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Nonstationary Axisymmetric Problem of the Impact of a Spherical Shell on an Elastic Half-Space (Initial Stage of Interaction)

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Abstract—The supersonic stage of interaction (where the rate of expansion of the contact region is no less than the speed of compression waves) between a Timoshenko-type spherical shell (indenter) and an elastic half-space (foundation) is studied. The expansion of the desired functions in series in Legendre polynomials and their derivatives are used to construct a system of resolving equations. An analytical-numerical algorithm for solving this system is developed. A similar problem was considered in [1], where the original problem was replaced by a problem with a periodic system of indenters.

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1. STATEMENT OF THE PROBLEM

At the initial time, a thin linearly elastic Timoshenko-type spherical shell (indenter), moving with an initial speed V_0 under the action of an external resultant force R_e directed along the axis of symmetry of the indenter, comes in contact with a homogeneous isotropic linearly elastic half-space. The vectors of the initial velocity and external force are directed normally to the unperturbed surface of the half-space. Initially, the shell and the half-space are in undeformed states.

The motion of the indenter is considered in a spherical coordinate system $r_0, \theta, \tilde{\vartheta}$ with the origin of the radius vector r_0 coinciding with the center of mass O_0 of the shell. To describe the motion of the half-space, we use a cylindrical coordinate system $z, r_1, \tilde{\vartheta}$ with origin at a point O_1 lying on its boundary and the axis z passing through the point O_0 and directed into the interior of the half-space (Fig. 1).

All variables and parameters are reduced to dimensionless form (a prime indicates a dimensionless quantity, the quantities with index $k = 1$ correspond to the half-space, and the quantities with index $k = 0$ correspond to the shell):

$$\varphi' = \frac{\varphi}{R^2}, \quad \psi' = \frac{\psi}{R^2}, \quad \eta_i = \frac{c_{11}}{c_{i1}} \quad (i = 1, 2), \quad z' = \frac{z}{R}, \quad \tau' = \frac{c_{11}\tau}{R}, \quad u'_k = \frac{u_k}{R},$$

$$w'_k = \frac{w_k}{R}, \quad \sigma'_{\alpha\beta} = \frac{\sigma_{\alpha\beta}}{\lambda_1 + 2\mu_1} \quad (\alpha, \beta = r_1, \vartheta, z), \quad \alpha_k = \frac{\lambda_k}{\lambda_k + 2\mu_k},$$

$$\beta_k = \frac{\mu_k}{\lambda_k + 2\mu_k}, \quad r' = \frac{r}{R}, \quad u'_c = \frac{u_c}{R}, \quad h' = \frac{h}{R}, \quad V'_0 = \frac{V_0}{C_{11}}, \quad b' = \frac{b}{R},$$

$$p' = \frac{p}{\lambda_0 + 2\mu_0}, \quad \gamma^2 = \frac{c_{11}^2}{c_{10}^2}, \quad c_{1k}^2 = \frac{\lambda_k + 2\mu_k}{\rho_k}, \quad c_{2k}^2 = \frac{\mu_k}{\rho_k}, \quad a' = \frac{h^2}{12R^2},$$

$$m'_0 = \frac{m_0}{\rho_1 R^3}, \quad R'_e = \frac{R_e}{\rho_1 c_{11}^2 R^2}, \quad R'_a = \frac{R_a}{\rho_1 c_{11}^2 R^2}, \quad \bar{\gamma} = \frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1},$$

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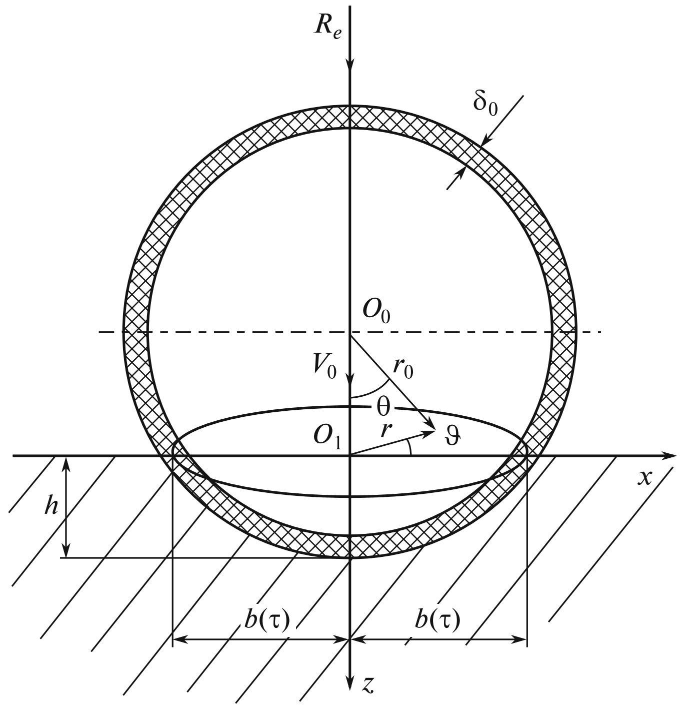

Fig. 1.

