

Nonsymmetric Askey-Wilson polynomials as vector-valued polynomials

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Symmetry, Separation, Super-integrability and Special Functions (S4),
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Work in collaboration with Fethi Bouzeffour (Bizerte, Tunisia)

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Some personal recollections

1969–1970

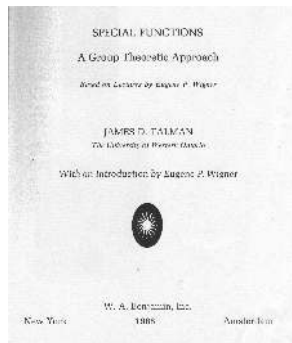
Dick Askey on sabbatical at Mathematical Centre Amsterdam. Inspired by him I started working on group theoretic interpretations of special functions.

I had three heroes in this area.

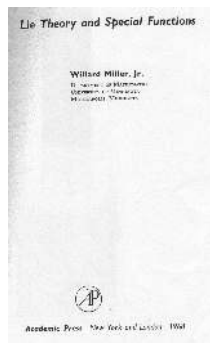
Three heroes



Vilenkin



Wigner (Talman)



Willard Miller, Jr.

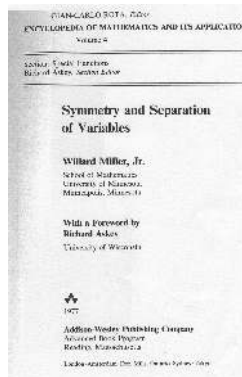
Different approaches

Vilenkin and Wigner started with the Lie group and identified matrix elements of irreducible representations with special functions. Only special parameter values came out.

But Miller started with the special functions and their differential recurrence relations, and he built from them a Lie algebra and next a local Lie group. This worked for all parameter values.

Symmetry and separation of variables

After a second book in 1972, Willard's third book appeared in 1977, on which I wrote a review for Bull. Amer. Math. Soc.



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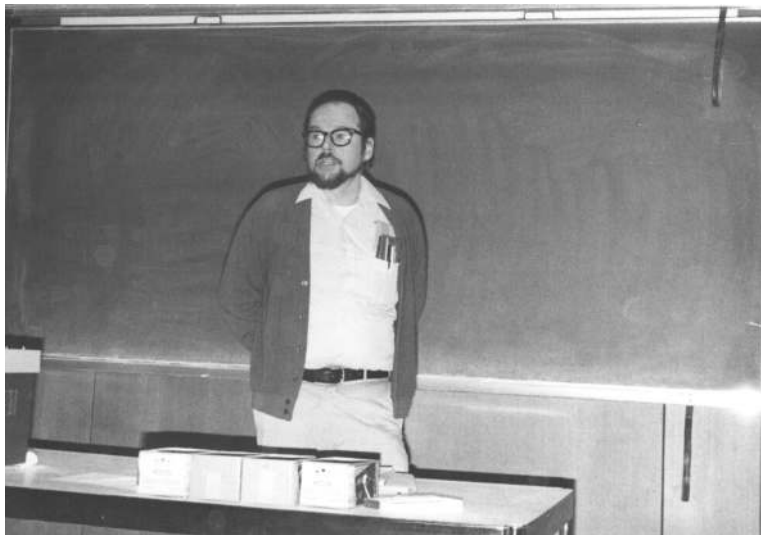
Symmetry and separation of variables, by Willard Miller, Jr., Addison-Wesley Publishing Company, Reading, Massachusetts, 1977, xxx + 285 pp., \$21.50.

Separation of variables is a technique for solving special partial differential equations. It is taught in elementary courses on partial differential equations, but the method usually does not achieve the status of a mathematical theory.

