school addition algorithm works. In fact, a student who holds both of these conceptions might be considered to have a rich understanding of addition, not an inconsistent one.

A nonstandard conception is also not simply a robust misconception. Researchers have often identified student conceptions that are very difficult to change, such as epistemological obstacles, which are described as misconceptions that are both tenacious and seemingly unavoidable (e.g., Brousseau, 1997; Sierpinska, 1987). Nonetheless, these are not nonstandard conceptions. They may be stubborn, but they are still perturbable, and must undergo restructuring in order for the student to have access to more powerful or general mathematics.

Third, nonstandard conceptions indicate a bifurcation in a learning trajectory, a place from which some learners may go one direction and some a different direction. At this crossroad, although there might yet be reasons for the student to take the well-worn path instead of the road less traveled, these reasons would not be purely mathematical ones. The existence of such a bifurcation would be interesting for its own sake and surely would be helpful for teachers to understand. But there is another reason for us to pay attention to it: its potential for informing the relationship between the way that our students learn and conceptualize mathematics and the way that mathematics developed historically. If there are nonstandard conceptions that can be built into structures that are as powerful, consistent, and viable as the standard one, then it would not be surprising if such nonstandard conceptions occurred in the historical development of mathematics as well. The implication goes both ways: If we see nonstandard conceptions in the reasoning of our students, this gives us something to look for in history; if we see nonstandard conceptions in history, it gives us something to look for in our students' reasoning. This would be one of many instances in which a learning theory suggests a significant relationship between student learning and historical development in mathematics (e.g., Brousseau, 1997; Kaput, 1994; Piaget & Garcia, 1983/1989; Sfard, 1992; Sierpinska, 1987; an overview of this body of research appears in Furinghetti & Radford, 2008).

Finally, the existence of nonstandard conceptions would provide insight into how students can have access to deep mathematical ideas that they are not explicitly taught, even ideas that are incommensurable with the ones they are taught. A student who maintains a nonstandard conception despite instruction to the contrary is likely to be a student who thinks clearly and mathematically, pursuing the consistency and coherence of her mental mathematical model rather than caving to the pressure of the classroom.

In this article, I show the existence of some nonstandard conceptions held by a student, Sarah, about the real number line, particularly conceptions that involve infinitesimal quantities and distances. These conceptions are nonstandard because (a) they are incommensurate with standard conceptions of the real number line; (b) they are robust, viable, and apparently not perturbable in the face of instruction that contradicts them; and (c) they could be used to build a cognitive structure as powerful and consistent as the standard conceptual structure of the real number line.

Before describing Sarah's conceptions, I first provide a historical and mathematical background about infinitesimals. Many of Sarah's nonstandard conceptions of infinitesimals are held by no less a mathematician than G. W. Leibniz, whose infinitesimal calculus led to some of the greatest mathematical discoveries of the 18th century. Leibniz' conceptions of infinitesimals were used in coherent, powerful systems of mathematical thought for more than a century.

Although history abandoned the course of infinitesimals, it was demonstrated about 50 years ago that a nonstandard mathematical system can be built from infinitesimals with the same power, consistency, and rigor as the standard system of real numbers taught in school today. I describe in some detail the development of this system, nonstandard analysis, because it provides strong evidence that Sarah's conceptions truly are nonstandard.

HISTORICAL BACKGROUND

Throughout the 1670s and 1680s, Gottfried Wilhelm Leibniz developed the infinitesimal calculus. His system of calculus was largely detailed in personal correspondences, and its first complete exposition was found in L'Hôpital's 1696 calculus textbook, *l'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes.*[1] His system is fundamentally based on infinitesimal quantities, positive quantities that are smaller than any finite positive number. He accomplished the basic operations of calculus by partitioning a curved line into an infinite number of straight "sides" that are infinitesimal in length. A derivative is then the slope of one of these straight sides, and an integral is an infinite sum of infinitesimal polygonal areas.

To calculate these derivatives and integrals, it was crucial for Leibniz to develop an algebra of operations on infinitesimal, finite, and infinite numbers. Leibniz' first published treatment in which he employed this algebra appeared in 1684 (Leibniz, 1684/1969, p. 272–280). For his calculus to work, Leibniz posited that there exist infinitely many infinitesimal numbers, a belief opposed by a number of mathematicians of the time who believed that there exist no infinitesimals or that there is only one (Mancosu, 1996). For Leibniz, the continuum is infinitely divisible, containing no atomic unit.[2] For example, if one raises an infinitesimal to successively larger powers, it will become smaller and smaller. In every neighborhood of a given number, there is a microcosmic world of numbers that looks like the larger continuum. "Since the continuum is divisible to infinity, any atom will be of infinite kinds like a sort of world, and there will be worlds within worlds to infinity [R. Ely, trans.]" (Leibniz 1663–72/1966, p. 241).

Leibniz' algebra of operations on infinitesimal numbers included rules such as $i \times f$ is infinitesimal; $f \div I$ is infinitesimal; $(f_1 + i_1)/(f_2 + i_2) \approx f_1/\mathrm{f}_2$; and $f + i \approx f$ (here i is an infinitesimal number; f, f_1, and f_2 are finite numbers with f_2 nonzero; and I is an infinite number). Although there are at least a dozen such rules and heuristics

[1] Mathematics historian Fred Rickey translates this as *Analysis of the Little-Bitty-Guys for the Study of Curved Lines* (Bressoud, 1994).

[2] Leibniz did describe indivisible units called *monads*, but these are metaphysical and are not directly related to the continuum. For a more complete treatment of monads vs. infinitesimals in Leibniz' metaphysics, see Ross, 1992.

for working with infinitesimals, the most important for calculus are like the latter two specified previously, which show how to simplify difference quotients to get derivatives.

This method of calculus was employed and extended by most of the great continental mathematicians of the 18th century, including the Bernoullis, Euler, and Legendre. During this time, there were vehement attacks on the idea of infinitesimal quantities (Grabiner, 1981), such as the famous paper by Bishop Berkeley in 1734, which specifically targeted Newton's fluxions and laid out foundational problems that were not resolved for over a century. What was an infinitesimal? Could one be produced and examined in the real world? Leibniz at times discussed an infinitesimal as the final term in a sequence of numbers approaching zero, but this did not particularly clarify matters. For example, what is the "final term"? Historian H. Bos (1974) suggested that this vagueness was a crucial element that led to the rapid development of calculus, but which also ultimately caused the system to be rejected on grounds of mathematical and philosophical rigor. Even though the idea was vague, mathematicians still readily worked with infinitesimals for two reasons: They were relatively intuitive and they kept producing important results. In the 18th century, Leibniz' "useful fictions" produced an explosion of mathematical discoveries in mechanics, calculus of variations, probability theory, astronomy, and more.

But in the beginning of the 1800s, mathematicians began encountering counterintuitive and even contradictory results arising from a cavalier treatment of convergence, many of which surfaced in a discussion of Fourier's 1807 paper on trigonometric series (Bressoud, 1994). The informal reliance on the intuition of infinitesimal quantities did nothing to resolve these debates, so mathematicians such as Cauchy and Bolzano independently worked to develop a foundational system of limits that could be used to ground the behavior of infinite series and functions. The rigorization of calculus culminated in the $\varepsilon - \delta$ definition of limit, ultimately formalized by Weierstrass in the 1860s, marking the general disappearance of infinitesimals from the foundations of advanced calculus (Grattan-Guinness, 1970).