$$M'_{\alpha\alpha} = \frac{M_{\alpha\alpha}}{Rh(\lambda_0 + 2\mu_0)}, \quad \tilde{T}'_{\alpha\alpha} = \frac{\tilde{T}_{\alpha\alpha}}{h(\lambda_0 + 2\mu_0)}, \quad Q' = \frac{Q}{h(\lambda_0 + 2\mu_0)}, \quad \kappa'_{\alpha\alpha} = \kappa_{\alpha\alpha} R \quad (\alpha = \theta, \vartheta).$$

Here R is the shell radius, c_{1k} and c_{2k} are the speeds of propagation of the compression and shear waves, φ and ψ are the scalar and vector potentials of elastic displacements of the half-space, $\sigma_{\alpha\beta}$ are the stress tensor components of the half-space, ρ_k is the density, $b(\tau)$ is the radius of the contract region, t is the time, h is the shell thickness, λ_k and μ_k are the Lamé elastic constants, w_k and u_k are the normal and tangential displacements, p is the normal contact stress, m_0 is the shell mass, R_a is the resultant contact force, $T_{\alpha\alpha}$, $\tilde{T}_{\alpha\alpha}$, $M_{\alpha\alpha}$, and $\kappa_{\alpha\alpha}$ are the nonzero components of the tensors of tangential tractions, their components, bending moments, and curvature variation, and Q is the shear force. In what follows, the primes are omitted everywhere.

The axially symmetric motion of the half-space is described by the well-known relations of elasticity (from now on, a dot over a symbol denotes a derivative with respect to the dimensionless time τ) which contain the equations of motion

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} = \ddot{\varphi}, \quad \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} = \eta_2^2 \ddot{\psi}, \tag{1.1}$$

the relations between the displacements and potentials

$$u_1 = \frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial z}, \quad w_1 = \frac{\partial \varphi}{\partial z} + \frac{1}{r}\left(\psi + r\frac{\partial \psi}{\partial r}\right), \tag{1.2}$$

and the relations between the stress tensor and displacement components

$$\begin{aligned} \sigma_{rz} &= \beta_1\left(\frac{\partial w_1}{\partial r} + \frac{\partial u_1}{\partial z}\right), \quad \sigma_{rr} = \frac{\partial u_1}{\partial r} + \alpha_1\left(\frac{\partial w_1}{\partial z} + \frac{u_1}{r}\right), \\ \sigma_{zz} &= \frac{\partial w_1}{\partial z} + \alpha_1\left(\frac{\partial u_1}{\partial r} + \frac{u_1}{r}\right), \quad \sigma_{\vartheta\vartheta} = \frac{u_1}{r} + \alpha_1\left(\frac{\partial w_1}{\partial z} + \frac{\partial u_1}{\partial r}\right). \end{aligned} \tag{1.3}$$

The corresponding relations for the shell contain [2] the equations of motion

$$\begin{aligned}
\gamma^2 \ddot{u}_0 &= \frac{\partial T_{\theta\theta}}{\partial\theta} + (T_{\theta\theta} - T_{\vartheta\vartheta})\cot\theta + Q,\\
\gamma^2 \ddot{w}_0 &= -T_{\theta\theta} - T_{\vartheta\vartheta} + \frac{\partial Q}{\partial\theta} + Q\cot\theta + \frac{p}{h},\\
\gamma^2 a\ddot{\chi} &= \frac{\partial M_{\theta\theta}}{\partial\theta} - (M_{\vartheta\vartheta} - M_{\theta\theta})\cot\theta - Q,
\end{aligned} \tag{1.4}$$

the geometric relations

$$\begin{aligned}
&\varepsilon_{\theta\theta} = \frac{\partial u_0}{\partial\theta} + w_0, \quad \varepsilon_{\vartheta\vartheta} = u_0\cot\theta + w_0, \quad \beta = \chi - \xi, \quad -\xi = \frac{\partial w_0}{\partial\theta} - u_0,\\
&\kappa_{\theta\theta} = \frac{\partial\chi}{\partial\theta} - \frac{\partial u_0}{\partial\theta} - w_0, \quad \kappa_{\vartheta\vartheta} = \cot\theta(\chi - u_0) - w_0,
\end{aligned} \tag{1.5}$$

and the physical relations

$$\begin{aligned}
&\tilde{T}_{\vartheta\vartheta} = \varepsilon_{\vartheta\vartheta} + \alpha_0\varepsilon_{\theta\theta}, \quad \tilde{T}_{\theta\theta} = \varepsilon_{\theta\theta} + \alpha_0\varepsilon_{\vartheta\vartheta},\\
&M_{\theta\theta} = a(\kappa_{\theta\theta} + \alpha_0\kappa_{\vartheta\vartheta}), \quad M_{\vartheta\vartheta} = a(\kappa_{\vartheta\vartheta} + \alpha_0\kappa_{\theta\theta}),\\
&T_{\theta\theta} = \tilde{T}_{\theta\theta} - M_{\theta\theta}, \quad T_{\vartheta\vartheta} = \tilde{T}_{\vartheta\vartheta} - M_{\vartheta\vartheta}, \quad Q = \beta_0 k^2\beta, \quad k^2 = \tfrac{5}{6}.
\end{aligned} \tag{1.6}$$

Here $\varepsilon_{\theta\theta}$ and $\varepsilon_{\vartheta\vartheta}$ are the nonzero components of the strain tensor and χ is the angle of rotation of the fiber that is normal to the shell midsurface.