Because most references do not give a precise definition of separation of variables, I invented a definition myself. Let us call a partial differential equation in n variables x_1, \dots, x_n *separable* if there are n ordinary differential equations in x_1, \dots, x_n , respectively, jointly depending on $n - 1$ independent parameters (the separation constants), such that, for each choice of the parameters and for each set of solutions (X_1, \dots, X_n) of the o.d.e.'s, the function $u(x_1, \dots, x_n) = X_1(x_1) \cdots X_n(x_n)$ is a solution of the p.d.e. Under the terms of this definition a converse implication often holds: If $u = X_1 \cdots X_n$ is a factorized solution of the p.d.e. then, for some choice of the parameters, the X_i 's are solutions of the o.d.e.'s. The most familiar cases of separability deal with a linear second order p.d.e. which separates into n

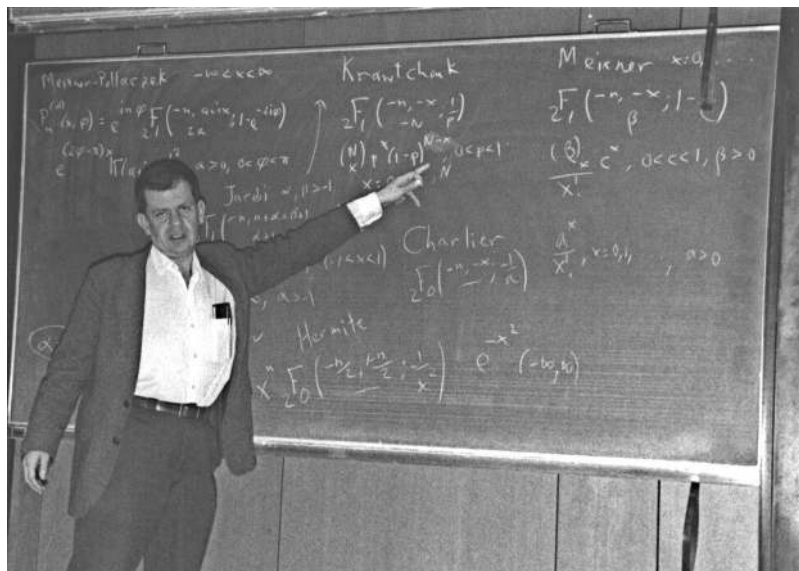
Oberwolfach 1983

Oberwolfach conference *Special functions and group theory*,
March 14–18, 1983.



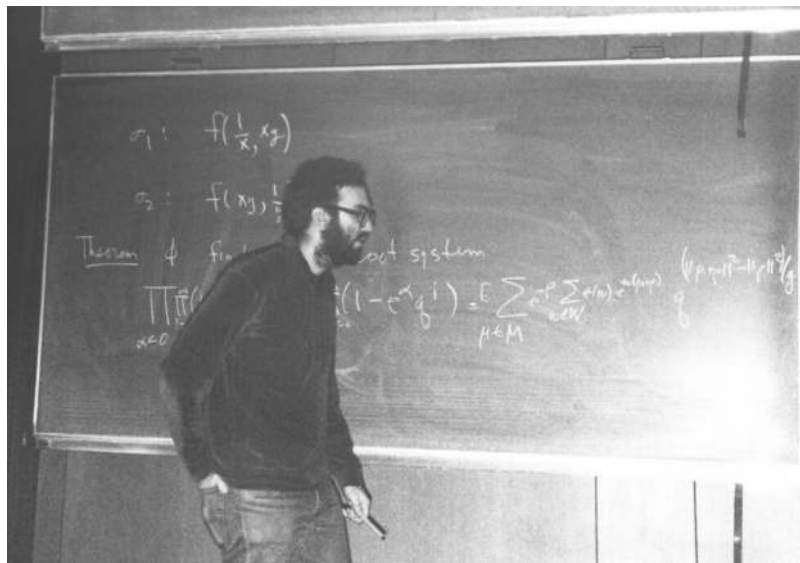
Willard Miller

Oberwolfach 1983 (cntd)



Dick Askey

Oberwolfach 1983 (cntd)



Dennis Stanton

Hankel transform

Normalized Bessel function:

$$\mathcal{J}_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{(\alpha+1)_k k!} = {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -\frac{1}{4}x^2\right).$$

$$\mathcal{J}_\alpha(x) = \mathcal{J}_\alpha(-x), \quad \mathcal{J}_\alpha(0) = 1, \quad \mathcal{J}_{-\frac{1}{2}}(x) = \cos x, \quad \mathcal{J}_{\frac{1}{2}}(x) = \frac{\sin x}{x}.$$