It was curious that infinitesimals, which supported a century of lively mathematical discovery, could not be formalized in such a way as to create a rigorous system of calculus. It seemed strange that these entities could be used so intuitively to discover calculus-based results, and yet would have to be jettisoned when it came time to check one's proofs. The reason became clear in another century: It was not infinitesimals that were at fault, but rather that the field of mathematical logic was not developed enough to allow for a rigorous treatment of infinitesimals until the work of Abraham Robinson (1961, as cited in Robinson, 1996).

The logicians of the early 1900s followed Hilbert's program to establish mathematical proof as a set of rigorous formal operations that do not result in paradoxes or contradictions. In order to view mathematical statements as abstract entities apart from any particular systems to which they referred, mathematicians such as Frege and Russell worked to develop (a) a formal language in which all mathematical statements could be written and (b) a set of purely syntactical rules according to which new mathematical statements could be produced from old ones. The goal was then to determine whether these abstract proof rules governed what was true or not in the real mathematical worlds (models) to which these statements apply.

In studying these various mathematical models, Lowenheim and Skolem discovered that it is possible to find two different models that look different from the outside, but the same from the inside. More precisely, all of the first-order logical statements that hold true about the objects in one model also hold true about the objects in the other model. Yet there are more global statements that can be made about the models themselves that are true about one of the models but are not true about the other model. In this way, it is possible to have substantively different models, but ones in which all the same statements are true within each model.

The first example was the "nonstandard" model of arithmetic (Skolem, 1934). The standard model of arithmetic is the set of natural numbers $\mathcal{N}$ (with operations such as +, −, ×, ÷, and relations such as =, <, >)—a world in which an abstract symbolic statement such as "$\forall v_o, R(v_o, v_o)$" can be interpreted meaningfully (e.g., "For every natural number *n*, *n* equals *n*.") and determined as true or false. The nonstandard model $\mathcal{M}$ developed by Skolem is the same as the standard model $\mathcal{N}$, but with one notable difference: It also contains infinite numbers. In other words, $\mathcal{M}$ contains all standard counting numbers, and it also has some numbers that are larger than any of these counting numbers. Within the two models, all statements that hold true in one model hold true in the other: "2 + 2 = 4," "every even number is divisible by 2," "every number has a successor," and so on. This means that both models are equally powerful and equally consistent and that both could be used to prove statements in arithmetic. Yet, from outside the models, we can see that the models are different; the nonstandard model contains infinite numbers and many of them. This difference between the two models is possible because there is no way to make a first-order logical statement that says, "*n* is an infinite number."

Using a similar technique, in 1961 Abraham Robinson developed the most important nonstandard model of a mathematical system. This was a nonstandard model of analysis, a mathematical world in which Leibniz' infinitesimal calculus can be formalized. By "analysis" I mean the set of formulas that characterizes advanced calculus, statements such as "every Cauchy sequence converges" and "every absolutely continuous function is an integral of its derivative." The standard model of analysis is the set of real numbers and its operations and relations. The nonstandard model of analysis is the same, except that it contains infinitely small and infinitely large numbers as well. The set with the real number line and these infinite and infinitesimal numbers is called the *nonstandard real numbers*, or *hyperreal numbers*.

Not only did Robinson show that there are infinitesimal and infinite numbers in his model, he also formalized how to understand these numbers in comparison to finite numbers and how to operate with them. For example, an infinitesimal number can be viewed as an infinite sequence of numbers that converges to 0 (actually an equivalence class of such sequences). Likewise an infinite number can be constructed as an infinite sequence of finite numbers that approaches infinity.

Remarkably, the same set of operations concerning combinations of finite, infinite, and infinitesimal numbers that worked in Leibniz' informal system also held in these nonstandard real numbers. In tribute to Leibniz, Robinson called an infinitesimal neighborhood of points a *monad*.

The most important of Robinson's results is that with this formalization of infinitesimal and infinite quantities, every first-order formula that holds true in the standard real numbers also holds true in the nonstandard real numbers, and vice versa. This "transfer principle" implies that these two systems are equivalently powerful and consistent. This fact has led to some important mathematical discoveries. In particular, if a theorem is difficult to prove using standard analysis, it might be easy to prove using nonstandard analysis. Since we know that the same theorem must be true in both models, it does not matter which model is used to prove the result. For example, the proof of the intermediate value theorem, which is surprisingly complicated in standard analysis, is quite simple using nonstandard analysis. Another example is the work of Albeverio, Fenstad, Høegh-Krohn, and Lindstrøm (1986), which uses nonstandard analysis to prove new results about stochastic processes.

The nonstandard real numbers provide a satisfying vindication of Leibniz' infinitesimal system. Robinson proved that the system that Leibniz used implicitly can be explicitly shown to be as consistent and powerful as the standard version of analysis. This means that calculus can be done using infinitesimals with a clean mathematical conscience, and, in fact, several calculus textbooks and other resources have been written that teach the subject using a simplified version of this approach (Henle & Kleinberg, 1979; Keisler, 1986, 2007). Nonetheless, although the nonstandard model of the real numbers marks a powerful, coherent, and mathematically correct mode of thought, it is one that is substantially different from the standard real number system taught in today's classrooms.

In the next sections, I show how a student uses elements of this nonstandard model in her own thinking about the real numbers, despite being taught the standard system in her classes. Her conceptions involve not only fledgling versions of intuitive Leibnizian infinitesimals but also include methods of visualization and manipulation that resemble Robinson's construction of the infinitesimal elements in his system of nonstandard analysis.

METHOD

Data Collection

This case study developed out of a larger study, in which 233 university calculus students completed a Calculus Conceptions Questionnaire on the 1st day of their 1st or 2nd semester of a yearlong differential and integral calculus sequence. Their responses were then used to catalogue their conceptions about various calculus concepts: limits, functions, continuity, and the real number line. Six of these students, two males and four females, participated in follow-up interviews. The main purpose of these interviews was to clarify some of their conceptions in order to refine the coding scheme for the larger study. I had identified several response patterns that were enigmatic and difficult to code but were displayed by multiple students, so I selected students to interview who had displayed these response patterns, based on their availability and willingness to be interviewed. Each interview was about 30 minutes long, and was audiorecorded. Students' written work as well as the interviewer's notes about the written referents for the students' comments supplemented the audiorecordings.

Since each of the six students that I interviewed had different questionnaire responses that required clarification, I used different interview protocols for each interview. However, there was one questionnaire item—Item 6—that I included in all six of the interview protocols (see Figure 1). I chose it because I was surprised that most of the students (83%) answered 6b as "true." Because it is only a true–false item, the results from the questionnaire were limited in showing me why students responded the way that they did, so I wanted to follow up on it in all the interviews. As is typical for interviews that are semistructured, these protocols were designed to be flexible in adapting to and pursuing the participants' responses (Denzin, 1989).

6. True or false:

a. T F It is possible to choose two different points on the real number line that are touching one another.

b. T F It is possible to choose two different points on the real number line that are infinitely close to one another.

Figure 1. Questionnaire Item 6.

The subject for this case study is Sarah, a sophomore whose high school calculus course excused her from taking Calculus I at the university. At the time of the interviews, she was taking Calculus II, and had taken a course on Finite Mathematics the previous year. At the university, Calculus I and Calculus II are each semester-long courses that do not require the students to construct proofs. The first includes calculating limits, derivatives, and integrals, and the second includes more integrals, sequences and series, parametric equations, and some differential equations.

