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NONUNIFORM CENTRAL LIMIT BOUNDS WITH APPLICATIONS TO PROBABILITIES OF DEVIATIONS

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For the distribution of the standardized sum of independent and identically distributed random variables, nonuniform central limit bounds are proved under an appropriate moment condition. From these theorems a condition on the sequence t_n, $n \in \mathbb{N}$, is derived which implies that $1 - F_n(t_n)$ is equivalent to the corresponding deviation of a normally distributed random variable. Furthermore, a necessary and sufficient condition is given for $1 - F_n(t_n) = o(n^{-c/2}t_n^{2+c})$.

1. Introduction. Consider a sequence of independent and identically distributed random variables $X_1, X_2, \cdots$ such that $EX = 0$ and $EX^2 = 1$.

Let F_n be the distribution function of $n^{-\frac{1}{2}}S_n = n^{-\frac{1}{2}} \sum_{i=1}^n X_i$. It follows from the central limit theorem that $1 - F_n(t_n) \to 0$ if and only if $t_n \to \infty$.

In applications we often are interested in obtaining bounds for the speed at which $1 - F_n(t_n)$ converges to zero. Obviously, these bounds should depend on n and t_n in a reasonably good way.

Assuming appropriate moment conditions we establish in Theorems 4 and 5 results on the speed of convergence to zero of $1 - F_n(t_n)$, which are deduced from two fundamental theorems related to two theorems of Esseen [4] and giving nonuniform central limit bounds. Using these two theorems once more we derive a nonuniform central limit result, which includes a weaker version of the corresponding result of Nagaev on the Berry–Esseen theorem as a special case. Finally, we also obtain a short proof of a result of von Bahr [1] on the approximation of the absolute moments of S_n.

2. The results. Since all theorems in this paper are proved under the same moment conditions, we do not state them explicitly in the formulation of the theorems. Throughout the paper we will assume the following conditions: $X_1, X_2, \cdots$ is a sequence of i.i.d. random variables such that

$$EX = 0, \qquad EX^2 = 1,$$

and $E|X|^{2+c} < \infty$ for some (fixed) $c > 0$.

THEOREM 1. *There exist constants $b, r > 0$ (depending on c) such that for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$ with $t^2 \leqq (c+1)\log n$,*

$$|F_n(t) - \Phi(t)| \leqq bn^{-c^*} \exp[-(1-\sigma)t^2/2] + nP(|X| > rn^{\frac{1}{2}}|t|),$$

where $c^* = \frac{1}{2}\min(c, 1)$ and $\sigma = c^*(c+1)^{-1}$.

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THEOREM 2. *There exist constants $b, r > 0$ (depending on c) such that for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$ with $t^2 \geqq (c+1)\log n$,*

$$|F_n(t) - \Phi(t)| \leqq bn^{-c/2}t^{-2(c+2)} + nP(|X| > rn^{\frac{1}{2}}|t|)\,.$$

The reader might wonder why the bounds given in Theorems 1 and 2 are written down in this "incomplete" way, i.e., why we did not use Markov's inequality to obtain $nP(|X| > rn^{\frac{1}{2}}|t|) \leqq bn^{-c/2}|t|^{-(2+c)}$. This is so because for $|t| \to \infty$ we lose a little, namely that in this case we have by the dominated convergence theorem, $nP(|X| > rn^{\frac{1}{2}}|t|) = o(n^{-c/2}|t|^{-(c+2)})$. Secondly, this form of the bounding terms is well suited for the proof of Theorem 6.

Theorem 1 is related to Theorem 2 in Esseen ([4], page 73), whereas Theorem 2 generalizes Theorem 3 of Esseen ([4], page 75), where $c \geqq 1$ is assumed to be an integer. This can easily be seen, since the conclusion in the remarks following both theorems of Esseen is equivalent to the corresponding assertion of the theorems.

We remark that Theorem 2 also has a certain relationship to Theorem 1 in Nagaev ([7], page 214).

In the following theorem we obtain a generalization of Theorem 3 in Nagaev ([7], page 215), where a nonuniform Berry–Esseen theorem is established under the moment condition $E|X|^3 < \infty$.