Eigenfunctions:

$$\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x} \frac{d}{dx} \right) \mathcal{J}_\alpha(\lambda x) = -\lambda^2 \mathcal{J}_\alpha(\lambda x).$$

Hankel transform pair:

$$\left\{ \begin{array}{l} \widehat{f}(\lambda) = \int_0^\infty f(x) \mathcal{J}_\alpha(\lambda x) x^{2\alpha+1} dx, \\ f(x) = \frac{1}{2^{2\alpha+1} \Gamma(\alpha+1)^2} \int_0^\infty \widehat{f}(\lambda) \mathcal{J}_\alpha(\lambda x) \lambda^{2\alpha+1} d\lambda. \end{array} \right.$$

Non-symmetric Hankel transform

Non-symmetric Bessel function:

$$\mathcal{E}_\alpha(x) := \mathcal{J}_\alpha(x) + \frac{ix}{2(\alpha+1)} \mathcal{J}_{\alpha+1}(x), \quad \text{so} \quad \mathcal{E}_{-\frac{1}{2}}(x) = e^{ix}.$$

Non-symmetric Hankel transform pair:

$$\begin{cases} \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) \mathcal{E}_\alpha(-\lambda x) |x|^{2\alpha+1} dx, \\ f(x) = \frac{1}{2^{2(\alpha+1)} \Gamma(\alpha+1)^2} \int_{-\infty}^{\infty} \widehat{f}(\lambda) \mathcal{E}_\alpha(\lambda x) |\lambda|^{2\alpha+1} d\lambda. \end{cases}$$

Differential-reflection operator:

$$(Yf)(x) := f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}$$

(Dunkl operator for root system A_1).

Eigenfunctions:

$$Y(\mathcal{E}_\alpha(\lambda \cdot)) = i\lambda \mathcal{E}_\alpha(\lambda \cdot).$$

Askey-Wilson polynomials

Assume $0 < q < 1$.

Monic Askey-Wilson polynomials as symmetric Laurent polynomials):

$$P_n[z] = P_n[z; a, b, c, d \mid q] = P_n\left(\frac{1}{2}(z + z^{-1})\right) \\ := \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

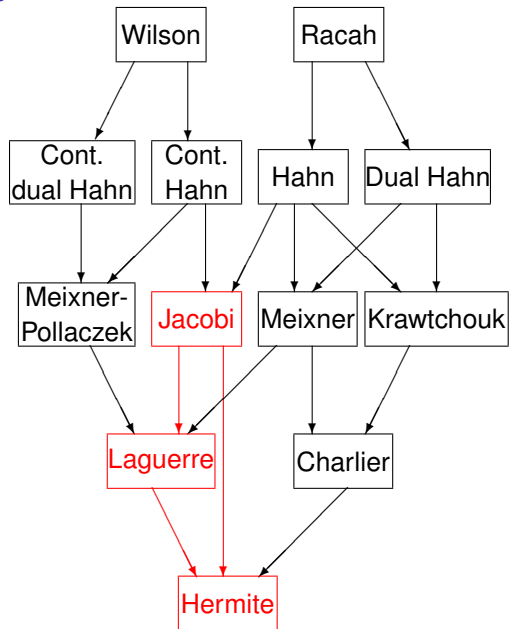
Eigenfunctions of second order q -difference operator L :

$$(LP_n)[z] := A[z] P_n[qz] + A[z^{-1}] P_n[q^{-1}z] - (A[z] + A[z^{-1}]) P_n[z] \\ = (q^{-n} - 1)(1 - abcdq^{n-1})P_n[z],$$

$$\text{where } A[z] := \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)}.$$

These are on the top level of the **q -Askey scheme**.

Askey scheme



Dick Askey



Jim Wilson

Non-symmetric Askey-Wilson polynomials

Further assume: $a, b, c, d \neq 0$, $abcd \neq q^{-m}$ ($m = 0, 1, 2, \dots$),
 $\{a, b\} \cap \{a^{-1}, b^{-1}\} = \emptyset$.