The reason that I chose Sarah to be one of the interviewees had more to do with the ambiguous way that she responded to two of the items about functions than with her answers to the questions about the real number line. In the interview, I again presented her with these items about functions, and she answered them and described her thoughts about them to my satisfaction. It was her responses about the real number line that are of interest in this article. I found these responses fascinating and surprising, and the follow-up questions by which I pursued them were for the most part unscripted.

I asked Sarah to answer Item 6, and then I followed up on it with several questions to clarify her answer, including:

- Can you provide an example of two different points that are infinitely close to one

another, but are not touching?
- What is between those two points? Can you write it down for me?
- Can you find a point that is infinitely close to both of those points?

Sarah's responses to these questions surprised me, and my prepared interview protocol was not detailed enough to pursue them. Because I wished to pursue her ideas about the real number line, I conducted a second interview with her 1 week later. I prepared another interview protocol that began with the same questions as listed previously but continued with questions such as:

- What would you get if you took this (infinitely small) number and squared it?
- What would you get if you took 1 over this number?
- If you took 1 over this other (infinitely small) number, how would that compare to 1 over this one?

The first interview revealed that Sarah believed in infinitesimal numbers. The purpose of the questions in the second interview was to further explore these conceptions and to determine whether she would pursue them coherently. Each of the two interviews lasted about 45 minutes.

Data Analysis

Because the purpose of this case is to provide insight into the issue of alternate mathematical structures found in student thinking, it is an *instrumental* rather than *intrinsic* case study (Stake, 2000). As such, any account of Sarah's thinking must be understood with respect to how my own concerns and conceptions inform the interview and my analysis of the interview. For example, I was not expecting conceptions relating to nonstandard analysis to emerge through the course of the interview. However, as I interpreted the interview data, my experience with the construction of the nonstandard real numbers and my knowledge of the infinitesimal calculus of the 17th century both served as lenses for interpretation.

I did not originally design this interview around a case study about infinitesimals, with a carefully devised interview protocol and coding scheme. Sarah's ideas about infinitesimals were surprising to me, and my follow-up questions were devised on the spot. This means that my coding scheme had to emerge gradually as I analyzed the interviews. For this reason, my coding scheme categories first developed very loosely through a process of open coding (Strauss & Corbin, 1990). First I looked at the specific notations and terminology that Sarah uses. An example of a notation she uses is "0.000. . .1," and a term that she consistently uses is "infinitely close." I then looked for other places in the interviews in which she used the same or similar notations or terms. Then I looked at the mathematical claims that she makes (true or untrue) about the real numbers. An example of such a claim is that 0.000. . .1 squared is infinitely close to zero. I looked for other places in the interview in which she makes the same or a very similar claim. After identifying the notations, terms, and claims she makes, I put them into categories of consistent usage. A consistent usage of a notation, term, or claim indicates a conception that is rather stable, for example, "Infinitesimal numbers can be written as infinite decimal expansions with some extra decimal digits at the end." Sarah uses such notation many times over the interviews and refers to these numbers as being "infinitely small" or "infinitely close."

Once the conceptions had been identified, I determined whether each was interpretable in, was conventionally used in, and was consistent with the other properties of the standard real numbers. For instance, in the standard real numbers, the notation 0.000. . .1 is not conventional, and the claim that infinitely small numbers exist is false. Then I determined whether each conception could be interpreted in, was conventionally used in, and was consistent with the other properties of the infinitesimal systems of Leibniz and Robinson. For instance, the claim that there exist infinitely many infinitely small numbers is consistent with these systems. I treated a conception to be nonstandard if it could not be interpreted as being true in the standard real number system, but if it could be interpreted to be consistent in a model such as Leibniz' or Robinson's that includes infinitesimals. Finally, I looked for logical relationships between the conceptions and mathematical implications of the conceptions to determine whether the conceptions were consistent with each other.

RESULTS

In this section I describe Sarah's nonstandard conceptions about the real number line, illustrated by excerpts from the interviews, and I detail the reasons for interpreting her remarks the way I do. Some of these conceptions are easily detected, because they are beliefs that she describes and defends explicitly. Other conceptions she does not explicitly state, but rather acts in accordance with them.

Sarah's Understanding of "Infinitely Close" and the Divisibility of the Number Line

I begin Sarah's first interview by asking her to respond to Item 6 (see Figure 1). To 6a, she answers "false," that it is not possible to find two different real numbers that are "touching" one another. She responds, "You can keep halving it, like cutting it in half and cutting it in half. You can get closer and closer and closer, but there's always gonna be like a little space between it." She had also responded "false" on this item on the original questionnaire, an answer shared by 90% of the other respondents. Her response and her interview comments about cutting in half again and again suggest that she believes that the number line does not have atomic units, but is rather arbitrarily divisible. Later in the interviews, as we shall see, she consistently extends this property to infinitely small spaces as well.

To 6b, Sarah answers "true," that it is possible to find two different real numbers that are "infinitely close" to one another. This is the same way that she answered on the original questionnaire, and the same way that 83% of the other students responded as well.

I then ask Sarah if she could provide an example of two numbers that are "infinitely close to one another," and she replies "3.999999 repeating forever" and "4." To get a sense of what she means, I pursue this by asking if she could find another number that is infinitely close to those two:

I: Okay. And can you find another number that's infinitely close to both of those numbers? Is it possible to find another number that's infinitely close to both of those two?

S: Yes, because the repeating is you don't know how long it's repeating. If it's repeating for infinity, there's always going to be one more to infinity. You can always add 1 to infinity. So you can always add another 9 to the infinity of 9s that are coming.

I: Okay. So how would you write down like the two different numbers that are like that? Or if you can, I don't know.

S: How would you write . . . ?

I: How would you explain it? Like . . .

S: You mean like . . . . Well, you could just say like three point nine nine nine whatever repeating forever to infinity, and 4. And then in between that is like three point nine nine nine to infinity plus [pause] an infinitely small number.

I: Plus an infinitely small number?

S: Yeah, uh-huh. [She laughs.] [I1 03:06-04:32]

In this section, Sarah describes two numbers that she claims to be infinitely close to each other: "3.999 repeating forever to infinity" and "4." She then says that these numbers are both infinitely close to "3.999 repeating forever plus an infinitely small number."

This conception that there exist numbers that are infinitely close together is quite definitely a misconception about the standard real numbers. In the standard real numbers, two numbers are either the same or else they are a particular finite distance apart; the term "infinitely close" does not even make sense.

So what does Sarah understand when she hears "infinitely close"? One view might be that she really is just envisioning the standard real numbers, and she hears "very close." Student misunderstandings certainly can arise from the fact that in colloquial usage "infinitely" often means "very." For instance, another interviewed student in this study claimed that 1.001 and 1.002 were "infinitely close" but 1.1 and 1.2 were not (Ely, 2007). But it is not clear that this simple misconception is Sarah's view. For instance, to illustrate "infinitely close" she does not choose numbers such as 1.001 and 1.002. Rather, she chooses examples with infinite decimal representations, which she consistently describes as having an "infinity of 9s" that are "repeating forever to infinity." Perhaps even more telling is that she admits throughout the interview that these numbers she writes that are "infinitely close" to other numbers, such as "3.999 repeating forever plus an infinitely small number," are not actually "real," but are numbers that she is inventing. For instance, she says that 0.000. . .1 (point zero repeating infinitely with a 1 after it) is "not really a number," and in the interviews she is sheepish about using these objects. If Sarah just takes "infinitely close" to mean "very close," then it seems that she would not describe this by using numbers that she admits to be fabricated.

