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On Nonuniform Principal Component Filter Banks: Definitions, Existence and Optimality

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ABSTRACT

The optimality of principal component filter banks (PCFB's) for data compression has been observed in many works to varying extents. Recent work by the authors has made explicit the precise connection between the optimality of uniform orthonormal filter banks (FB's) and the principal component property: The PCFB is optimal whenever the minimization objective is a concave function of the subband variances of the FB. This gives a unified explanation of PCFB optimality for compression, denoising and progressive transmission. However not much is known for the case when the optimization is over a class of nonuniform FB's. In this paper we first define the notion of a PCFB for a class of nonuniform orthonormal FB's. We then show how it generalizes the uniform PCFB's by being optimal for a certain family of concave objectives. Lastly, we show that existence of nonuniform PCFB's could imply severe restrictions on the input power spectrum. For example, for the class of unconstrained orthonormal nonuniform FB's with any given set of decimators that are not all equal, there is no PCFB if the input spectrum is strictly monotone.

Keywords: Principal Component Filter-Banks, nonuniform filter banks, optimal filter banks, majorization, convex objectives

1. INTRODUCTION

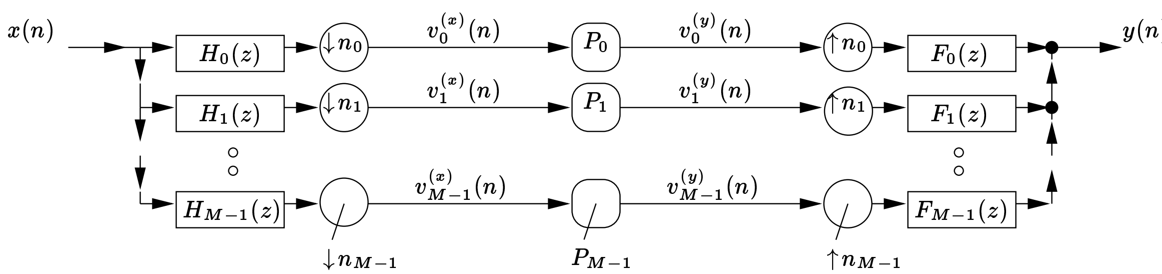

Fig. 1. General subband signal processing scheme using M channel filter bank.

Figure 1 shows a general subband signal processing scheme using an M channel filter bank (FB). We always assume the FB to be *maximally decimated*, i.e. the channel decimation rates n_i satisfy

$$\sum_{i=0}^{M-1} \frac{1}{n_i} = 1. \tag{1}$$

The FB is uniform if all decimators n_i are equal, i.e. (by (1)) $n_i = M$ for all i. The FB has the perfect reconstruction (PR) property if the input and output are identical in absence of subband processing, i.e. in Fig. 1, $y(n) \equiv x(n)$ when all the P_i are identity systems. We will only study *orthonormal* FB's, i.e. those having PR with $F_i(e^{j\omega}) = H_i^*(e^{j\omega})$.

The subband processors P_i are usually aimed at producing a certain desired signal $d(n)$ at the FB output. For example in data compression, the P_i are quantizers and $d(n) = x(n)$. In noise suppression, $x(n) = s(n) + \mu(n)$ where

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$\mu(n)$ is the additive noise, and $d(n) = s(n)$, the pure signal. Now let us be given a class of FB's, all having the same set of decimators n_i. The question arises as to which is the best FB in the class for a given application. In other words, given the processors P_i and some well defined measure of the error signal $e(n) = y(n) - d(n)$ between the true and desired FB output, which FB in the class minimizes the error measure? The answer of course depends on the error measure and the given FB class.

To obtain a well defined error measure, it is common to employ statistical models of the input using stationary random processes. In many situations the error signal $e(n)$ is cyclostationary, and the error measure used is its average mean–square value. Now suppose the given FB class is *uniform*, i.e. all decimators have same value M. For such classes, it often happens that the above mentioned mean square value is a concave function of the subband variances. Recently it has been shown[1,2] that in all such cases, a *principal component* filter bank (PCFB)[10–12,1] for the given FB class and input spectrum will be optimal (if it exists). This leads to a unified explanation of PCFB optimality for several applications such as progressive transmission, compression, noise suppression, and as shown more recently, for optimal DMT (discrete multitone modulation) communications.[13]

The present work seeks to generalize this basic PCFB optimality result to classes of nonuniform FB's. In[3] a preliminary study of nonuniform PCFB's has been carried out by the authors. This paper elaborates on these results. We compare our definitions of nonuniform PCFB's with previously attempted definitions[8,14] that are superficially similar to that of uniform PCFB's. We point out that these earlier works do not involve the central concept of *majorization*[1,7] which is crucial to uniform PCFB optimality. Thus the defined nonuniform PCFB's could not be shown to have any interesting optimality properties analogous to uniform ones. Using our definitions, we can show certain general optimality properties for concave objectives. The form of the objective is somewhat more restricted here as compared to that for uniform FB classes, however the results clearly show that our definitions are the more natural generalization of PCFB's to classes of nonuniform FB's. We will then prove an important result stating that for strictly monotone input power spectra, PCFB's do not exist for the class of unconstrained nonuniform FB's with any given set of decimators that are not all equal. In contrast, the class of unconstrained uniform M-channel orthonormal FB's always has a PCFB for any input spectrum. Thus PCFB existence for nonuniform FB classes is much more delicate than that for uniform ones.