THEOREM 3. *There exists a constant $b > 0$ (depending on c) such that for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$,*

$$|F_n(t) - \Phi(t)| \leqq bn^{-c^*}(1 + |t|^{2+c})^{-1}\,,$$

where $c^* = \frac{1}{2}\min(c, 1)$.

REMARK 1. Since for $Y \geqq 0$ we have $EY = \int_0^\infty P(Y > t)\,dt$, a nonuniform bound of the form $|F_n(t) - \Phi(t)| \leqq b_n(1 + |t|^\alpha)^{-1}$, $t \in \mathbb{R}$, implies that $E|S_n|^\beta$ and therefore $E|X|^\beta$ exists for all $\beta \in (0, \alpha)$. Hence, the power of $|t|$ in our Theorem 3 cannot be increased in general.

When the moment generating function exists, Cramér [3] has shown that

$$1 - F_n(t_n) \sim (2\pi)^{-\frac{1}{2}}t_n^{-1}\exp[-t_n^2/2]\,,$$

if $t_n \to \infty$ and $n^{-\frac{1}{6}}t_n \to 0$.

For moderate deviations, i.e., for the case $t_n = (c\log n)^{\frac{1}{2}}$ this result has been obtained by Michel [6] under the much less restrictive moment condition $E|X|^{2+c} < \infty$.

Using Theorem 1 we prove that—under the same moment condition—this result holds true for all sequences t_n, $n \in \mathbb{N}$, with $t_n \to \infty$ such that $t_n^2 - c\log n - (c+1)\log\log n$ is bounded from above.

THEOREM 4. *Let t_n, $n \in \mathbb{N}$, be a sequence with $t_n \to \infty$ such that $t_n^2 - c\log n - (c+1)\log\log n$ is bounded from above. Then,*

$$1 - F_n(t_n) \sim (2\pi)^{-\frac{1}{2}}t_n^{-1}\exp[-t_n^2/2]\,.$$

The preceding theorem gives an approximation of $1 - F_n(t_n)$ for certain sequences t_n, $n \in \mathbb{N}$. In the remaining case such a strong result cannot be expected under our mild moment conditions. It is shown in Theorem 5 that in this case one obtains $1 - F_n(t_n) = o(n^{-c/2} t_n^{-(2+c)})$.

THEOREM 5. *Let t_n, $n \in \mathbb{N}$, be a sequence with $t_n \to \infty$ and $t_n^2 - c \log n - (c+1) \log\log n \to \infty$. Then,*

$$1 - F_n(t_n) = o(n^{-c/2} t_n^{-(2+c)}) . \tag{1}$$

We remark that the result of Theorem 5 has been obtained by Chibisov ([2], Lemma 4.2, page 158) for the case $t_n/\log n \to \infty$.

REMARK 2. It will be proved below that—under the assumed moment conditions—the property $t_n^2 - c \log n - (c+1) \log\log n \to \infty$ characterizes the sequences t_n, $n \in \mathbb{N}$, for which (1) holds true.

There has been much work done in deriving upper bounds for the absolute moments of S_n. In von Bahr [1] expansions of these moments are given. For the approximation of the absolute moments of order greater than 2 by the corresponding absolute moments of a standard normal distribution these results follow immediately from our Theorems 1 and 2.

THEOREM 6. *There exists a constant $b > 0$ (depending on c) such that for all $n \in \mathbb{N}$,*

$$|E|n^{-\frac{1}{2}} S_n|^{2+c} - \pi^{-\frac{1}{2}} 2^{(2+c)/2} \Gamma((3+c)/2)| \leqq b n^{-c^*} ,$$

where $c^* = \frac{1}{2} \min(c, 1)$.

3. Proofs.

PROOF OF THEOREM 1. For $|t| \leqq 1$ the assertion follows from the Katz–Petrov theorem (see Katz [5]).

For $|t| \geqq 1$ the proof is an obvious modification of the proof of Theorem 1 in Michel [6]: Truncate the random variables X_i, $i = 1, \cdots, n$, at $rn^{\frac{1}{2}}|t|$, where $r > 0$ is sufficiently small, and replace $c(\log n)^{\frac{1}{2}}$ by t.