These relations are supplemented with the equation of motion of the shell as a rigid body

$$m_0\ddot{u}_c = R_e + R_a, \quad R_a(\tau) = 2\pi\tilde{\gamma}\int_0^{b(\tau)} p(r,\tau)r\,dr, \tag{1.7}$$

where u_c is the penetration depth of the shell as a rigid body.

The initial conditions in the problem under study are

$$\begin{aligned}
&u_c\big|_{\tau=0} = 0, \quad \dot{u}_c\big|_{\tau=0} = V_0, \quad u_0\big|_{\tau=0} = 0, \quad w_0\big|_{\tau=0} = 0, \quad \dot{u}_0\big|_{\tau=0} = -V_0\sin\theta,\\
&\dot{w}\big|_{\tau=0} = V_0\cos\theta, \quad \varphi\big|_{\tau=0} = 0, \quad \dot{\varphi}\big|_{\tau=0} = 0, \quad \psi\big|_{\tau=0} = 0, \quad \dot{\psi}\big|_{\tau=0} = 0.
\end{aligned} \tag{1.8}$$

There are no perturbations at the infinitely remote point of the half-space.

The linearization of the boundary conditions consists in referring them to the undeformed boundary surfaces and taking account of the smallness of the contact region. Assuming that the contact occurs under the free slip conditions (there is no friction between the interacting surfaces) and, outside the interaction region, the half-space and shell surfaces are free from stresses, we obtain the following conditions:

$$\begin{aligned}
&\sigma_{zz}\big|_{z=0} = \tilde{\gamma}p \quad (|r| \le b(\tau)), \quad \sigma_{zz}\big|_{z=0} = 0 \quad (|r| > b(\tau)),\\
&\sigma_{z\vartheta}\big|_{z=0} = 0 \quad (r \in (-\infty,\infty)), \quad w_1 = (w_0 + 1)\cos\theta - 1 \approx w_0 \quad (|r| \le b(\tau)).
\end{aligned} \tag{1.9}$$

Neglecting the deformation of the indenter and half-space free surfaces, we see that the contact region is a circle of radius

$$b(\tau) = \sqrt{u_c(2 - u_c)}. \tag{1.10}$$

Relations (1.1)–(1.10) form a closed initial boundary-value problem.

2. SYSTEM OF RESOLVING EQUATIONS

We restrict ourselves to the initial (supersonic) stage of interaction at which, because of the indenter convexity, the rate of expansion of the contact region is no less than the speed of compression waves in the elastic medium [3]. Therefore, the displacements of the indenter and half-space boundary surfaces stay within the contact region. In this case, we have the following integral representation of the contact stress as a two-dimensional convolution, with respect to time and the radius, of the derivative of the influence function Γ for a half-space with the speed of the normal displacement of the shell [3]:

$$p = \tilde{\gamma}^{-1} \dot{w}_0 {}^{**} \dot{\Gamma}. \tag{2.1}$$

Taking account of the axial symmetry of the problem, we rewrite (2.1) as

$$p(r,\tau) = \frac{1}{\gamma}[p_1(r,\tau) + p_2(r,\tau) + p_3(r,\tau)], \quad p_1(r,\tau) = -\dot{w}_0 H(\tau) H[b(\tau) - r],$$

$$p_2(r,\tau) = \int_0^\tau \dot{w}_0[b(\tau), t]\vartheta[r, b(\tau), \tau - t]\, dt, \quad p_2(r,\tau) = -\int_0^\tau dt \int_0^{b(\tau)} \frac{\partial \dot{w}_0}{\partial \rho}\vartheta(r,\rho,\tau - t)\, d\rho, \tag{2.2}$$

$$\vartheta(r,\rho,\tau) = \sum_{q=1}^{2} \vartheta_{rq}(r,\rho,\tau) + \frac{1}{\tau}\vartheta_s(r,\rho,\tau), \tag{2.3}$$

where the terms in (2.3) have the form

$$\vartheta_{rq}(r,\rho,\tau) = \frac{1}{r-\rho}\vartheta_{rq1}(r,\rho,\tau) + \frac{1}{(r-\rho)^3}\vartheta_{rq2}(r,\rho,\tau) + \frac{1}{\tau}\frac{1}{r-\rho}\vartheta_{rq3}(r,\rho,\tau), \tag{2.4}$$

$$\vartheta_{rq1}(r,\rho,\tau) = \frac{d_q}{\pi\eta^4}[F(\delta_q,m)H(\varphi_{1q}(r,\rho,\tau)) + K(m)H(\varphi_{2q}(r,\rho,\tau))]H(\varphi_q(r,\rho,\tau)),$$

$$\vartheta_{rq2}(r,\rho,\tau) = \frac{b_q}{\pi\eta^4}[F(\delta_q,m)H(\varphi_{1q}(r,\rho,\tau)) + E(m)H(\varphi_{2q}(r,\rho,\tau))]H(\varphi_q(r,\rho,\tau)),$$