2. PROBLEM FORMULATION

Consider first the case when the FB is uniform, i.e. $n_i = M$ for all i in Fig. 1. Denote by $v_i^{(s)}(n)$ the i-th subband signal when the FB input is $s(n)$. We are given a class of (uniform M-channel) orthonormal FB's, and the power spectrum (psd) of the wide sense stationary (WSS) FB input $x(n)$. As explained earlier, we seek to find the best FB in the class for the given psd. Suppose the subband error signals $v_i^{(e)}(n)$ are jointly WSS, or equivalently the full band error $e(n)$ is wide sense cyclostationary with period M (CWSS(M)). From FB orthonormality it can be shown that the chosen minimization objective, i.e. time averaged expected mean square value ε of $e(n)$, has the form

$$\varepsilon = \frac{1}{M} \sum_{i=0}^{M-1} E[|v_i^{(e)}(n)|^2]. \tag{2}$$

Often the subband error variance is related to the subband signal variance $\sigma_i^2 \triangleq E[|v_i^{(x)}(n)|^2]$, as

$$E[|v_i^{(e)}(n)|^2] = h_i(\sigma_i^2) \tag{3}$$

where h_i is some function depending purely on the nature of the subband processing (i.e. on P_i in Fig. 1), and independent of the choice of FB.* Thus in this case, the minimization objective g is purely a function of the *subband variance vector* $\mathbf{v} \triangleq (\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)$, i.e.

$$g(\mathbf{v}) = \frac{1}{M} \sum_{i=0}^{M-1} h_i(\sigma_i^2) \tag{4}$$

Now if all the functions h_i are concave on the interval $[0, \infty)$ then $g(\mathbf{v})$ is a concave function of the variance vector $\mathbf{v}$. In this case, a principal component filter bank (PCFB) for the given class of FB's and given input psd will be optimal (if it exists). Uniform PCFB's and this central result on their optimality will be reviewed in the next section.

*The term 'variance' here should technically be replaced with 'energy' or expected mean square value; we will often make this convenient misuse of terminology. The distinction vanishes if the relevant signals are assumed to be zero mean.

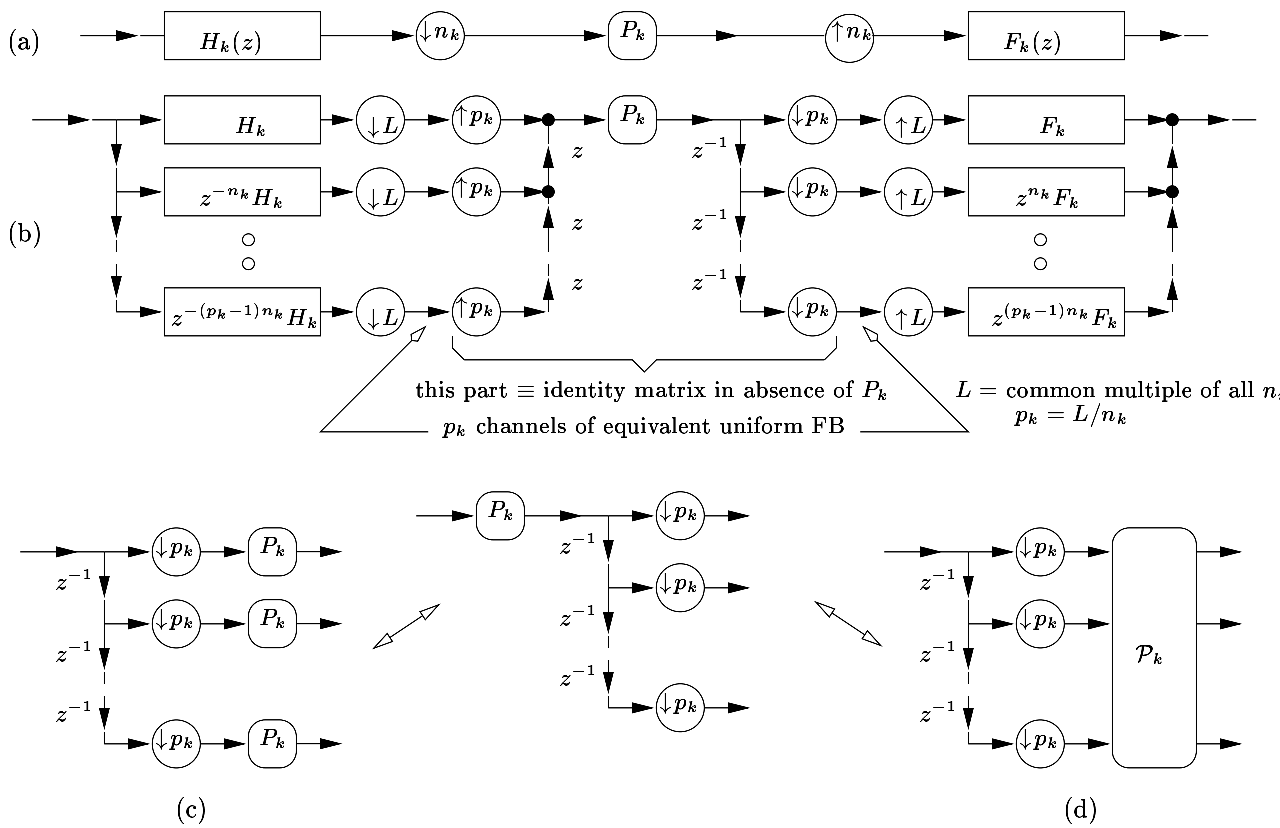

Fig. 2. Transforming nonuniform FB's to equivalent uniform ones.
(a) A single subband, (b) corresponding subbands of equivalent uniform FB,
(c) moving processor P_k from nonuniform to uniform subbands – special case, (d) general case.