In terms of

$$P_n[z] = P_n[z; a, b, c, d \mid q],$$

$$Q_n[z] := a^{-1}b^{-1}z^{-1}(1 - az)(1 - bz)P_{n-1}[z; qa, qb, c, d \mid q]$$

the nonsymmetric Askey-Wilson polynomials are defined by:

$$E_{-n} := \frac{ab}{ab - 1} (P_n - Q_n) \quad (n = 1, 2, \dots), \quad E_0[z] := 1,$$
$$E_n := \frac{(1 - q^n ab)(1 - q^{n-1} abcd)}{(1 - ab)(1 - q^{2n-1} abcd)} P_n$$
$$- \frac{ab(1 - q^n)(1 - q^{n-1} cd)}{(1 - ab)(1 - q^{2n-1} abcd)} Q_n \quad (n = 1, 2, \dots).$$

Eigenfunctions of q -difference-reflection operator

Let

$$\begin{aligned}(Yf)[z] := & \frac{z(1 + ab - (a + b)z)((c + d)q - (cd + q)z)}{q(1 - z^2)(q - z^2)} f[z] \\ & + \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)} f[qz] \\ & + \frac{(1 - az)(1 - bz)((c + d)qz - (cd + q))}{q(1 - z^2)(1 - qz^2)} f[z^{-1}] \\ & + \frac{(c - z)(d - z)(1 + ab - (a + b)z)}{(1 - z^2)(q - z^2)} f[qz^{-1}],\end{aligned}$$

Then

$$YE_{-n} = q^{-n} E_{-n} \quad (n = 1, 2, \dots),$$

$$YE_n = q^{n-1} abcd E_n \quad (n = 0, 1, 2, \dots).$$

These come from I. Cherednik's theory of double affine Hecke algebras associated with root systems, extended by S. Sahi to the type (C_l^\vee, C_l) . Here his case $l = 1$ is considered.

Double affine Hecke algebra of type (C_1^\vee, C_1)

This is the algebra $\tilde{\mathfrak{h}}$ generated by Z, Z^{-1}, T_1, T_0 with relations $ZZ^{-1} = 1 = Z^{-1}Z$ and

$$\begin{aligned}(T_1 + ab)(T_1 + 1) &= 0, & (T_0 + q^{-1}cd)(T_0 + 1) &= 0, \\ (T_1Z + a)(T_1Z + b) &= 0, & (qT_0Z^{-1} + c)(qT_0Z^{-1} + d) &= 0.\end{aligned}$$

This algebra acts faithfully on the space of Laurent polynomials:

$$\begin{aligned}(Zf)[z] &:= z f[z], \\ (T_1f)[z] &:= \frac{(a+b)z - (1+ab)}{1-z^2} f[z] + \frac{(1-az)(1-bz)}{1-z^2} f[z^{-1}], \\ (T_0f)[z] &:= \frac{q^{-1}z((cd+q)z - (c+d)q)}{q-z^2} f[z] \\ &\quad - \frac{(c-z)(d-z)}{q-z^2} f[qz^{-1}].\end{aligned}$$

Then $Y = T_1 T_0$.

Eigenspaces of T_1

- ▶ T_1 acting on Laurent polynomials has eigenvalues $-ab$ and -1 .
- ▶ $T_1 f = -ab f \iff f$ is symmetric.
- ▶ $T_1 f = -f \iff f[z] = z^{-1}(1 - az)(1 - bz)g[z]$ for some symmetric Laurent polynomial g .

Let A be an operator acting on the Laurent polynomials. Write $f[z] = f_1[z] + z^{-1}(1 - az)(1 - bz)f_2[z]$ (f_1, f_2 symmetric Laurent polynomials). Then we can write

$$(Af)[z] = (A_{11}f_1 + A_{12}f_2)[z] + z^{-1}(1 - az)(1 - bz)(A_{21}f_1 + A_{22}f_2)[z],$$

where the A_{ij} are operators acting on the symmetric Laurent polynomials. So

$$f \leftrightarrow (f_1, f_2), \quad A \leftrightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Rewriting the eigenvalue equation for E_n in matrix form

$$\left(\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - q^{-n} \right) \begin{pmatrix} P_n[z; a, b, c, d | q] \\ -a^{-1}b^{-1}P_{n-1}[z; qa, qb, c, d | q] \end{pmatrix} = 0,$$
$$\left(\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - q^{n-1}abcd \right) \times \begin{pmatrix} (1 - q^n ab)(1 - q^{n-1}abcd) P_n[z; a, b, c, d | q] \\ -(1 - q^n)(1 - q^{n-1}cd) P_{n-1}[z; qa, qb, c, d | q] \end{pmatrix} = 0.$$