$$\vartheta_{rq3}(r,\rho,\tau) = \frac{1}{\pi\eta^4}c_q H(\varphi_{1q}(r,\rho,\tau))H(\varphi_q(r,\rho,\tau)),$$

$$\vartheta_s(r,\rho,\tau) = \vartheta_{s0}(r,\rho,\tau)H(\varphi_{s1}(r,\rho,\tau))H(\varphi_{s2}(r,\rho,\tau)),$$

$$\vartheta_{s0}(r,\rho,\tau) = -\frac{(\eta^2-2)^2}{\pi\eta^4}\frac{c_0}{\sqrt{r+\rho-\tau}\sqrt{\tau^2-(r-\rho)^2}},$$

$$c_0 = \frac{r^2-\rho^2-\tau^2}{\sqrt{r+\rho+\tau}}, \quad m = \frac{4r\rho}{(r+\rho)^2}, \quad \sin^2\delta_q = \frac{(r+\rho)^2\left[(\tau/\eta_q)^2 - (r-\rho)^2\right]}{4r\rho(\tau/\eta_q)^2},$$

$$\varphi_q(r,\rho,\tau) = \tau - \eta_q|r-\rho|, \quad \varphi_{1q}(r,\rho,\tau) = \eta_q(r+\rho) - \tau, \quad \varphi_{2q}(r,\rho,\tau) = \tau - \eta_q(r+\rho),$$

$$\varphi_{s1}(r,\rho,\tau) = r+\rho-\tau, \quad \varphi_{s2}(r,\rho,\tau) = \tau - |r-\rho|,$$

$$d_1 = \frac{2}{r+\rho}\left[\frac{\tau^2}{r+\rho} - (r-\rho)(2\eta^2-1)\right], \quad d_2 = \frac{2}{r+\rho}\left[(r-\rho)\eta^2 - \frac{\tau^2}{r+\rho}\right],$$

$$c_1 = -2\frac{\sqrt{(r+\rho)^2-\tau^2}\sqrt{\tau^2-(r-\rho)^2}}{r+\rho}, \quad c_2 = -2\eta\frac{\sqrt{\eta^2(r+\rho)^2-\tau^2}\sqrt{\tau^2-\eta^2(r-\rho)^2}}{r+\rho},$$

$$b_1 = 2\left\{\tau^2\left[\frac{3(r-\rho)}{r+\rho} - \frac{2(r-\rho)^2}{(r+\rho)^2} - 2\right] + (2\eta^2-1)(r-\rho)^2\right\},$$

$$b_2 = 2\left\{\tau^2\left[-\frac{3(r-\rho)}{r+\rho} + \frac{2(r-\rho)^2}{(r+\rho)^2} + 2\right] - \eta^2(r-\rho)^2\right\},$$

for $r > 0$ and the form

$$\begin{gathered}\vartheta_{r1}(0,\rho,\tau) = \frac{2}{\eta^4\rho^3}[3\tau^2 - (2\eta^2-1)\rho^2], \quad \vartheta_{r2}(0,\rho,\tau) = -\frac{2}{\eta^4\rho^3}(3\tau^2 - \eta^2\rho^2),\\ \vartheta_s(0,\rho,\tau) = \frac{(\eta^2-2)^2}{\eta^4}\delta(\tau-\rho).\end{gathered} \tag{2.5}$$

for $r = 0$.

In the last relations, $H(x)$ is the Heaviside unit-step function, $F(\delta,m)$, $E(\delta,m)$, $K(m)$, and $E(m)$ are the incomplete and complete elliptic integrals of the first and second kind [4].

With this approach, the system of resolving equations contains relations (2.1) and (1.4)–(1.10). To solve this system, we expand the desired functions in the Legendre polynomials $P_n(\cos\theta)$ and their derivatives:

$$\left\| \begin{matrix} w_0(\theta,\tau) \\ p(\theta,\tau) \\ \vartheta(\theta,\theta_*,\tau-t) \end{matrix} \right\| = \sum_{n=0}^{\infty} \left\| \begin{matrix} w_{0n}(\tau) \\ p_n(\tau) \\ \vartheta_n(\theta_*,\tau-t) \end{matrix} \right\| P_n(\cos\theta), \qquad \left\| \begin{matrix} u_0(\theta,\tau) \\ \chi(\theta,\tau) \end{matrix} \right\| = \sum_{n=0}^{\infty} \left\| \begin{matrix} u_{0n}(\tau) \\ \chi_n(\tau) \end{matrix} \right\| \frac{dP_n(\cos\theta)}{d\theta}. \quad (2.6)$$

Here and henceforth, we take account of the smallness of the angle θ and use the approximate relations $r \approx \sin\theta$ and $\rho \approx \sin\theta_*$.