Now consider the case when all FB's in the given class of orthonormal M-channel FB's are *nonuniform* with a *fixed* set of decimators n_i (not necessarily all equal). In this section we will (a) generalize the form of the objective (4) to cover this case, and (b) illustrate the generalization using specific signal processing problems where the assumptions leading to (4) are satisfied for uniform FB classes.

The key step in dealing with nonuniform FB classes is the transformation[6,4,9] from the nonuniform FB to a uniform L-channel FB where L is any common multiple of the decimators n_i (usually $L = \text{lcm}\{n_i\}$). This transformation is shown in Fig. 2a,b. The k-th channel of the nonuniform FB, with decimator n_k, corresponds to $p_k = L/n_k$ channels of the uniform one. The filters in these p_k channels are delayed versions of each other. Due to these dependencies between the filters of the equivalent uniform FB, it is not possible to redraw every L-channel uniform FB as a nonuniform FB with the given decimators n_i (unless we allow the nonuniform FB to have filters that are *periodically time varying*[4] with period L). Note from Fig. 2 that many properties such as maximal decimation, perfect reconstruction and orthonormality are shared in common by the nonuniform FB and its equivalent uniform FB (i.e. each has the property if and only if the other does).

The above transformation makes it easy to formulate the minimization objective for nonuniform FB's starting from the earlier formulation for uniform ones. Let $w_i^{(s)}(n)$ denote the signals in the equivalent uniform FB subbands when the FB input is $s(n)$. The earlier notation $v_i^{(s)}(n)$ is used for the nonuniform FB subbands. Both (2) and (3) can be generalized to nonuniform FB's; the issue to be resolved is whether the subband signals in these equations should now refer to the uniform subbands $w_i^{(e)}(n)$ or to the nonuniform ones $v_i^{(e)}(n)$. Evidently (2) generalizes as $\varepsilon = \sum_{i=0}^{L-1} E[|w_i^{(e)}(n)|^2]$, under the assumption that the signals $w_i^{(e)}(n)$ are jointly WSS. On the other hand, (3) reflects the action of processor P_i on its WSS input (a subband signal), and hence generalizes as it is, under the assumption that the signal $v_i^{(x)}(n)$ is WSS with variance σ_i^2. Finally, examine the signals $w_i^{(e)}(n)$ for the group of p_k uniform FB subbands in Fig. 2b. By linearity of the systems in Fig. 2, these signals are obtained by passing the signal $v_k^{(e)}(n)$ of the corresponding nonuniform subband through the p_k channel delay-chain. Thus if $v_k^{(e)}(n)$ is WSS, all these $w_i^{(e)}(n)$ have the same variance equal to that of $v_k^{(e)}(n)$. Combining these observations yields the minimization objective for nonuniform FB classes:

$$g(\mathbf{v}) = \sum_{i=0}^{M-1} \frac{p_i}{L} h_i(\sigma_i^2) = \sum_{i=0}^{M-1} \frac{1}{n_i} h_i(\sigma_i^2) \tag{5}$$

This is a very natural generalization of (4), obtained by appropriate generalization of the choice of stationarity assumptions.

Notice that (5) is exactly the objective that would result for the equivalent uniform FB optimization if the processor P_k in Fig. 2b were replaced by p_k identical copies of itself, one in each of the corresponding uniform FB subbands as demonstrated in Fig. 2c. For a general processor P_k, this operation would be changing the system – if the system is to be preserved a p_k-input p_k-output processor is needed in the uniform FB subbands as shown in Fig. 2d. However the two systems are identical from the point of view of the minimization objective. We now mention specific signal processing schemes satisfying the assumptions leading to (5), the corresponding processors P_i and 'error functions' h_i. For all these P_k the two systems in Fig. 2c in fact happen to be equivalent. Also the h_i turn out to be concave on $[0, \infty)$ so that in the special case of uniform FB classes ($n_i = M$ for all i), PCFB's are optimal (Section 3.1). The later sections seek to generalize this to nonuniform FB classes.

Compression: Here the processor P_i is a quantizer, modelled as an additive noise source with variance proportional to the variance σ_i^2 of the (WSS) quantizer input. The proportionality constant $f_i(b_i)$ depends on the number of bits b_i alloted to the quantizer, and is assumed to be independent of choice of FB. Thus $h_i(x) = f_i(b_i)x$ in (5). For uniform FB's, we need the assumption that the noises from different quantizers are all jointly WSS. For nonuniform FB's, we require the corresponding noises generated in the subbands of the equivalent uniform FB to be jointly WSS. The stronger assumption of WSS and *uncorrelated* quantizer noises implies both these.

The special case where the quantizers are assumed to satisfy the high bit-rate assumption is worth mention: Here $f_i(b_i) = c_i 2^{-2b_i}$ where the constant depends on the i-th subband probability density function (pdf), and hence on the choice of FB. This is not allowed under our formulation of (5) (the functions h_i are assumed to be independent of choice of FB); however we circumvent the problem by assuming a Gaussian input. This forces all subband pdf's to be Gaussian independent of the FB, so that $c_i = c$ for all i, where c is a constant independent of the FB. In this case, under optimal allocation of the bits b_i (subject to a total bitrate constraint $\sum_{i=0}^{M-1} b_i = B$), it can be shown (using the arithmetic mean - geometric mean inequality) that minimizing (5) is equivalent to minimizing the weighted geometric mean $GM = \prod_{i=0}^{M-1} (\sigma_i^2)^{(1/n_i)}$ of the subband variances σ_i^2 (with the decimators n_i as weights). Equivalently, we can minimize $\log(GM)$, which again has the form of (5) with $h_i(x) = \log(x)$ for all i.