At this point in the interview, I choose to interpret Sarah's remarks from the perspective that she is envisioning things that are infinitely close, rather than very close. This immediately raises the question about the features she envisions in a number system that supports a meaningful interpretation for "infinitely close." At this point, it would be too early to say whether or not Sarah's conception that there exist numbers that are infinitely close to each other is a misconception or a nonstandard conception. Only as she develops notation for describing infinitely close and infinitely small numbers, and describes the features of these numbers, does it become apparent that her conceptions are indeed nonstandard.

The following is a nonstandard conception evidenced in Sarah's responses:

- There exist numbers that are infinitely close to each other.

Sarah's Idea of and Emergent Notation for Infinitesimals

As I investigate Sarah's conceptions of the number line, she begins to develop a notation for expressing what she means by "infinitely small" numbers and distances. In the following excerpt, I have just asked her what is between $0.99\overline{9}$ and 1:

S: If there was a way to express something like . . . I mean, this is not real at all . . . but like, zero repeating forever and then 1, then I would say that. Like an infinitely minuscule . . . hmm . . .

. . .

I: Is there . . . are there any numbers between 0.9 repeating and 1?

S: Um . . . I mean, this isn't really a number. [She writes 0.000. . .1.] This isn't real, so . . .

I: Point 0 repeating with a 1?

S: Yeah. But um . . . I don't know. I mean . . . There is numbers, because no matter how small you get it there's still going to be some kind of space, and even in that tiny infinitely small space you can still cut that into infinity too and put numbers in there. So I would say yes but I don't know how to express that.

I: So how to express a number that's in there?

S: Yeah. [I1 08:25-10:10]

Although I am asking about a number that resides between 0.999. . . and 1, Sarah seems to be describing the distance between the two numbers. In the second interview, when she does this again, I ask her about it and she corrects herself [I2 03:50-04:24].

In this transcript, Sarah writes a number that she admits "isn't real": 0.000. . .1. This number is an example of what she calls an "infinitely small number," as she mentions in the earlier transcript. The fact that she acknowledges that it "isn't really a number" suggests that she is not simply thinking of a very small number with a large, but finite, number of zeroes. She is, in fact, envisioning an infinite number of zeroes. She consistently affirms this stance in the second interview as well, in which she again claims that between 0.999. . . and 1 there exists "an infinitely small space," and that between these two numbers there are "infinitely small numbers." It is important to notice that even though I am referring to her "infinitely small numbers" as infinitesimals, she never uses the word *infinitesimal*.

Solely from her comments about infinitely small spaces and numbers, it would be difficult to determine what Sarah means. However, she develops an explicit notation for expressing these infinitesimal numbers and distances, which she often uses throughout both interviews. In this notation, it is possible to represent numbers using an infinite string of decimal digits, which can be followed by yet more digits. This notation not only enables her to represent infinitesimal numbers but also to generate new infinitesimal numbers from old ones. For example, in the second interview I ask her again about what is between 0.999. . . and 1, noting that we had discussed this briefly in the first interview:

I: I asked you what, are there any numbers between 0.9 repeating and one?

S: Um . . . yes.

I: And what did you, what did you, yeah . . .

S: Infinitely small numbers.

I: Infinitely small numbers, okay, and what was, what would be an example of a number like that?

S: Like . . .

I: You can write it if you want.

S: Like . . . point zero repeating one [writes 0.000. . .1].

I: With a one? Okay. And how many zeroes are here?

S: An infinite number of zeroes.

I: Okay, infinite number of zeroes. And then a one. Okay, so, I think, yeah, this is what you said last time, and I just wanted to make sure that . . .

S: Yeah, as many nines that are here [points to 0.999. . .] are as many zeroes are there [points to 0.000. . .1].

I: Right, and if there are infinitely many, then they're infinitely many there?

S: Yeah.

I: Okay. Um, so now I ask you, are there, are there other numbers between here and here [between 1 and 0.9 repeating]? Like . . .

S: Um, maybe an even smaller one, like . . . hahaha [writes 0.000. . .01].

I: Okay . . .

S: If you can do that.

I: No, that, how many numbers are between here and here? [Points to 0.999. . . and 1.]

S: An infinitely small, an infinite amount of infinitely small numbers. [I2 00:48-02:00]

She claims that in her notation of "infinitely small numbers" there are infinitely many digits, followed by more digits. In this notation, it is important to attend to the numbers that come after the infinite string of digits: 0.000. . .1 and 0.000. . .01 are different numbers. Her notation, which is a natural extension of the standard decimal notation, serves to make her ideas about infinitesimals more explicit—although at this point she is still conflating numbers with distances.

Sarah appears to be generating this notation, and the conclusions that it affords, through the course of the interviews. She already held a belief that there are infinitely small distances and numbers, as evidenced in her responses on the initial questionnaire, but her way of describing them is emergent. In the beginning of the first interview, she just states "an infinitely small number." By the end of the second interview, she has developed a notation for such numbers and has explored their properties extensively.

Her notation seems suggestive to her. By simply adding a different digit at the end of the infinite decimal expansion, she is able to describe another infinitely small number. In the preceding excerpt, this apparently enables her to generalize that there must be infinitely many such numbers, presumably because she could generate arbitrarily many such nonstandard expansions.

The following additional nonstandard conceptions were identified in Sarah's responses:

- There exist infinitesimal numbers and infinitesimal distances.
- Infinitesimal numbers can be written as infinite decimal expansions with some extra decimal digits at the end.

Affordances of Sarah's Notation: Infinitely Many Infinitesimals

Sarah's emergent notation for infinitesimal numbers allows her to posit properties of infinitesimal numbers that she otherwise may not have been able to describe. One such property is that infinitesimal numbers can be ordered. For instance, in the following transcript, she orders two infinitesimal numbers. Both 0.000. . .01 and 0.000. . .1 are "infinitely close to zero," but the latter is still "a little bigger than" the former. I ask her where the number 0.000. . .1 would live on the number line, not as the distance between 0.999. . . and 1 but as the number itself:

S: Oh, I would say that this number [points to 0.000. . .1] would be infinitely close to zero.

I: Oh, okay.

S: So would this number [points to 0.000. . .01].

I: And so would that number? Now, how would they relate to each other compared to zero? Like . . . like . . .

S: Like oh . . . .

I: Where would you put both of these two numbers?

S: This one [points to 0.000. . .1] would just be a little bigger than that one [points to 0.000. . .01]. [I2 04:58-05:19]

Her emergent notation has also enabled her to claim that there exist infinitely many infinitesimal numbers. This can be seen in the previous excerpts. When I ask her if there are any numbers between 0.999. . . and 1, she replies that “even in that tiny infinitely small space you can still cut that into infinity, too, and put numbers in there.” When I ask her how many, she responds, “An infinitely small, an infinite amount of infinitely small numbers.” When I ask her what the numbers in there look like, she says, “Well, they would look like, they would look [like] the big number line except, they would look exactly like the big number line except they wouldn’t be called the same numbers.”

These conceptions are misconceptions with respect to the standard real numbers. In the standard real numbers, there are no numbers between 0.999. . . and 1. For Sarah, there are infinitely many numbers, even though it is an infinitesimal space, and this miniature number line looks like the big number line. These conceptions are all features of Leibniz’ system of infinitesimals and of Robinson’s nonstandard real numbers as well. In particular, in both Leibniz’ and Robinson’s system, infinitesimals are generative—once we posit that an infinitesimal number exists, we can generate, and order, infinitely many infinitesimal numbers from this one. In his correspondence with the early 18th-century mathematician Nieuwentijt, who believed that there exists only one order of infinitesimal number, Leibniz asserted that any consistent system of infinitesimals must contain infinitely many infinitesimals, and that these must be orderable and manipulable by means of arithmetic operations (Mancosu, 1996).