Substituting the series (2.6) into (1.5) and (1.6) and then into (1.4), we obtain the following infinite system of integro-differential equations for the coefficients of these series:

$$\begin{gathered} \gamma^2 \ddot{\mathbf{U}}_n = \mathbf{L}_n \mathbf{U}_n + \mathbf{P}_n \quad (n = 0, 1, 2, \ldots), \\ \mathbf{L}_n = \|L_{ijn}\|_{3\times 3}, \quad \mathbf{U}_n = \|u_{0n}, w_{0n}, \chi_n\|^T, \quad h\tilde{\gamma}\mathbf{P}_n = \left\| 0, \sum_{i=1}^{3} p_{in}, 0 \right\|^T, \end{gathered} \quad (2.7)$$

$$\begin{gathered} L_{11n} = (1+a)[(1-\alpha_0) - n(n+1)] - \beta_0 k^2, \quad L_{12} = (1+\alpha_0)(1+a) + \beta_0 k^2, \\ L_{13n} = a[\alpha_0 - 1 + n(n+1)] + \beta_0 k^2, \quad L_{21n} = n(n+1)[(1+a)(1+\alpha_0) + \beta_0 k^2], \\ L_{22n} = -[2(1+\alpha_0)(1+a) + \beta_0 k^2 n(n+1)], \quad L_{23n} = -n(n+1)[(\alpha_0+1)a + \beta_0 k^2], \\ L_{32n} = -2 - a^{-1}\beta_0 k^2, \quad L_{31n} = -L_{33n} = n(n+1) - 1 + \alpha_0 + a^{-1}\beta_0 k^2. \end{gathered} \quad (2.8)$$

In this case, the coefficients of the series expansions of the contact stress components take the form

$$\begin{gathered} p_{1n}(\tau) = -\frac{2n+1}{2} H(\tau) \sum_{k=0}^{\infty} \dot{w}_{0k}(\tau) \int_0^{b(\tau)} P_k(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta, \\ p_{2n}(\tau) = \frac{2n+1}{2} \sum_{k=0}^{\infty} \int_0^{\tau} \dot{w}_{0k}(t) P_k(\cos b(t)) \, dt \int_0^{\pi} \vartheta(\theta, b(t), \tau - t) P_n(\cos\theta) \sin\theta \, d\theta, \\ p_{3n}(\tau) = -\frac{2n+1}{2} \sum_{k=0}^{\infty} \int_0^{\tau} \dot{w}_{0k}(t) \, dt \int_0^{b(t)} \frac{dP_k(\cos\theta_*)}{d\theta_*} \, d\theta_* \int_0^{\pi} \vartheta(\theta, \theta_*, \tau - t) P_n(\cos\theta) \sin\theta \, d\theta. \end{gathered} \quad (2.9)$$

It follows from formulas (2.3) and (2.4) that $p_{2n}(\tau)$ and $p_{3n}(\tau)$ contain integrals with nonintegrable singularities of order -1 and -3 and integrable singularities of order $-1/2$.

Taking (2.6) into account, we write the equation of motion of the shell as a rigid body in the form

$$m_0 \ddot{u}_c = R_e + \pi \sum_{i=1}^{3} \sum_{n=0}^{\infty} p_{in}(\tau) \int_0^{b(\tau)} P_n(\cos\theta) \sin(2\theta) \, d\theta. \quad (2.10)$$

3. METHOD AND ALGORITHM OF SOLUTION

To solve the system of Eqs. (2.7)–(2.10), we use the modified fourth-order Runge–Kutta method and the principle of truncation of an infinite system of equations [5].

We replace the series (2.6) by finite sums with the upper summation limit equal to N. We reduce the system of Eqs. (2.7)–(2.10) to a first-order system and obtain a system of $6(N+1)+2$ ordinary differential equations supplemented with the algebraic equation (1.10). The first $6(N+1)$ equations can be represented in matrix form with block structure:

$$\gamma^2 \dot{\mathbf{W}} = \mathbf{M}\mathbf{W} + \mathbf{Q}, \quad (3.1)$$

$$\mathbf{W} = \|\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_N\|^T, \quad \mathbf{W}_n = \|u_{0n}, w_{0n}, \chi_n, \tilde{u}_{0n}, \tilde{w}_{0n}, \tilde{\chi}_n\|^T$$
$$\tilde{u}_{0n} = \dot{u}_{0n}, \quad \tilde{w}_{0n} = \dot{w}_{0n}, \quad \tilde{\chi}_{0n} = \dot{\chi}_{0n},$$
$$\mathbf{Q} = \|\mathbf{Q}_0, \mathbf{Q}_1, \ldots, \mathbf{Q}_N\|^T, \quad \mathbf{Q}_n = \frac{1}{h\tilde{\gamma}} \left\| 0, 0, 0, 0, \sum_{i=1}^{3} p_{in}, 0 \right\|^T,$$
$$\mathbf{M} = \begin{Vmatrix} \mathbf{M}_0 & & & \\ & \mathbf{M}_1 & \mathbf{0} & \\ & & \ddots & \\ & \mathbf{0} & & \mathbf{M}_N \end{Vmatrix}, \quad \mathbf{M}_n = \|M_{ijn}\|_{6\times 6} = \begin{Vmatrix} & & & | & 1 & 0 & 0 \\ & \mathbf{0} & & | & 0 & 1 & 0 \\ & & & | & 0 & 0 & 1 \\ - & - & - & | & - & - & - \\ L_{11n} & L_{12n} & L_{13n} & | & & & \\ L_{21n} & L_{22n} & L_{23n} & | & & \mathbf{0} & \\ L_{31n} & L_{32n} & L_{33n} & | & & & \end{Vmatrix}.$$

We supplement them with the system of equations of motion of the shell as a rigid body, which follows from (2.10):

$$\dot{u}_c = \tilde{u}_c, \quad \dot{\tilde{u}}_c = \frac{1}{m_0}\left[R_e + \pi \sum_{i=1}^{3}\sum_{n=0}^{N} p_{in}(\tau) \int\limits_0^{b(\tau)} P_n(\cos\theta)\sin(2\theta)\, d\theta\right] \tag{3.2}$$

and with an equation for determining the contact region radius (1.10).