Noise reduction: Here the FB input is $x(n) = s(n) + \mu(n)$ where $s(n)$ is the pure input and $\mu(n)$ is white noise uncorrelated to $s(n)$. Each processor P_i is a multiplier k_i that is either constant or adapted to its input statistics as (a) a Wiener filter: $k_i = \frac{\sigma_i^2}{\sigma_i^2+\eta^2}$, or (b) a subband hard threshold operator: $k_i = 0$ if $\sigma_i^2 < \eta^2$ and $k_i = 1$ otherwise. Here σ_i^2 is the variance of the pure signal component $v_i^{(s)}(n)$ in the i-th subband, and η^2 is the noise variance. Equation (5) applies in all cases, with

$$h_i(x) = \begin{cases} |1-k_i|^2 x + |k_i|^2 \eta^2 & \text{for constant subband multiplier } k_i \\ \frac{x\eta^2}{x+\eta^2} & \text{for zeroth order Wiener filter } k_i \\ \min(x, \eta^2) & \text{for hard threshold } k_i \end{cases} \tag{6}$$

3. NONUNIFORM PCFB'S: DEFINITIONS AND OPTIMALITY

3.1. Review of uniform PCFB's

Definition: Majorization. Given two (equal length) vectors $\mathbf{a} = (a_0, a_1, \ldots, a_{M-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{M-1})$, we say that $\mathbf{a}$ majorizes $\mathbf{b}$ if after rearranging the entries of the vectors in decreasing order we have

$$\sum_{i=0}^{P} a_i \geq \sum_{i=0}^{P} b_i \quad \text{for } 0 \leq P \leq M-1, \qquad \text{with equality holding for } P = M-1 \tag{7}$$

Note that if **a** majorizes **b**, any permutation of **a** majorizes any permutation of **b**.

Definition: Uniform PCFB's. Let $\mathcal{C}$ be a class of uniform orthonormal M-channel FB's. A FB in $\mathcal{C}$ is said to be a PCFB for $\mathcal{C}$ for the given input power spectrum if its subband variance vector $(\hat{\sigma}_0^2, \hat{\sigma}_1^2, \ldots, \hat{\sigma}_{M-1}^2)$ **majorizes** the subband variance vector $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)$ of every FB in $\mathcal{C}$.

The PCFB and its existence depends both on $\mathcal{C}$ and on the input psd. The requirement of majorization in the PCFB definition in particular demands that the average (or the sum) of the subband variances be FB–independent. This follows automatically from FB orthonormality: The average equals the variance of the WSS input to the FB.

PCFB optimality. PCFB's are optimum orthonormal FB's if the minimization objective is a concave function of the subband variance vector. As noted earlier, this basic result leads to PCFB optimality for many FB based signal processing schemes, motivating generalization of the PCFB concept to nonuniform FB's. Detailed proofs of (uniform) PCFB optimality[1,2] are based on the geometric meaning of majorization. For the present purpose of generalization to nonuniform FB's, it suffices to note that uniform PCFB optimality is solely dependent on the following facts:

1. The objective function has form (4) where all h_i are *FB–independent* functions concave on the interval $[0, \infty)$.
2. The PCFB subband variance vector majorizes the subband variance vectors of all FB's in the given class $\mathcal{C}$. (Definition of PCFB's)
3. The entries of the variance vector are allowed to be inserted in the objective (4) in any order. In other words, if **v** is a variance vector realizable by the given class $\mathcal{C}$, so is any permutation of **v**. This is true because permuting **v** simply corresponds to permuting the subbands, or to redistributing the processors among them. The subbands have no intrinsic ordering – their numbering purely denotes their association with the subband processor. Clearly all permutations of the PCFB are PCFB's. In general they do not all have the same value for the objective (4); in fact a bad permutation could even give the worst possible performance. However the best one of these (finitely many) permutations will be optimal for the entire class $\mathcal{C}$.

3.2. Nonuniform PCFB's: Definition

We seek to generalize the definition of PCFB's to a general class $\mathcal{N}$ of nonuniform M-channel orthonormal FB's with a fixed set of decimators n_i. It is desirable that the generalization reduce to the usual definition for uniform FB's if all the n_i are equal, and that the defined PCFB's be optimal for some reasonable class of objectives of the form (5). From the discussion of uniform PCFB optimality, a natural approach to generalization is to define some suitable variance vector for all FB's in the class, and define the PCFB to be the one whose variance vector majorizes all the variance vectors of all FB's in the class $\mathcal{N}$. This PCFB would then be optimal for all objectives of the form $\sum_i f_i(\alpha_i)$ where α_i are the entries of the variance vector, provided conditions analogous to facts 1,3 mentioned in the earlier section hold. The majorization requirement demands that $\sum_i \alpha_i$ be FB–independent; this should follow automatically from FB orthonormality just as it happens for uniform FB's. Otherwise it would impose too severe a constraint on the FB class $\mathcal{N}$, and PCFB's would hardly ever exist.