Here

$$Y_{11} = q^{-1}abcd - \frac{ab}{1-ab} L_{a,b,c,d;q},$$

$$Y_{22} = \frac{1 - abcd - abq + abcdq + L_{aq,bq,c,d;q}}{q(1-ab)},$$

where $L_{a,b,c,d;q}$ is the second order q -difference operator L having the $p_n[z; a, b, c, d | q]$ as eigenfunctions.

On the next sheet the shift operators Y_{21}, Y_{12} .

The shift operators

$$(Y_{21}g)[z] = \frac{z(c-z)(d-z)(g[q^{-1}z] - g[z])}{(1-ab)(1-z^2)(1-qz^2)} + \frac{z(1-cz)(1-dz)(g[qz] - g[z])}{(1-ab)(1-z^2)(1-qz^2)},$$

$$(Y_{12}h)[z] = \frac{ab(a-z)(b-z)(1-az)(1-bz)}{(1-ab)z(q-z^2)(1-qz^2)} \times ((cd+q)(1+z^2) - (1+q)(c+d)z) h[z] \\ - \frac{ab(a-z)(b-z)(c-z)(d-z)(aq-z)(bq-z)}{q(1-ab)z(1-z^2)(q-z^2)} h[q^{-1}z] \\ - \frac{ab(1-az)(1-bz)(1-cz)(1-dz)(1-aqz)(1-bqz)}{q(1-ab)z(1-z^2)(1-qz^2)} h[qz].$$

An equivalent form for the eigenvalue equations

The eigenvalue equations for E_n and for E_{-n} are equivalent to the four equations

$$L_{a,b,c,d;q}P_n[\cdot; a, b, c, d | q]$$

$$= (q^{-n} - 1)(1 - abcdq^{n-1})P_n[\cdot; a, b, c, d | q],$$

$$L_{qa,qb,c,d;q}P_{n-1}[\cdot; qa, qb, c, d | q]$$

$$= (q^{-n+1} - 1)(1 - abcdq^n)P_{n-1}[\cdot; qa, qb, c, d | q],$$

$$Y_{21}P_n[\cdot; a, b, c, d | q]$$

$$= - \frac{(q^{-n} - 1)(1 - cdq^{n-1})}{1 - ab} P_{n-1}[\cdot; qa, qb, c, d | q],$$

$$Y_{12}P_{n-1}[\cdot; qa, qb, c, d | q]$$

$$= - \frac{ab(q^{-n} - ab)(1 - abcdq^{n-1})}{1 - ab} P_n[\cdot; a, b, c, d | q].$$

Non-symmetric Bessel functions in vector-valued form

We can rewrite the equations

$$\mathcal{E}_\alpha(x) = \mathcal{J}_\alpha(x) + \frac{ix}{2(\alpha+1)} \mathcal{J}_{\alpha+1}(x),$$

$$(Yf)(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

$$Y(\mathcal{E}_\alpha(\lambda \cdot)) = i\lambda \mathcal{E}_\alpha(\lambda \cdot).$$

in the form

$$\left(\begin{pmatrix} 0 & x \frac{d}{dx} + 2\alpha + 2 \\ x^{-1} \frac{d}{dx} & 0 \end{pmatrix} - i\lambda \right) \begin{pmatrix} \mathcal{J}_\alpha(\lambda x) \\ \frac{i\lambda}{2(\alpha+1)} \mathcal{J}_{\alpha+1}(\lambda x) \end{pmatrix} = 0.$$

Applications of the vector-valued approach

By writing the nonsymmetric Askey-Wilson polynomials in vector-valued form, we can obtain two results which would have been impossible or much harder in the Laurent polynomial form:

- ▶ Orthogonality relations with positive definite inner product
- ▶ Limit to nonsymmetric little q -Jacobi polynomials