The fifth equation in system (3.1) in each block with number n is an integro-differential equation, because its right-hand side contains unknown functions $\tilde{w}_{0n}(\tau) = \dot{w}_{0n}(\tau)$ in the integrand (see (2.9)). Then the right-hand side contains all $N+1$ functions $\tilde{w}_{0n}(\tau)$, $n = 0, 1, \ldots, N$, and hence, the system can be solved simultaneously for all $6(N+1)+3$ equations.

The modified method is different from the classical one in that, along with the use of the classical scheme [5], it is necessary to construct and use quadrature formulas to calculate the integrals in the integro-differential equations of system (3.1), (3.2). In the construction of the quadrature formulas, we use the explicit representation for the Legendre polynomials [4]:

$$P_n(x) = 2^{-n} \sum_{m=0}^{[n/2]} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m)!} x^{n-2m}. \tag{3.3}$$

The time coordinate τ is associated with discrete time moments $\tau_m = \delta_m m$, where δ_m is the time increment. The desired coefficients of the series of the displacements and their speeds, the radius of the contact region, the penetration depth of the indenter as a rigid body and the rate of penetration are replaced by their discrete analogues, i.e., the values at discrete time moments: $u_{0nm} = u_{0n}(\tau_m)$, $w_{0nm} = w_{0n}(\tau_m)$, $\chi_{nm} = \chi_n(\tau_m)$, $\tilde{u}_{0nm} = \tilde{u}_{0n}(\tau_m)$, $\tilde{w}_{0nm} = \tilde{w}_{0n}(\tau_m)$, $\tilde{\chi}_{nm} = \tilde{\chi}_n(\tau_m)$, $b_m = b(\tau_m)$, $u_{cm} = u_c(\tau_m)$, and $\tilde{u}_{cm} = \tilde{u}_c(\tau_m)$.

The quadrature formulas for the integrals appearing on the right-hand sides of Eqs. (3.1) and (3.2) are constructed with the use of representation (3.3), the method of weight coefficients, and the canonical regularization [6] for the obtained finite values of the singular integrals (see (2.3) and (2.4)):

$$p_{1nm} = p_{1n}(\tau_m) \approx -\frac{2n+1}{2}\sum_{k=0}^{N} \dot{w}_{0km} 2^{-n-k} \sum_{m_3=0}^{[k/2]}\sum_{m_2=0}^{[n/2]} (-1)^{m_2+m_3} \frac{(2k-2m_3)!}{m_3!(k-m_3)!(k-2m_3)!}$$
$$\times \frac{(2n-2m_2)!}{m_2!(n-m_2)!(n-2m_2)!}\left[\frac{1-(\cos b_m)^{k+n-2(m_2+m_3)+1)}}{k+n-2(m_2+m_3)+1}\right],$$
$$p_{2nm} = p_{2n}(\tau_m) \approx \frac{2n+1}{2}\sum_{k=0}^{N}\sum_{i=0}^{m-1} \dot{w}_{0ki} P_k(\cos b_i) \sum_{j=1}^{i} P_n(\cos\theta_j)\sin\theta_j$$

$$\times [\delta_m\{\omega_{1j}(b_i)f_1(\theta_j, b_i, \tau - t_i) + \omega_{3j}(b_i)f_2(\theta_j, b_i, \tau - t_i)\}$$
$$+ \tilde{\omega}_i\{\omega_{1j}(b_i)f_3(\theta_j, b_i, \tau - t_i) + \delta_l f_4(\theta_j, b_i, \tau - t_i)\}],$$

$$p_{3nm} = p_{2n}(\tau_m) \approx -\frac{2n+1}{2}\sum_{k=0}^{N}\sum_{i=0}^{m-1}\dot{w}_{0ki}\sum_{i_1=1}^{m_1}[P_k(\cos\theta_{*i_1}) - P_k(\cos\theta_{*i_1-1})]\sum_{j=1}^{i}P_n(\cos\theta_j)\sin\theta_j$$
$$\times [\delta_m\{\omega_{1j}(\theta_{*i_1})f_1(\theta_j, \theta_{*i_1}, \tau - t_i) + \omega_{3j}(\theta_{*i_1})f_2(\theta_j, \theta_{*i_1}, \tau - t_i)\}$$
$$+ \tilde{\omega}_i\{\omega_{1j}(\theta_{*i_1})f_3(\theta_j, \theta_{*i_1}, \tau - t_i) + \delta_l f_4(\theta_j, \theta_{*i_1}, \tau - t_i)\}],$$