The fact that Sarah’s conceptions accord with Leibniz rather than Nieuwentijt on this point suggests that she may be motivated by a desire to be consistent with the affordances of her notation. The notational process by which she came to posit the existence of one infinitesimal number allows also for the existence of many other infinitesimals, so consistency forces her to acknowledge these infinitesimals as well.

I also ask Sarah whether there are any rational numbers between 0.999. . . and 1. After puzzling for a while, she replies:

S: I know that there has to be a rational nu—my principles say there has to be a rational number in there. Because it’s a tiny tiny space and there’s gotta be a rational nu . . . like the tiniest little space . . . [pause]. Because in infinity there’s still an infinitely small space, and in that infinitely small space . . . I mean, it’s an infinitely small space, but it’s infinity. So in infinity there is rational numbers. [I1 43:41-44:22]

Sarah resorts to her mathematical “principles” when she is presented with a question about her model that she has never entertained. She seems to rely on her intuition that an “infinitely small space” has the same characteristics as a finite space, and so it must also contain rational numbers.

The following can be added to the list of nonstandard conceptions identified in Sarah’s responses:

- There exist infinitely many infinitesimal numbers—in particular, there are infinitely many infinitesimal numbers in an infinitely small space.
- Infinitesimal numbers can be ordered and compared.
- In an infinitesimal space there exist rational numbers.

Operations on Infinitesimal Numbers: Squaring and Reciprocating

Also, like Leibniz, Sarah is happy to manipulate these infinitesimal numbers. In the second interview, I ask her to operate on her infinitesimal numbers in several ways:

I: What did, what would you get if you took this number and squared it? Point zero repeating infinitely with a one after it. [0.000. . .1]

S: A smaller number.

I: A smaller number, okay. Like, how, talk to me more, like, could you write it? Like how would you, you know what I mean?

S: I, I don't know how you would write it because it'd be kind of like exactly the same as that. It would look the same as that, but it would have more zeroes.

I: But it would have more zeroes?

S: Mm-hm.

I: How many more zeroes would it have?

S: Um, I don't know. But it would be, it would be smaller.

I: Okay, okay. Yeah. Um, so okay.

S: Since this [0.000. . .1] is infinitely small, it would be infinitely smaller than that.

I: It would be infinitely smaller than that?

S: Yeah.

I: So it's like, so could you zoom in? Like, tell me what you mean by infinitely smaller.

S: Like, I still think, I mean, I think maybe you said this example, like if you start cutting like, cutting something, you're never going to get it to be nothing. But it's going to be infinitely smaller than that, like maybe like a miniscule little thing, you can't measure it but it's not gone. [I2 06:36-07:54]

It is worth noting that it is not I, but Sarah, who had previously said something about "cutting." In the first interview, her comments about cutting were in response to Item 6a, described at the beginning of the Results section.

In the first interview she said that you could zoom in infinitely to see the difference between 0.999. . . and 1. To pursue what she means by saying that the new number is "infinitely smaller" than the original one, I ask her where the new number would be on a number line that was "zoomed in" around 0:

I: Um, now, where would this number squared be? [0.000. . .1]

S: Closer to the zero, so . . .

I: Like how close to the zero?

S: I, I don't really know. Maybe like infinitely close to the zero.

I: Okay.

S: Because you're squaring infinity, so it's going to be, or, you're squaring this. So it's going to be infinitely more closer.

I: Oh, so could you zoom in then again infinitely much and then see that?

S: Yeah, yeah.

I: Okay.

S: I mean you could *never*, but in theory, yeah.

I: Okay.

S: Like most of the, like this could never, you could never be able to calc, like, calculate it if you're talking about infinity and then infinity smaller, and then infinity smaller. But like, if infinity goes on forever, there's gotta be something a little bit smaller. More infinity. [08:39-09:36]

Sarah appears to have not thought about squaring an infinitesimal number before now. She prefaces her answers with "I don't know," and she never specifies exactly how to represent 0.000. . .1 squared in her notation. Nonetheless, she is willing to assert several things about what happens if you square an infinitesimal number: (a) squaring the infinitesimal 0.000. . .1 generates a new number, one with more zeroes in its decimal representation; (b) this new number is "infinitely smaller" than the original infinitesimal number; and (c) the new number is "infinitely closer" to the number 0 than the original infinitesimal. These views seem to be an extension of some properties of the finite real numbers: Just as squaring a small finite number gives you a slightly smaller finite number, squaring an infinitesimal number gives you an infinitely smaller infinitesimal number. These views accord also with Leibniz' conception that you can perform the same operations on infinitesimal numbers as you can on finite numbers, even though Sarah's notation allows her to describe only a few of the features of such operations, without the detail that one might use to describe such operations on regular finite numbers.

In her last comment in the transcript above, she justifies her conception that one can square an infinitesimal and thus generate a smaller infinitesimal, even though you might not be able to calculate the result. Her reason is "If infinity goes on forever, there's gotta be something a little bit smaller. More infinity." This accords with her earlier justification that even an infinitely small space can still be cut and cut, and it will never be gone. These justifications suggest that infinitesimal numbers are arbitrarily divisible, a view that is consistent with her belief that infinitely many numbers live in a small, even infinitesimal, space. As long as she continues to reject an atomist belief that there exist units that cannot be divided into smaller pieces, then the operations of dividing and squaring will always generate new numbers, even when applied to infinitesimals.

I then ask Sarah to perform another operation on her infinitesimal numbers, namely, to take the reciprocal of them. The reason I ask her this is that I want to ascertain whether she will consistently pursue her system. Up to this point, her answers are consistent with the systems of infinitesimals of Leibniz and Robinson, although her system is much more rudimentary and she appears to be investigating many of its features for the first time. In Leibniz' and Robinson's systems, the reciprocal of an infinitesimal number is an infinite one. Both Leibniz and Robinson realized that in order for a model that includes infinitesimals to be closed under the operation of division, the model must include infinite numbers as well, and many of them. I ask Sarah about reciprocals of infinitesimal numbers to determine whether she will realize the same thing and continue to pursue her conception of infinitesimal numbers in a consistent manner.

I: Okay. Okay, let me ask you a different question. What is one over this number? [Points to 0.000. . .1.]

S: [Writes 1/0.000. . .1.] Oh yeah, this one doesn't, well this is kind of like the one, an infinitely small number, that, that you, that we talked about before. If this [0.000. . .1] is smaller and smaller, then . . . [pauses] this [1/0.000. . .1] is just going to get bigger and bigger and bigger and bigger.

I: The quotient is going to get bigger and bigger and bigger. Okay. Um . . . what about one over this number again here? [Points to 0.000. . .01.]

S: Bigger infinity than that. [Indicates 1/0.000. . .1.]

I: Bigger infinity than that?

S: Yeah, I know that's really bad, heh.

I: No, you tell me what you think, you're telling me what you think, not what, not what you're going to write down on your test next week.

S: [Laughing] Yeah.

I: So, a bigger infinity than that?

S: Yeah.

I: Um . . .

S: I just think of it like, if it's infinity, like yeah, like, what I'm saying, it's kind of like not really making sense, like I know it doesn't really sound right, but I feel like if there's an infinity, then there has to be something *bigger* than the infinity, which is still infinity or, you know, small infinity. [I2 09:38-11:28]

Here Sarah claims that 1/0.000. . .1 is "going to get bigger and bigger and bigger and bigger," and that 1/0.000. . .01 is a "bigger infinity than that." Again Sarah's answers are consistent with the features of Leibniz' and Robinson's systems, in which the reciprocals of infinitesimal numbers are infinite numbers.