Orthogonality relations: scalar case

Let $\langle \cdot, \cdot \rangle_{a,b,c,d;q}$ be the Hermitian inner product on the space of symmetric Laurent polynomials such that the $P_n[\cdot; a, b, c, d | q]$ are orthogonal in the familiar way:

$$\langle P_n[\cdot; a, b, c, d | q], P_m[\cdot; a, b, c, d | q] \rangle_{a,b,c,d;q} = h_n^{a,b,c,d;q} \delta_{n,m},$$

where

$$h_n^{a,b,c,d;q} := \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n} (q^{n-1} abcd; q)_n}.$$

(Assume that a, b, c, d are such that $\langle \cdot, \cdot \rangle_{a,b,c,d;q}$ and $\langle \cdot, \cdot \rangle_{qa,qb,c,d;q}$ are positive definite.) Then

$$\begin{aligned} \frac{h_n^{a,b,c,d;q}}{h_{n-1}^{qa,qb,c,d;q}} &= \frac{(1 - q^n)(1 - q^{n-1}cd)}{(1 - q^n ab)(1 - q^{n-1}abcd)} \\ &\times \frac{(1 - ab)(1 - qab)(1 - ac)(1 - ad)(1 - bc)(1 - bd)}{(1 - abcd)(1 - qabcd)}. \end{aligned}$$

Orthogonality relations: vector-valued case

For g_1, h_1, g_2, h_2 symmetric Laurent polynomials define an hermitian inner product

$$\begin{aligned} \langle (g_1, h_1), (g_2, h_2) \rangle &:= \langle g_1, g_2 \rangle_{a,b,c,d;q} - ab(1-ab)(1-qab) \\ &\quad \times \frac{(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)} \langle h_1, h_2 \rangle_{qa,qb,c,d;q}. \end{aligned}$$

Then the E_n ($n \in \mathbb{Z}$) in vector-valued form are orthogonal with respect to this inner product. (Need only to check that E_n is orthogonal to E_{-n} .)

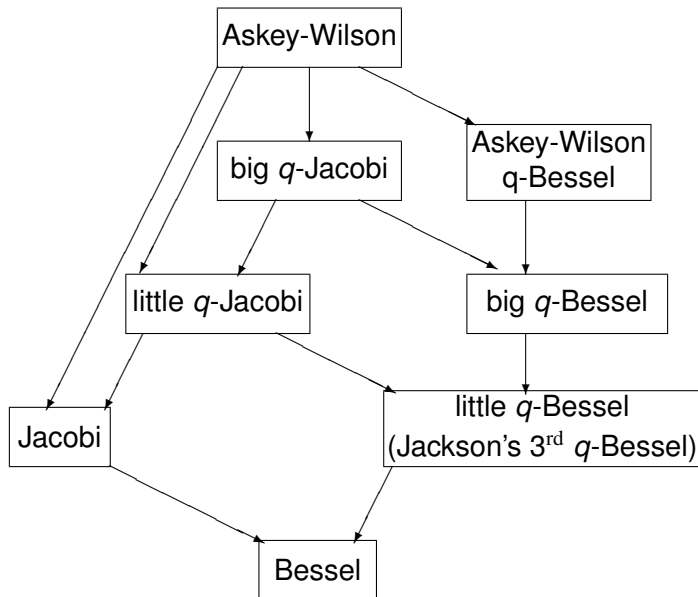
Theorem

If moreover $ab < 0$ then the inner product is positive definite.

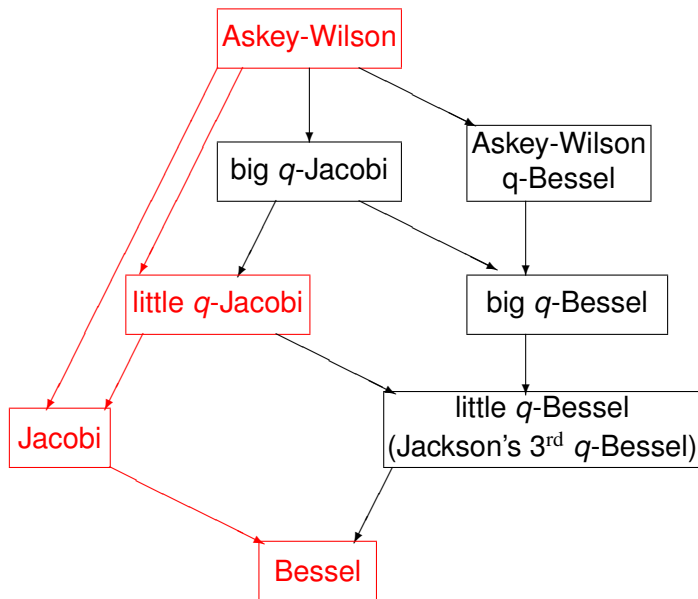
In earlier papers (Sahi, Noumi & Stokman, Macdonald's 2003 book) a biorthogonality was given in the form of a contour integral, and there were no results on positive definiteness of the inner product.