Let σ_i^2 be the variance of the i-th subband (with decimator n_i). Now FB orthonormality implies that $\sum_i \frac{\sigma_i^2}{n_i}$ is FB–independent, but in general $\sum_i \sigma_i^2$ is not (unless all the n_i are equal). Thus, the usual definition of subband variance vector (as one with entries σ_i^2) is not suitable (unless the FB's are uniform). In fact the above considerations leave us only two reasonable definitions of the variance vector:

- The *normalized* subband variance vector defined as $(\frac{\sigma_0^2}{n_0}, \frac{\sigma_1^2}{n_1}, \ldots, \frac{\sigma_{M-1}^2}{n_{M-1}})$.
- The *equivalent uniform* subband variance vector, defined as the (usual) subband variance vector of the equivalent uniform L-channel FB (for any fixed L that is a multiple of all the n_i). From the construction of the equivalent uniform FB's (Fig. 2), with $p_i = L/n_i$, this vector has the form

$$(\underbrace{\sigma_0^2, \sigma_0^2, \ldots, \sigma_0^2,}_{p_0 \text{ elements}} \quad \underbrace{\sigma_1^2, \sigma_1^2, \ldots, \sigma_1^2,}_{p_1 \text{ elements}} \quad \ldots \;\ldots, \quad \underbrace{\sigma_{M-1}^2, \sigma_{M-1}^2, \ldots, \sigma_{M-1}^2}_{p_{M-1} \text{ elements}})^T \tag{8}$$

Definition: Nonuniform PCFB's. Let $\mathcal{N}$ be a class of orthonormal M-channel FB's having a fixed set of decimation rates n_i, not necessarily equal. A FB in $\mathcal{N}$ is said to be a PCFB for $\mathcal{N}$ for the given input power spectrum if its suitably defined subband variance vector **majorizes** the subband variance vector of every FB in $\mathcal{C}$. The variance vector in this definition could be either the normalized or the equivalent uniform subband variance vector (defined earlier). Thus we have two definitions of nonuniform PCFB's, one based on each of these variance vectors.

Remarks on the nonuniform PCFB definitions.

1. *Reduction to uniform PCFB's.* For the special case of uniform FB's, where $n_i = M$ for all i, both the above definitions reduce to the usual one by choosing $L = M$ for the equivalent uniform variance vector and omitting the constant scale M for the normalized variances.

2. *Ambiguity in L does not matter.* Let $L_1 = \text{lcm}\{n_i\}$ and $L_2 = kL_1$ for any positive integer k. For two FB's in the given class, let $\mathbf{v}_1^*$ and $\mathbf{v}_1$ be the variance vectors of their equivalent L_1 channel uniform FB's, and let $\mathbf{v}_2^*$ and $\mathbf{v}_2$ be the respective variance vectors for the equivalent L_2 channel uniform FB's. From (8), if the entries of $\mathbf{v}_1$ in descending order are $(a_0, a_1, \ldots, a_{L_1-1})$ then the entries of $\mathbf{v}_2$ in descending order are obtained by arranging the a_i in the same order and repeating each a_i k times. Thus by definition (7) of majorization, $\mathbf{v}_1^*$ majorizes $\mathbf{v}_1$ if and only if $\mathbf{v}_2^*$ majorizes $\mathbf{v}_2$. Hence, the definition of PCFB's using the equivalent L channel uniform subband variance vector is independent of which common multiple of the n_i we fix L to be.

3. *The two PCFB definitions are distinct.* It may seem that extending the above argument will also prove that using normalized variance vectors is equivalent to using the equivalent uniform ones. However this is not true. The reason is that the definition of majorization (7) demands arranging the entries of the vectors in decreasing order. This order could be different for the σ_i^2 (i.e. for the entries of (8)) and for the $\frac{\sigma_i^2}{n_i}$. The distinctness of the two definitions is shown in[3] by an example of a specific class $\mathcal{N}$ with exactly two FB's with decimators $2, 4, 4$, for a carefully chosen family of input spectra. Each FB is the unique PCFB by one definition but not by the other. In fact in this example the input psd can be chosen so that the PCFB defined by the equivalent uniform variances is a PCFB not just for the two element class $\mathcal{N}$, but for *any* class of FB's with decimators $2, 4, 4$ to which it belongs. In other words it is a PCFB for the class of unconstrained orthonormal FB's with $2, 4, 4$ as decimators.† Even then, the other FB in $\mathcal{N}$ is the unique PCFB for $\mathcal{N}$ by the alternative definition.

3.3. Nonuniform PCFB's: Optimality

For uniform FB's, there is no intrinsic ordering of the subbands. Permuting the subbands, or redistributing the processors P_i among the subbands, was simply represented by evaluating the same objective function $g(\mathbf{v})$ of (4) for a different argument $\mathbf{v}$, i.e. a permutation of the original subband variance vector. On the other hand, nonuniform FB subbands are indexed by their decimation rates n_i. If we interchange the processors for two subbands with different decimators, the new performance measure must be computed by interchanging not just the corresponding variances but also the corresponding decimators in (5). This changes the functional form of the objective, since the objective in (5) is viewed as a function of the variances σ_i^2 with the n_i as parameters. Free redistribution of processors among subbands without changing the functional form of the objective is possible only among groups of subbands with equal decimators. Thus, it is necessary to distinguish two different nonuniform FB optimization problems:

1. Finding the best FB for a fixed ordering of the subbands (i.e. of the n_i). We refer to this problem as $OP1$.

2. Optimizing both the FB (i.e. its subband variances) and the decisions as to which subband should have which processor P_i, i.e the choice of ordering of the subbands (or the n_i). We refer to this problem as $OP2$.