Here Sarah talks about the first of these infinite numbers in a dynamic way. Up to this point, she has not used much dynamic language to describe infinitesimals, but here she says that the number 0.000. . .1 is smaller and smaller, so its reciprocal is going to get bigger and bigger. Until now her language suggests that she views these numbers to be static objects, not dynamic processes. She talks about 0.000 . . .1 being, not getting, infinitely close to 0. She says that 0.999. . . and 1 *are* infinitely close, not that they are *getting* infinitely close. Why does her language change here, so that now she says 0.000. . .1 is "smaller and smaller"?

One account for her dynamic language here, and not elsewhere, might be that here she is in the middle of determining properties of a brand new thing. She determines the properties by appealing to the finite elements of the process that produced the infinite thing. If she envisions 0.000. . .1 to be the final result of the sequence 0.1, 0.01, 0.001, . . . , then she might try to determine 1/0.000. . .1 as the final result of the sequence 1/0.1, 1/0.01, 1/0.001, ... The former sequence is getting "smaller and smaller," and the latter one is getting "bigger and bigger." Therefore she projects that the number 1/0.000. . .1 is infinite. Once she has determined this, she may now be able to view the object statically, as a final product of the process that produced it. This enables her to say, for instance, that 1/0.000. . .01 is a "bigger infinity" than 1/0.000. . .1.

This interpretation is in keeping with prior work about how students transition between processes and objects, put forth in the most detail in the ideas of encapsulation (e.g., Dubinsky, 1991; Dubinsky, Weller, McDonald, & Brown, 2005) and reification (Sfard, 1991). Sarah talks about 0.000. . .1 as an object and is comfortable speaking about it in static terms throughout the interview. But here she unencapsulates it, thinking about it instead in terms of the process by which it was constructed. It is crucial that she is able to do this, because it enables her to make another infinite process to construct a new object. The result of this new process is the object 1/0.000. . .1. It is not surprising that Sarah's infinitesimals and infinite numbers might be constructed as the products of infinite processes; in Robinson's nonstandard model of the real numbers, infinitesimals and infinite numbers are constructed in the same way. The initial creation of new infinitesimal and infinite numbers involves the encapsulation and unencapsulation of infinite processes.

What is unusual is that Sarah proposes that 1/0.000. . .01 is a "bigger infinity" than 1/0.000. . .1. In particular, she views that taking the reciprocal of each of these two infinitesimals generates two different infinite numbers, rather than treating all infinite numbers as being the same. She admits that this is "really bad," and indeed it is an unorthodox view, to say the least. She describes why she holds it by saying, "I feel like if there's an infinity, then there has to be something *bigger* than the infinity, which is still infinity or, you know, small infinity." Her vocal emphasis in this last statement makes her meaning clearer: She is suggesting that the fact that the second infinity is bigger than the first does not mean that the first is not infinite. The first is "still infinity"; it simply is "small infinity." By maintaining that these two different infinitesimals generate two different infinite numbers when reciprocated, she now must consistently assert that there is not just one single infinity, but that there are different sizes of infinity. This is nothing as grandiose as cardinalities of infinite sets; it simply indicates that she is treating these infinities as numbers that can be operated on and compared, as with finite numbers. This is consistent with her treatment of infinitesimal numbers, and it is a feature of the nonstandard real numbers. Unfortunately, in the interview I do not ask her to describe or notate these infinite numbers further, so I do not know how far she would pursue operations on these infinite numbers.

Sarah says that these conceptions of hers about infinite numbers are "really bad" and she says, "I know it doesn't really sound right." Such comments are in keeping with earlier remarks that the numbers she is discussing are not "real," that she knows that her answers are wrong, but that is just how she thinks about it. This is why I tell her here that her answers do not have to be the ones that she would record on a math test next week, but that I want to know what she thinks. Although she believes her answers to be incorrect, she nonetheless pursues them in a way that preserves their internal consistency, rather than responding in a way that she believes will be viewed as "correct."

The following complete the list of nonstandard conceptions identified in Sarah's responses:

- Infinitesimals can be operated on exactly like ordinary real numbers; in particular, they can be squared and their reciprocals can be taken. These operations generate new infinitesimal and infinite numbers, respectively.

- There exist more than one infinite number; these are reciprocals of infinitesimal numbers.

A Resilient System

Although Sarah often admits that these conceptions of hers are probably wrong, she still maintains them despite being shown the "correct" conceptions. At the end of the first interview she says that she has seen a proof in high school that 0.999. . . equals 1, but she never believed the proof. She asks me why she was supposed to believe that 0.999. . . = 1.

I oblige with a standard explanation (although not a rigorous proof, because it avoids the issue of convergence): I write $N = 0.999...$, so $10N = 9.999...$, so by subtracting the equations, $9N = 9$, so $N = 1$. She still objects, saying that if 0.999. . . really equals 1, then why do you even have 0.999. . . ? "The only reason to have 0.9 repeating is to show that it's not 1!"

Then I ask her whether she believes that 0.333. . . equals 1/3. To my surprise, she says "no," that she never really believed that either, even though "it's what they make you memorize." When she says that she does not really understand how to divide 1 by 3, I ask her whether she is comfortable with 1/2 = 0.5. She says that is fine, because she could see that 5/10 is the same as 1/2. This is reasonable, because this was "a nice number," as opposed to the infinite repeating decimal expansion for 1/3.

This particular episode takes place at the end of the first interview [I1 48:45-52:11]. Because, as seen previously, she continues to assert in the second interview that 0.999. . . does not equal 1, it is clear that my explanation does not perturb her thinking. For instance, what follows is part of the initial portion of the second interview:

I: Like, the question I asked last time was what is between 0.9 repeating and one?

S: Right.

I: How would you answer that?

S: Um . . . an infinitely small space. If, if I were to, well now I know that they equal each other, I know that that's right, but . . .

I: But does that, is that how you think about that?

S: I think about a little, an infinitely small space. [I2 00:25-00:43]

The resiliency of Sarah's conceptions despite instruction to the contrary is illuminated by her views about how mathematics is learned in school. She believes that knowing the "correct" mathematical results, but still not believing or understanding them, is part of what doing mathematics in school is all about:

S: . . . a lot of the things that I think, like, about math are just rules and memorizing rules that someone made up, and that's why I think this is important to have, because what math students do, and this is what I do too: Studying for a math test is just like memorizing the rules, doesn't matter why there's rules there, you just memorize it

> so you can use the formula on your test and get an A. You know? But it's important for kids to, like, think of the concepts of it. That's why this is good for me, see, because I don't know the concepts . . . . [I1 46:23-46:54]

It is well known that many calculus-level students do not believe that 0.999. . . equals 1 (e.g., Tall & Vinner, 1981). In response to the question about 0.999. . . and 1 on my original questionnaire, only 12% of the students said that 0.999. . . = 1 (Ely, 2007). Furthermore, many students are not convinced by explanations such as the one I provide for Sarah for why the two numbers are equivalent (Sierpinska, 1987). Sierpinska's study shows examples of students that hold different kinds of robust conceptions about 0.999. . . , such as the *heuristic dynamic* conception of infinity (0.999. . . never equals 1, "no matter how many nines you put"), the *potential actualist* conception of infinity (". . . unless it really goes to the very infinity, then it may be 1"), and a *discursive empiricist* attitude toward mathematics in general (mathematics is memorized, not discussed; thus, proofs or explanations that 0.999. . . = 1 are not convincing). Sarah's comments here about mathematics being something "you just memorize" are in keeping with a discursive empiricist viewpoint. But Sarah's other conceptions reveal an interesting reason for this.