$$\delta_l = \frac{\pi}{l}, \quad \delta_{m_1} = \frac{b_i}{m_1}, \quad \theta_j = j\delta_l, \quad t_i = i\delta_m, \quad \tau_m = m\delta_m, \quad \theta_{*i_1} = i_1\delta_{m_1},$$

$$f_1(\theta_j, x, \tau - t_i) = \sum_{q=1}^{2}\vartheta_{rq1}(\theta_j, x, \tau - t_i), \quad f_2(\theta_j, x, \tau - t_i) = \sum_{q=1}^{2}\vartheta_{rq2}(\theta_j, x, \tau - t_i),$$

$$f_3(\theta_j, x, \tau - t_i) = \sum_{q=1}^{2}\vartheta_{rq3}(\theta_j, x, \tau - t_i), \quad f_4(\theta_j, x, \tau - t_i) = \vartheta_s(\theta_j, x, \tau - t_i).$$

The weight coefficients are expressed as

$$\omega_{n_1 j} = \int_{\theta_{j-1}}^{\theta_j}\frac{d\theta}{(\sin\theta - x)^{n_1}}, \quad n_1 = 1, 3, \quad \tilde{\omega}_i = \int_{t_i}^{t_{i+1}}\frac{dt}{\tau_m - t}.$$

The system of Eqs. (3.1), (3.2), (1.10) is supplemented with the initial conditions

$$\mathbf{W}_n(0) = \begin{cases}\|0,0,0,0,0,0\|^T & \text{if } n \neq 1,\\ \|0,0,0,V_0,V_0,0\|^T & \text{if } n = 1,\end{cases}, \quad u_c(0) = 0, \quad \tilde{u}_c = V_0, \quad b(0) = 0. \tag{3.4}$$

The proposed calculation algorithm was implemented within the Delphi environment. The values of the total elliptic integrals were calculated by using their approximation by polynomials ($|\varepsilon(m)| \leq 2\times 10^{-8}$, $0 \leq m < 1$, $m_1 = 1 - m$):

$$K(m) = [a_0 + a_1 m_1 + \cdots + a_4 m_1^4] + [b_0 + b_1 m_1 + \cdots + b_4 m_1^4]\ln\frac{1}{m_1} + \varepsilon(m),$$
$$E(m) = [1 + a_1 m_1 + \cdots + a_4 m_1^4] + [b_1 m_1 + \cdots + b_4 m_1^4]\ln\frac{1}{m_1} + \varepsilon(m).$$

The values of the coefficients of these approximations are given in [4]. The incomplete elliptic integrals were calculated by the Simpson method [5].

4. EXAMPLE

As an example, we consider the problem with the following values of the dimensionless parameters (the shell and half-space materials are the same): $m_0 = 0.62832$, $h = 0.05$, $V_0 = 0.01$, $R_e = 0.1$, $\gamma^2 = 1$, $\lambda/\mu = 2$, and $\tilde{\gamma} = 1$. Figures 2–5 display the graphs of the time variation of the contact region radius $b(\tau)$ and its derivative $\dot{b}(\tau)$, the normal contact stress $p(\tau)$ at the frontal point, and the displacement of the indenter as a rigid body $u_c(\tau)$. Figures 6 and 7 show the normal displacements $w_0(\theta)$ and the normal contact stress $p(\theta)$ against the angle θ at the final time instant of the supersonic stage of interaction $\tau = 0.076$. In the calculations, we retained four terms of the series expansions in the Legendre polynomials and their derivatives, because retaining a greater number of terms did not improve the results significantly.

CONCLUSION

The use of the superposition principle has allowed us to obtain the resolving system of functional equations for the coefficients of the series expansion of the shell displacements, the indenter penetration

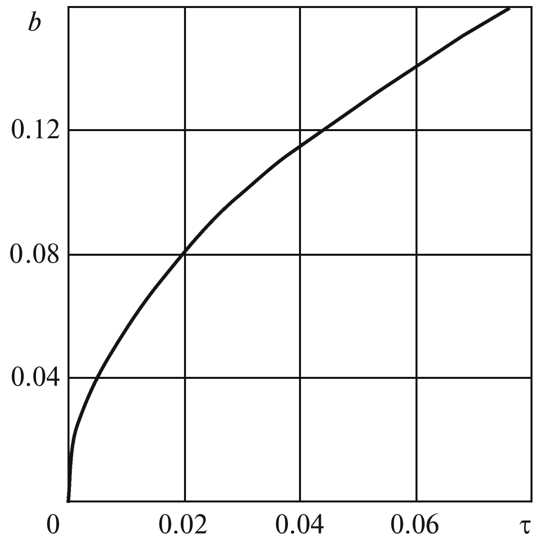

Fig. 2.