The problem $OP1$ does not exist for uniform FB's since there is no intrinsic ordering of the subbands. If $OP1$ is solvable for all the (finitely many) possible orderings of the subbands, then clearly $OP2$ is solved by picking from these solutions the one with the best value of the performance measure. Note that $OP2$ is usually of much more interest than $OP1$, since fixing apriori which channel decimator should be associated with which processor P_i is quite restrictive. However $OP1$ is easier to analyze, due to the fixed functional form of its objective. A summary of all the

†This is done by setting $p = b$ in Fig. 3a of[3] and using Theorem 3 in that reference.

optimality results for nonuniform PCFB's (to be derived next) can be found at the end of the section, and optimality there refers only to the (more interesting) problem $OP2$. The derivations involve study of both problems.

Optimality of PCFB defined using equivalent uniform variance vector: We generically denote the entries of this variance vector by α_i. The objective (5) has the form $\sum_i f_i(\alpha_i)$ with the f_i chosen as follows: All the $p_k = L/n_k$ equivalent uniform FB subbands derived from the k-th nonuniform FB subband (with decimator n_k) have the corresponding f_i equal to h_k of (5) (upto constant scale L). Thus this PCFB is a possible candidate for solving $OP1$ when all h_k (and hence all f_i) are FB–independent and concave. However for optimality, we must be allowed to insert the α_i into the f_i in any order in forming the sum $\sum_i f_i(\alpha_i)$ (condition 3 in Section 3.1). In actual fact we are only allowed certain permutations of the α_i, corresponding to groups of nonuniform subbands with equal decimator value. Thus the PCFB solves $OP1$ if the best ordering of the α_i happens to be one that is actually allowed.

Recall that solving $OP1$ for all permutations of the n_k solves $OP2$. For a given permutation of the n_k, suppose the PCFB does not actually solve $OP1$, i.e. the best ordering of the α_i above is not allowed. Then this ordering of the α_i gives an unattainable upper bound on the performance for the problem $OP1$. Sometimes we may be able to solve $OP2$ using just these bounds (instead of the true optima). Indeed if a known solution to $OP1$ for one permutation of the n_k outperforms all these bounds computed for all the other permutations, then it also solves $OP2$.

In the noteworthy special case when $h_k = h$ for all i, i.e. all h_k and hence all f_i are identical, we see that all orderings of the α_i give the same performance measure $\sum_i f_i(\alpha_i)$. Thus in this case the PCFB will solve $OP1$ if the function h is FB–independent and concave. This is true for all permutations of the n_k, and the corresponding optimal performance measures are identical too. Thus the PCFB also solves $OP2$. In fact the distinction between $OP1$ and $OP2$ vanishes in this case. Identical h_k usually results from similar or identical subband processors P_k.

Optimality of PCFB defined using normalized variance vector: We generically denote the entries of this variance vector by $\beta_i = \sigma_i^2/n_i$. The objective (5) has the form $\sum_i \hat{f}_i(\beta_i)$ where $\hat{f}_i(x) = (1/n_i)h_i(n_i x)$. Thus, this PCFB is a candidate for solving $OP1$ when the h_i (and hence the $\hat{f}_i$) are FB–independent and concave. However, again optimality is assured only if the β_i can be inserted in the sum $\sum_i \hat{f}_i(\beta_i)$ in any order. In actual fact we are only permitted to permute the β_i within groups of channels having the same decimator n_i. Thus, as with the earlier definition, these PCFB's solve $OP1$ if the best ordering of the β_i turns out to be one that is permitted. The special case where all $\hat{f}_i$ are identical (which would make all orderings of the β_i equally good) is not of much interest here: It does not commonly happen since $\hat{f}_i(x) = (1/n_i)h_i(n_i x)$ which depends on n_i.

However, another special case is of interest, namely that where $h_i(x) = k_i x$ for some constants k_i (for all i). Here $\hat{f}_i(x) = k_i x$ too, which has the speciality of being (concave and) independent of the decimators n_i. In general, to compute the performance measure after permuting two nonuniform subbands required not just permuting of the corresponding β_i, but also corresponding modifications of the $\hat{f}_i$ since they depended on the n_i. (This changed the functional form of the objective, and gave rise to the distinction between problems $OP1$ and $OP2$.) However if $\hat{f}_i$ are independent of the n_i, permuting the nonuniform subbands is fully equivalent to permuting the β_i. Thus, the nonuniform FB defined using the variances β_i will solve $OP2$ (for a suitable permutation of its subbands, i.e. ordering of the β_i).

Nonuniform PCFB optimality: Summary.

- Nonuniform PCFB's defined using the equivalent uniform subband variance vector are optimal for objectives of the form (5) when $h_i = h$ for all i, where h is a FB–independent concave function. Such objectives occur with similar processing in the subbands, e.g. in the high bitrate coding problem with optimal bit allocation ($h(x) = \log(x)$) or in the noise suppression problem using either zeroth order Wiener filters ($h(x) = \frac{x\eta^2}{x+\eta^2}$) or hard thresholds ($h(x) = \min(x, \eta^2)$) in *all* subbands (see Section 2).
- Nonuniform PCFB's defined using the normalized variance vector are optimal for objectives of the form (5) where for all i, $h_i(x) = k_i x$ for constant k_i. Such objectives occur in the coding problem with fixed bit allocation when the high bitrate assumption is not necessarily used ($k_i = f_i(b_i)$, see Section 2).
- For general objectives of the form (5) where h_i are FB–independent and concave, it may be possible to show optimality of either of the PCFB's in a specific case, using the actual specifications of the h_i and numerical

values of the PCFB subband variances. Finite procedures to do this follow from the earlier discussion. However in general the procedure will be unsuccessful, and PCFB optimality is not assured.