Sarah's discursive empiricist viewpoint may be a result of her conceptions about 0.999. . . , rather than a cause. If she were able to pursue her nonstandard conceptions in a consistent way, then it would be illogical for her to accept that 0.999. . . = 1. To accept that 0.999. . . = 1, she would have to go against the mechanisms of consistency by which she generates and grounds her conceptions about infinitesimal numbers. Instead, to maintain the cognitive consistency of her own conceptions, she adopts a discursive empiricist attitude toward classroom mathematics: Such mathematical ideas do not cohere with her own conceptions, but she must still be able to access them in order to get good grades. Only by assuring her many times that I want to hear what she thinks, not what she would say to get the grades in class, does she reveal in these interviews the resilient conceptions she holds beneath the memorized answers.

Sarah is maintaining an underlying set of intuitions that remains unaltered in spite of her classroom mathematics. After the interview, when I asked her where she learned her ideas about infinitely small and infinite numbers, she said it was not from any of her classrooms but that it was her own way of making sense of things. Although we have only her report of her classroom instruction, the resiliency of her conceptions suggests that opposing proofs, explanations, and classroom instruction are not perturbing her conceptions. This is in keeping with the interpretation that her conceptions are genuinely nonstandard, surviving unperturbed because they are in fact unperturbable.

Sarah's conceptions about infinitesimal and infinite numbers are consistent with the features of Leibniz' system of infinitesimal and infinite numbers and Robinson's nonstandard real numbers. A summary of the similarities among these three systems can be seen in Figure 2 (see pp. 140–141). Although the similarities are extensive, there is one enormous difference: Sarah has no "system"—she is exploring the entailments of her conceptions as the interview proceeds. Her ideas are emergent, yet because they are motivated by a need to maintain self-consistency, they develop along the lines of Leibniz' and Robinson's systems. But there is no reason to believe that she would readily be able to use these emergent ideas for a different purpose, such as for doing calculus or analysis, as Leibniz and Robinson did.

Sarah certainly makes mistakes, such as conflating distance and number for a time. But her nonstandard conceptions are not mistakes, even though they contradict the features of the standard real numbers. By recognizing that hers are nonstandard conceptions, not misconceptions, we can account for why they are so resilient even though they contradict the classroom mathematics that she says she memorizes in order to get good grades on the tests. Without an understanding of Leibniz' system of infinitesimals, or of Robinson's formalization of this system, it would be tempting to dismiss Sarah's ideas as being quirky but stubbornly wrong. With this understanding, it becomes evident that Sarah is developing a coherent framework and an emergent notation for thinking about infinitesimal quantities. Not only have her ideas not yet contradicted themselves mathematically for Sarah, the work of Robinson indicates that she could continue to pursue these ideas without their being perturbed.

DISCUSSION

Sarah's case has several implications and suggests some promising lines of further study. First, Sarah's story brings a new perspective to how students may be thinking about infinite processes and the real numbers. Although this may be the first time that this particular set of conceptions has been identified in a student, a belief in infinitesimal numbers and distances is not unusual among calculus students by any means. In the broader study out of which Sarah's case emerged, 31% of the calculus students claimed consistently over multiple questionnaire items that there exist infinitely small numbers and/or distances (Ely, 2007). Numerous studies have examined the difficulties that students have with the ideas of infinite processes and the real numbers (e.g., Cornu, 1991; Cottrill et al., 1996; Davis & Vinner, 1986; Przenioslo, 2004; Schwarzenberger & Tall, 1978; Sierpinska, 1987; Szydlik, 2000; Tall & Vinner, 1981; Weller, Brown, Dubinsky, McDonald, & Stenger, 2004; Williams, 1991). These studies show instances in which students may have been appealing to intuitions about infinitesimal numbers also. For example, in Tall and Vinner's study (1981), students described 0.999. . . as being "just less than one, because the difference between it and one is infinitely small" (p. 159). Sarah's story suggests that students who speak of infinitely small distances and numbers may not be simply "struggling" with ideas of infinity and the real numbers, but may be holding nonstandard conceptions.

If students have such intuitions about infinitesimals that they are not explicitly taught, then what is the origin of these intuitions? One answer may be the decimal notations used to represent real numbers. A notation such as 0.999. . . is seen by

Leibniz' foundational system (ca. 1690)	**Nonstandard analysis (ca. 1961)**	**Sarah's conceptions**
There exist infinitesimal, finite, and infinite numbers.	There exist infinitesimal, finite, and infinite numbers.	There exist infinitesimal, finite, and infinite numbers.
There are infinitely many infinitesimal and infinite numbers. Operating on them produces other numbers: for example, squaring an infinitesimal produces a smaller infinitesimal; squaring an infinite number produces a larger infinite number. This can be done infinitely. (Here Leibniz disagrees with Nieuwentijt (Mancosu, 1996).) Any number, even an infinitesimal, can be divided infinitely many times.	There are infinitely many infinitesimal and infinite numbers. The nonstandard real numbers are a field extension of the real numbers, and are thus closed under the arithmetical operations, which gives rise to infinitely many infinitesimal and infinite numbers. An infinitesimal is the multiplicative inverse of an infinite number.	Sarah's conception is quite similar to Leibniz'. One can keep dividing any length into infinitely smaller and smaller segments. Infinitesimal numbers can be operated upon: squaring an infinitesimal produces a smaller infinitesimal, and the reciprocal of an infinitesimal is an infinite number.
An infinitesimal is not strictly defined or represented (other than as, say, *dx*); an infinite series has an infinitesimal term at its end.	An infinitesimal can be precisely represented by an equivalence class of sequences of positive rational numbers $\{a_n\}$ that converges to 0.	An infinitesimal can be represented by a decimal expansion that has digits extending past the "infinityth" decimal place. For instance, 0.0000. . . (infinite 0s) with a 1 at the end.
One can find two different numbers infinitely close to each other. In particular, $f + i \approx f$ (f is a finite number, i is an infinitesimal).	One can find two different numbers infinitely close to each other. Any nonstandard number is infinitely close to but not necessarily equal to its standard part ($n \approx s(n)$).	One can find two different numbers infinitely close to each other. For instance, 0.000. . .1 and 0.000. . .01 are both "infinitely close" to 0.

Leibniz' foundational system (c. 1690)	Nonstandard analysis (c. 1961)	Sarah's conceptions
If one zooms in enough on part of a thing, one will see that this part looks like a microcosm of the thing itself. There are worlds within worlds. This can only be done finitely in the real world, but in the mathematical one it can be done infinitely, or at least it is a "useful fiction" to pretend that it can. On the other hand, monads are indivisible and spiritual units, as different from infinitesimals (which are infinitely divisible) as the limitlessness of God is from mere infinite quantities.	Keisler's "infinitesimal microscope" allows one to zoom in infinitely. A monad is defined to be a set of all numbers infinitely close to one another (this is not definable in first-order logic). This is sort of a microcosm of the standard real numbers. (Note that this differs dramatically from Leibniz' use of the word "monad.")	Sarah's conception is similar to Leibniz', at least with respect to the number line: ". . . even in that tiny infinitely small space [between 0.999. . . and 1] you can still cut that into infinity too and put numbers in there," and this "would look exactly like the big number line except they wouldn't be called the same numbers."
I found no reference to rationality of infinitesimal numbers by Leibniz.	Rational and irrational infinitesimals exist.	There exist rational numbers in every small space, including infinitesimally small spaces.
The purpose of infinitesimals is to provide a method for doing calculus.	The purpose is to produce a rigorous model for analysis that uses infinitesimals.	The purpose of infinitesimals is sense making, not some additional functional purpose.
Gradually discarded in the 19th century in favor of the limit-based formulation of the standard real numbers.	Established in 1961 (Robinson) as logically equivalent to the standard real numbers.	She refers to her conceptions as her own strange way of thinking.