PROOF OF THEOREM 2. W.l.o.g. we may assume $t > 0$ (recall that $EX = 0$). For $k = 1, \cdots, n$ let $\bar{X}_k$ denote X_k truncated at $rn^{\frac{1}{2}}t$, where $r = 1/(2(c+1)(c+2))$, and let $\bar{S}_n = n^{-\frac{1}{2}} \sum_{k=1}^n \bar{X}_k$. Since $1 - F_n(t) \leqq P(\bar{S}_n > t) + nP(|X| > rn^{\frac{1}{2}}t)$, and since $t^2 \geqq (c+1) \log n$ implies $1 - \Phi(t) \leqq b_1 \exp[-t^2/2] \leqq b_1 n^{-c/2} \exp[-t^2/(2(c+1))] \leqq b_2 n^{-c/2} t^{-2(c+2)}$, we have to show that

$$P(\bar{S}_n > t) \leqq b_3 n^{-c/2} t^{-2(c+2)} .$$

Let $h = t^{-1} n^{-\frac{1}{2}} (c \log n + 2(c+1)(c+2) \log t)$. Then,

$$P(\bar{S}_n > t) \leqq \beta^n \exp[-hn^{\frac{1}{2}}t] = \beta^n n^{-c} t^{-2(c+1)(c+2)} ,$$

where $\beta = E \exp[h\bar{X}]$.

Using $E\bar{X}^2 \leqq 1$ and $|hE\bar{X}| + \frac{1}{6} h^3 \exp[hrn^{\frac{1}{2}}t] E|\bar{X}|^3 \leqq b_4 n^{-1}$, where $b^4 > 0$ is an appropriately chosen constant, we obtain

$$\beta \leqq 1 + h^2/2 + b_4 n^{-1} \leqq \exp[h^2/2 + b_4 n^{-1}] .$$

Hence,

$$\beta^n \leqq \exp[nh^2/2 + b_4] .$$

The assertion now follows immediately, since $t \geqq ((c+1)\log n)^{\frac{1}{2}}$ implies

$$nh^2/2 \leqq \tfrac{1}{2}c \log n + 2c(c+2)\log t + b_5 ,$$

where $b_5 > 0$ is an appropriately chosen constant.

Proof of Theorem 3. Follows in the case $|t| \geqq 1$ immediately from Theorems 1 and 2. For $|t| < 1$ it is a consequence of the Katz–Petrov theorem.

Proof of Theorem 4. Let t_n, $n \in \mathbb{N}$, be a sequence with $t_n \to \infty$ such that $t_n^2 - c\log n - (c+1)\log\log n \leqq M$, where M is a positive constant. W.l.o.g. we may assume $t_n^2 \leqq (c+1)\log n$. From Theorem 1 we obtain

$$\begin{aligned} t_n \exp[t_n^2/2]|1 - F_n(t_n) - \Phi(-t_n)| \\ \leqq b t_n n^{-c^*} \exp[\sigma t_n^2/2] + t_n^{-(c+1)} n^{-c/2} r_n \exp[t_n^2/2] , \end{aligned}$$

where $r_n = r^{-(2+c)} E|X|^{c+2} 1_{\{|X| > rn^{\frac{1}{2}} t_n\}} = o(1)$.

Concerning the first term of the r.h.s. of the inequality $t_n^2 \leqq (c+1)\log n$ implies

$$t_n n^{-c^*} \exp[\sigma t_n^2/2] \leqq t_n n^{-c^*/2} = o(1) .$$

Furthermore, since $t \to t^{-(c+1)} \exp[t^2/2]$ is increasing for $t \geqq (c+1)^{\frac{1}{2}}$, we have

$$\begin{aligned} & t_n^{-(c+1)} n^{-c/2} \exp[t_n^2/2] \\ & \quad \leqq \exp[M/2](c\log n + (c+1)\log\log n + M)^{-(c+1)/2}(c\log n)^{(c+1)/2} = O(1) . \end{aligned}$$

Since $\Phi(-t_n) \sim (2\pi)^{-\frac{1}{2}} t_n^{-1} \exp[-t_n^2/2]$, the assertion follows.