3.4. Previous attempts at defining nonuniform PCFB's

In the authors' knowledge, the works[8,14] are the only earlier attempts to define nonuniform PCFB's. These can be described as follows: Let $\mathcal{N}$ be the given class of M-channel orthonormal FB's, all having the same set of decimators n_i. First an ordering of the subbands of all FB's (i.e. of the n_i) is decided upon. Then with σ_i^2 as the variance of the i-th subband (with decimator n_i), the PCFB is defined as one whose variances $\hat{\sigma}_i^2$ satisfy

$$\sum_{i=0}^{P} \frac{\hat{\sigma}_i^2}{n_i} \geq \sum_{i=0}^{P} \frac{\sigma_i^2}{n_i} \quad \text{for } P = 0, 1, \ldots, M-1, \quad \text{with (automatic) equality for } P = M-1 \tag{9}$$

While this equation is superficially similar to (7) that defines majorization, there is a very important difference: The a_i and b_i of (7) are arranged in decreasing order, whereas the analogous quantities $\frac{\hat{\sigma}_i^2}{n_i}$ and $\frac{\sigma_i^2}{n_i}$ in (9) are arranged according to the predefined ordering of the n_i. In[14] the PCFB is defined only for classes of dyadic (or 'wavelet style') tree–structured FB's, and the n_i are chosen in decreasing order. A construction procedure is outlined for the PCFB thus defined, when the FB class $\mathcal{N}$ is the unconstrained one. In[8] the PCFB is defined with respect to the ordering of the n_i, i.e. there are several PCFB's defined as above, one for each ordering of the n_i.

Consider the case when the minimization objective is as in (5) with $h_i(x) = k_i x$ for constant k_i (for all i). Consider the FB optimization problem $OP1$ defined in Section 3.3, where the n_i are ordered apriori and the association between the n_i and k_i is thus forcibly fixed. Let j_i be the permutation such that $k_{j_0} \leq k_{j_1} \leq \ldots k_{j_{M-1}}$. Then elementary algebra shows that the PCFB defined in[8] as above, for the permutation j_i of the decimators n_i, is a solution for $OP1$. Indeed it suffices to prove this assuming $j_i = i$. This amounts to showing that $\sum_i k_i \hat{\sigma}_i^2/n_i \leq \sum_i k_i \sigma_i^2/n_i$ given (9) and $k_i < k_{i+1}$. The proof follows from the fact that

$$\sum_{i=0}^{M-1} k_i \frac{\sigma_i^2}{n_i} = \sum_{i=0}^{M-2} (k_i - k_{i+1}) \sum_{l=0}^{i} \frac{\sigma_l^2}{n_l} + k_{M-1} \sum_{i=0}^{M-1} \frac{\sigma_i^2}{n_i}. \tag{10}$$

In summary, the earlier definitions result in several PCFB's, one for each permutation of the decimators n_i, and in optimality for the problem $OP1$ for linear objectives (i.e. $h_i(x) = k_i x$ in (5)). However, in general optimality for the more interesting problem $OP2$ cannot be claimed unless *all* the PCFB's (for all permutations of n_i) exist (in which case the best one of these would solve $OP2$). Apart from the very special result in[14] (specific to unconstrained dyadic tree–structured FB classes and using a particular ordering of subbands), there are no general existence results known for these PCFB's. Further, since the above PCFB definition does not use the concept of majorization, there are no optimality results for more general concave functions h_i besides $h_i(x) = k_i x$. These observations suggest that our definitions of Section 3.2 are more natural generalizations of the PCFB concept to nonuniform FB classes.

4. EXISTENCE OF NONUNIFORM PCFB'S

For the case of uniform FB's, PCFB's are known to exist for all input power spectra for only three special FB classes: Any class of two channel FB's, the class of M-channel orthogonal transform coders, and the class $\mathcal{C}^u$ of unconstrained M-channel orthonormal FB's (where there are no constraints on the filters besides those imposed by orthonormality, e.g. ideal brickwall filters are allowed). For the class $\mathcal{C}^u$, the PCFB is the one satisfying the two properties called spectral majorization and total decorrelation.[12] For classes of DFT and cosine–modulated FB's, PCFB's do not exist for large families of input spectra.[2]

Much less is known about existence of nonuniform PCFB's. For nonuniform PCFB's defined using normalized variances, no general existence result is currently known. Thus, in this section nonuniform PCFB's always mean those defined using equivalent uniform ones. Some existence results are known for these.[3] Let $\mathcal{N}$ be any class of nonuniform FB's with a given set of decimators n_i, and let $\mathcal{E}$ be the corresponding class of equivalent uniform FB's derived from FB's in $\mathcal{N}$. Then by definition, PCFB's for $\mathcal{N}$ and $\mathcal{E}$ are equivalent. Now if the PCFB for some FB class $\mathcal{C} \supset \mathcal{E}$ lies within $\mathcal{E}$, then it is also a PCFB for $\mathcal{E}$. For example we could choose $\mathcal{C} = \mathcal{C}^u$, the unconstrained class, whose PCFB has been well studied. This gives rise to the result of[3] stating that any FB in $\mathcal{N}$ having white and totally decorrelated subbands is a PCFB for $\mathcal{N}$. The paper[3] also gives examples of such PCFB's.