Figure 2. A summary of the similarities among Leibniz' system, the system of nonstandard analysis, and Sarah's conceptions.

students long before they learn about series and convergence, so students may generate misconceptions as they try to make sense of the number. Certainly some of these misconceptions will not lead to a belief in infinitesimals, such as the "heuristic dynamic" conception (Sierpinska, 1987). For a student who believes that 0.999. . . "never reaches" 1, the process of adding 9s is never completed, never encapsulated (Dubinsky et al., 2005). This conception, which is quite common among students, should not lend itself to a belief in numbers that are infinitely small.

However, Sarah does not seem to be making sense of the notation in this way. She is treating the number 0.999. . . as being infinitely close to 1. Rather than treating the ". . ." as an unfinished process, she understands it as a completed object, one upon which we can then act by putting other digits beyond it to generate objects such as 0.000. . .01. This view is just as reasonable of a way of making sense of the notation as the standard way, which would have students believe that "0.999. . ." is just another way of writing "1." The "0.999. . ." notation may thus be quite suggestive for students, lending itself to the creation of new objects.

A second implication of this case study is that nonstandard conceptions can help to explain why students have certain beliefs about how to learn mathematics. One corollary of the view that students learn by accommodating perturbations is that students do not simply learn the mathematics that they are taught. If an idea from the classroom conflicts with a student's prior knowledge, then sometimes the classroom idea may be rejected or merely overlaid upon the underlying conceptions in some superficial way. But if the prior knowledge is nonstandard, and is unperturbable, then this rejection or overlaying of classroom mathematics might happen again and again, becoming a regular mode of operating in the classroom for the student. This may cause the belief that mathematics is something that does not really make sense, but which one memorizes to get good grades.

Being aware of the issues of nonstandard conceptions may help a teacher successfully address them in the classroom. By discussing issues of infinity with his students, a teacher could introduce the idea of infinitesimals and the standard real numbers as two different viable routes outlined by history—an example of how mathematics is influenced by human history. He could indicate that in this class it is necessary to choose some set of rules and that they will choose the standard route, but this route is not chosen because the other route is incorrect. By directly addressing the issue in this way, students with nonstandard conceptions may be able to learn the standard system without continually encountering contradictions.

A third implication of the existence of a student's nonstandard conceptions of infinitesimals adds a new dimension to the conversation about the relationships between historical thinking and student thinking. Why do Sarah's conceptions reflect those of mathematicians whose ideas she has not seen, and which are not part of the standard curriculum?

The idea of nonstandard conceptions suggests that the reason for these parallels is that the construction of Sarah's knowledge is governed by mechanisms for cognitive consistency, and that these mechanisms are related to the mechanisms for publicly establishing mathematical consistency. Radical constructivism claims that there are mechanisms for cognitive consistency at work that govern the construction of student knowledge. Because inconsistencies in a schema allow for perturbations, cognition is a locally self-regulating system. Thinking can be thought of as a game with rules, and these rules govern the construction of consistent cognitive structures. "Consistency, in maintaining semantic links and in avoiding contradictions, is an indispensable condition of what I would call our 'rational game' " (von Glasersfeld, 1990, p. 25).

Just as there are internal mechanisms for cognitive consistency, there are also sociomathematical rules and mechanisms for publicly establishing mathematical consistency, namely, proofs. So mathematics, too, can be thought of as a system with rules for establishing consistency, although these rules and practices change over time (Lakatos, 1976). Sarah's structure, the formation of which was governed by mechanisms for cognitive consistency, is strikingly similar to a nonstandard mathematical structure, the formation of which was governed by the public rules for mathematical consistency.

This raises a broader question: What is the relationship between the internal cognitive rules that govern the construction of (locally) consistent sets of conceptions and our external sociomathematical rules that govern the construction of consistent mathematics? Is it that our culture's criteria for mathematical consistency and coherence are an externalization of our innate internal cognitive criteria for the consistency and coherence of constructed knowledge? Or is it perhaps that our personal cognitive mechanisms for constructing consistent knowledge are an internalization of the criteria for mathematical consistency (more generally, an internalization of the norms of consistency in our discursive social practices)? Or, perhaps our internal cognitive mechanisms and our public mathematical rules influence one another in complex and iterative ways.

This question is not a new one, and our further attempts to answer it should be informed by the more general debate in educational psychology whether individuals' patterns of thinking have consistent structures from which sociocultural structures derive (e.g., Chomsky, 1967), whether sociocultural structures inform individuals' patterns of thinking (e.g., Hutchins, 1993; Vygotsky, 1962), or whether there is a complex interaction between the two (e.g., Piaget & Garcia, 1983/1989).

Piaget and Garcia's approach may be particularly promising. Unlike this study, it is not specifically focused on the relationship between the cognitive and historical mechanisms by which consistency is maintained. Rather, it focuses on several common mechanisms by which new conceptions are made from old ones. Piaget and Garcia's goal was to show that "the mechanisms mediating transitions from one historical period to the next are analogous to those mediating the transition from one psychogenetic stage to the next" (1983/1989, p. 28). By focusing in future work on the ways that students such as Sarah progress from one psychogenetic stage to the next, we could use Piaget and Garcia's work to discern more about why the ideas of a modern student could so closely resemble those of a 17th-century mathematician.

Several simpler areas of future inquiry also present themselves based on this article. Certainly one area is to look for other examples of nonstandard student conceptions of infinite and infinitesimal numbers. For instance, Skolem's nonstandard model of arithmetic contains actually infinite numbers. The Tennis Ball problem and Ping-Pong® Ball problem (Dubinsky et al., 2005; Mamolo & Zaskis, 2008) may provide rich and promising contexts for investigations into whether students find appeal in such a nonstandard model in their understandings of infinity.

Another area for further research is to see whether students who believe in infinitesimals are able to build a system of calculus using these conceptions as a basis. There is no evidence that a student such as Sarah is connecting her conceptions about infinitesimals to calculus concepts such as derivatives. Could these connections be made and supported for such a student? Conversely, are students who have a robust understanding of derivatives and integrals involving images of infinitesimal slopes and sums of infinitesimally wide rectangles more inclined to think of the number line using infinitesimals?

These lines of inquiry are important for one to understand the implications of this research for the teaching of calculus and higher mathematics. For example, textbooks that take an infinitesimal approach to calculus (e.g., Henle & Kleinberg, 1979; Keisler, 1986) might provide a rigorous conceptual base for students such as Sarah. Such books avoid the perennially difficult idea of limits, but may instead require students to learn difficult ideas in mathematical logic.

By recognizing that some student conceptions that appear to be misconceptions are in fact nonstandard conceptions, we can see meaningful connections between cognitive structures and mathematical structures of the present and past that otherwise would have been overlooked. In this case, this adds a new dimension to our understanding of student ideas about infinity and the real numbers, and provides a lens for interpreting the import of those ideas.

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