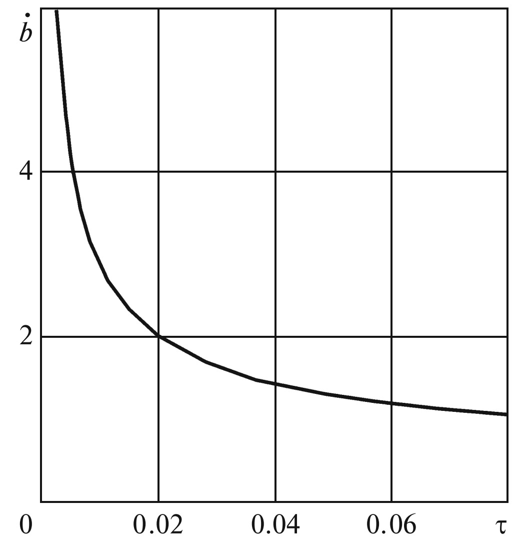

Fig. 3.

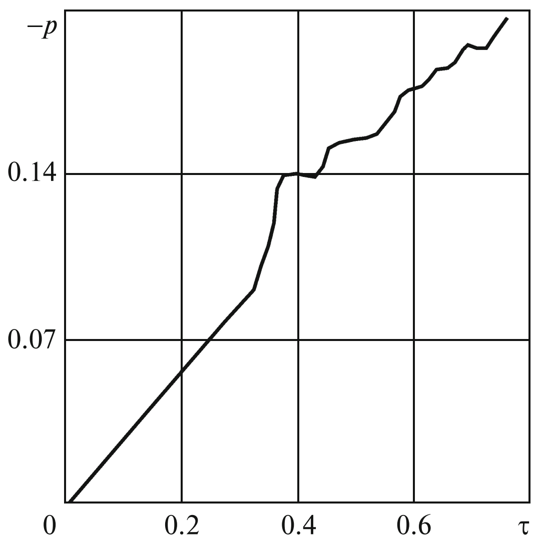

Fig. 4.

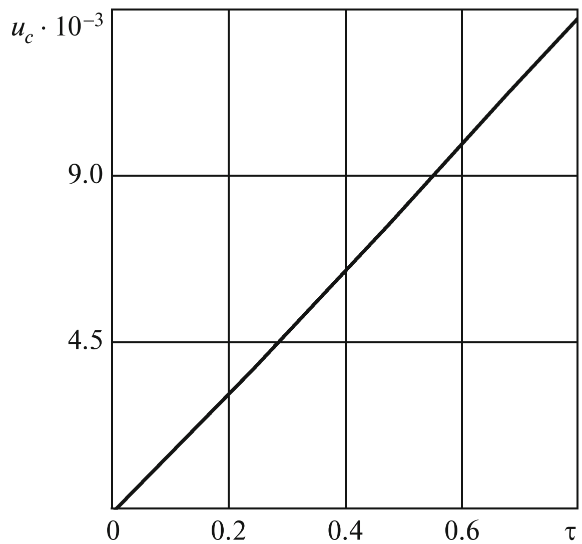

Fig. 5.

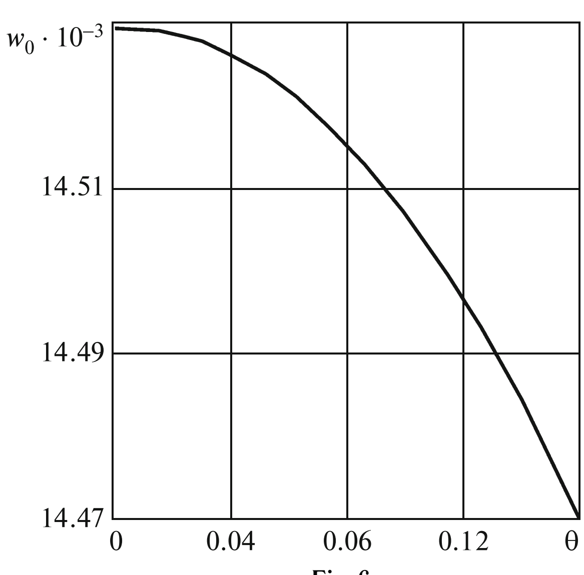

Fig. 6.

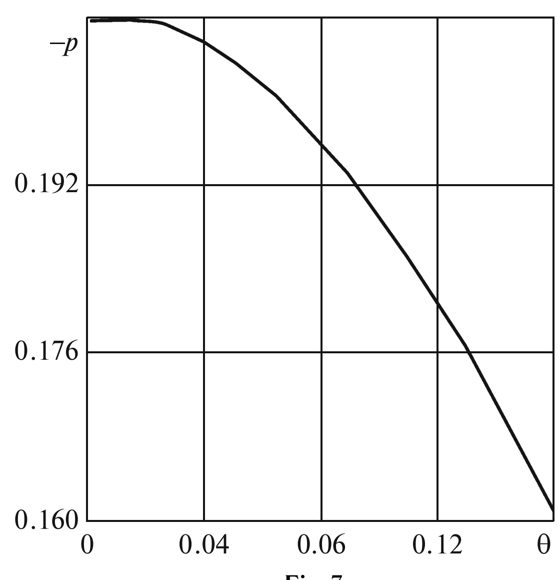

Fig. 7.

depth, and the radius of the contact region. At each discrete time instant, we have determined the distribution of the contact pressure over the interaction domain by using specially developed quadrature formulas, which take account of the nonintegrable and integrable singularities of the kernels of the integral representations.

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