The (q -)Askey-Bessel scheme

See Koelink & Stokman, NATO, Tempe, 2000.



Already done: red boxes and arrows



Little q -Jacobi polynomials

Monic little q -Jacobi polynomials as ordinary polynomials:

$$P_n^{\text{lqJ}}(x; a, b; q) := \frac{(-1)^n q^{n(n-1)/2} (aq; q)_n}{(abq^{n+1}; q)_n} {}_2\phi_1 \left(\begin{matrix} q^n, abq^{n+1} \\ aq \end{matrix}; q, qx \right).$$

They are limits of Askey-Wilson polynomials:

$$P_n^{\text{lqJ}}(x; a, b; q) = \lim_{\lambda \downarrow 0} \lambda^n P_n^{\text{AW}}(\lambda^{-1}x; -q^{1/2}a, qb\lambda, -q^{1/2}, \lambda^{-1} \mid q),$$

$$P_{n-1}^{\text{lqJ}}(x; qa, qb; q) = \lim_{\lambda \downarrow 0} \lambda^{n-1} P_{n-1}^{\text{AW}}(\lambda^{-1}x; -q^{3/2}a, q^2b\lambda, -q^{1/2}, \lambda^{-1} \mid q)$$

Nonsymmetric little q -Jacobi polynomials

$$\left(\begin{pmatrix} Y_{11}^{AW} & Y_{12}^{AW} \\ Y_{21}^{AW} & Y_{22}^{AW} \end{pmatrix} - q^{-n} \right) \begin{pmatrix} P_n^{AW}[z; a, b, c, d | q] \\ -a^{-1}b^{-1}P_{n-1}^{AW}[z; qa, qb, c, d | q] \end{pmatrix} = 0,$$

$$\left(\begin{pmatrix} Y_{11}^{AW} & Y_{12}^{AW} \\ Y_{21}^{AW} & Y_{22}^{AW} \end{pmatrix} - q^{n-1}abcd \right) \times \begin{pmatrix} (1 - q^n ab)(1 - q^{n-1}abcd) P_n^{AW}[z; a, b, c, d | q] \\ -(1 - q^n)(1 - q^{n-1}cd) P_{n-1}^{AW}[z; qa, qb, c, d | q] \end{pmatrix} = 0.$$

Substitute: $a \rightarrow -q^{1/2}a$, $b \rightarrow qb\lambda$, $c \rightarrow -q^{1/2}$, $d \rightarrow \lambda^{-1}$,
 $z \rightarrow \lambda^{-1}x$ and let $\lambda \downarrow 0$:

$$\left(\begin{pmatrix} Y_{11}^{lqJ} & Y_{12}^{lqJ} \\ Y_{21}^{lqJ} & Y_{22}^{lqJ} \end{pmatrix} - q^{-n} \right) \begin{pmatrix} P_n^{lqJ}(x; a, b; q) \\ a^{-1}b^{-1}q^{-3/2}P_{n-1}^{lqJ}(x; qa, qb; q) \end{pmatrix} = 0,$$

$$\left(\begin{pmatrix} Y_{11}^{lqJ} & Y_{12}^{lqJ} \\ Y_{21}^{lqJ} & Y_{22}^{lqJ} \end{pmatrix} - q^{n+1}ab \right) \begin{pmatrix} (1 - q^{n+1}ab) P_n^{lqJ}(x; a, b; q) \\ -(1 - q^n)q^{n-1/2} P_{n-1}^{lqJ}(x; qa, qb; q) \end{pmatrix} = 0.$$

Some literature

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