Proof of Theorem 5. Let $t_n^2 = c\log n + (c+1)\log\log n + s_n$, where $s_n \to \infty$.

(i) For all $n \in \mathbb{N}$ with $t_n^2 \geqq (c+1)\log n$ we obtain from Theorem 2 and from

$$\Phi(-t_n) \leqq \exp[-t_n^2/2] \leqq n^{-c/2} \exp[-t_n^2/(2(c+1))]$$

that

$$n^{c/2} t_n^{2+c}(1 - F_n(t_n)) = o(1) .$$

(ii) Using $\Phi(-t_n) \leqq (2\pi)^{-\frac{1}{2}} t_n^{-1} \exp[-t_n^2/2]$ we obtain from Theorem 1 for all $n \in \mathbb{N}$ with $t_n^2 \leqq (c+1)\log n$ that

$$\begin{aligned} n^{c/2} t_n^{c+2}(1 - F_n(t_n)) \leqq n^{c/2} t_n^{c+1} \exp[-t_n^2/2]((2\pi)^{-\frac{1}{2}} + b n^{-c^*} t_n \exp[\sigma t_n^2/2]) \\ + r^{-(c+2)} E|X|^{c+2} 1_{\{|X| > rn^{\frac{1}{2}} t_n\}} . \end{aligned}$$

Since

$$n^{c/2} t_n^{c+1} \exp[-t_n^2/2] \leqq (c+1)^{(c+1)/2} \exp[-s_n/2] = o(1)$$

and

$$n^{-c^*} t_n \exp[\sigma t_n^2/2] \leqq (c+1)^{\frac{1}{2}} (\log n)^{\frac{1}{2}} n^{-c^*/2} ,$$

the assertion follows.

PROOF OF REMARK 2. Let t_n, $n \in \mathbb{N}$, with $t_n \to \infty$ be a sequence containing a subsequence t_k, $k \in \mathbb{N}$, such that $t_k^2 \leqq c \log k + (c+1) \log\log k + M$, where $M > 0$ is a constant. From Theorem 4 we then obtain for $k \in \mathbb{N}$,

$$k^{c/2} t_k^{2+c}(1 - F_k(t_k)) \sim (2\pi)^{-\frac{1}{2}} k^{c/2} t_k^{1+c} \exp[-t_k^2/2] .$$

Since

$$k^{c/2} t_k^{1+c} \exp[-t_k^2/2] \geqq (c \log k + (c+1) \log\log k + M)^{(1+c)/2} (\log k)^{-(1+c)/2} \exp[-M/2] ,$$

we have

$$\liminf k^{c/2} t_k^{2+c}(1 - F_k(t_k)) > 0 .$$

Hence,

$$\limsup n^{c/2} t_n^{2+c}(1 - F_n(t_n)) > 0 .$$

PROOF OF THEOREM 6. Since $Y \geqq 0$ implies $EY = \int_0^\infty P(Y > t)\,dt$, we obtain from Theorems 1 and 2

$$\begin{aligned} |E|n^{-\frac{1}{2}} S_n|^{c+2} - \pi^{-\frac{1}{2}} 2^{(c+2)/2} \Gamma((c+3)/2)| &\leqq \int_0^\infty |P(|n^{-\frac{1}{2}} S_n| > t^{1/(c+2)}) - 2\Phi(-t^{1/(c+2)})|\,dt \\ &\leqq 2b_1 n^{-c^*} \int_0^{r_n} \exp[-(1-\sigma) t^{2/(c+2)}/2]\,dt \\ &\quad + 2b_2 n^{-c/2} \int_{r_n}^\infty t^{-2}\,dt + 2n \int_0^\infty P(|X| > r n^{\frac{1}{2}} t^{1/(c+2)})\,dt , \end{aligned}$$

where $r_n = ((c+1)\log n)^{(c+2)/2}$.

Since the integrals in the first two terms of the r.h.s. of this inequality are finite, the conclusion follows from

$$\int_0^\infty P(|X| > r n^{\frac{1}{2}} t^{1/(c+2)})\,dt = r^{-(c+2)} n^{-(c+2)/2} E|X|^{c+2} .$$

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