This section is devoted to a discussion and proof of an important *nonexistence* result for nonuniform PCFB's. The result is mentioned without proof in[3] and is as follows:

Theorem 1: Let $\mathcal{N}^u$ be the class of unconstrained nonuniform FB's with a given set of decimators n_i *not all equal*. Let the input spectrum $S(e^{j\omega})$ be *strictly monotonic* (increasing or decreasing) for $a \leq \omega < a + 2\pi$ for some real a. Then there is no PCFB for the class $\mathcal{N}^u$.

4.1. Discussion on Theorem 1

Existence of nonuniform PCFB's is very delicate. As shown in Fig. 3, a slight perturbation of a monotone input spectrum for which a PCFB for $\mathcal{N}^u$ exists (eg, the flat or white spectrum) can change it to a spectrum that is strictly monotone (on an interval of form $[a, a + 2\pi)$), destroying existence of the PCFB. In contrast, for the *uniform* unconstrained FB class $\mathcal{C}^u$, PCFB's exist for all input spectra, and moreover usually several perturbations are possible on the spectrum without even changing the PCFB. Of course, most objective functions defined on the FB class have their values for each FB perturbed only slightly by small perturbations of the input spectrum. Thus, consider the FB of Fig. 3. Being a PCFB for spectrum A, it optimizes many concave objectives as explained in Section 3.3. For the perturbed spectrum B, the FB is no longer a PCFB, and so it will not optimize all these objectives. However it will be quite close to optimal if the perturbation is small enough.

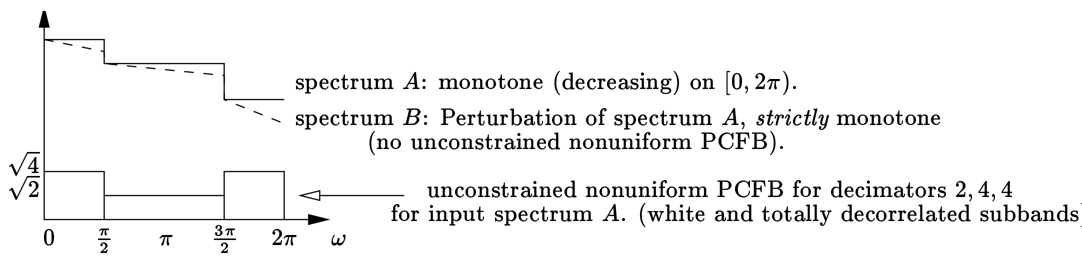

Fig. 3. Delicateness of existence of nonuniform PCFB for unconstrained class.

Total decorrelation and spectral majorization do not imply nonuniform PCFB's. Total decorrelation of subbands is evidently defined for nonuniform FB's too, and is achieved for example if the analysis filters have nonoverlapping supports. The notion of spectral majorization[12] can also be defined for nonuniform FB's. (It is the condition when their subband spectra do not 'cross each other', just as for uniform FB's.) Thus, as shown in Fig. 4, if the input spectrum is monotone on $[a, a+2\pi)$ then any contiguous–stacked brickwall FB with a as a filter band-edge has subbands satisfying both these conditions. However the FB is not a PCFB for $\mathcal{N}^u$ if the spectrum is strictly monotone (since there is no PCFB in this case). Thus, unlike the situation for the class $\mathcal{C}^u$, FB's with subbands satisfying spectral majorization and total decorrelation need not be PCFB's for $\mathcal{N}^u$.

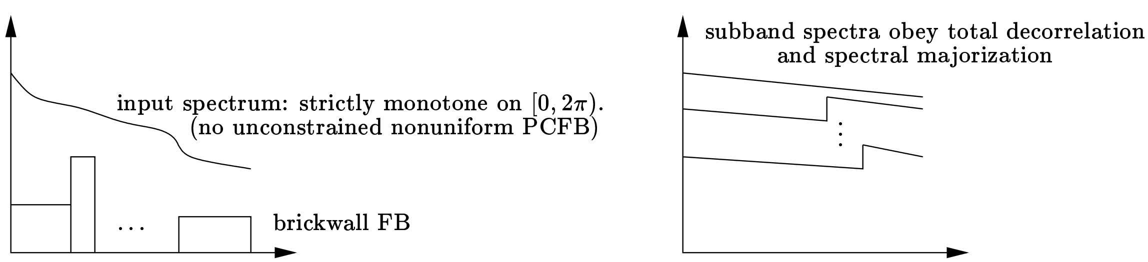

Fig. 4. Total decorrelation and spectral majorization do not imply nonuniform PCFB's.

4.2. Proof of Theorem 1

Without loss of generality, let $n_0 \geq n_1 \geq \ldots \geq n_{M-1}$. We prove the theorem for $a = 0$ assuming that $S(e^{j\omega})$ is strictly *decreasing* and that $n_0 \neq n_1$ (i.e. $n_0 > n_1$). The proof will show that this does not lose generality. The basic idea of the proof is as follows: Let $\mathcal{E}^u$ be the class of equivalent uniform FB's derived from FB's in $\mathcal{N}^u$.

1. We apply the sequential compaction algorithm described in[5] to the uniform FB class $\mathcal{E}^u$, and show that this results uniquely in the FB denoted by FB_A in Fig. 5 (more precisely, in the equivalent uniform FB derived from FB_A). From[5] this means that if the class $\mathcal{N}^u$ has a PCFB for the psd $S(e^{j\omega})$, the PCFB must be FB_A.

2. We then show that FB_A is not the PCFB by proving that its equivalent uniform subband variance vector does not majorize the variance vector of the distinct FB in $\mathcal{N}^u$ denoted by FB_B in Fig. 5. This means that there is no PCFB.

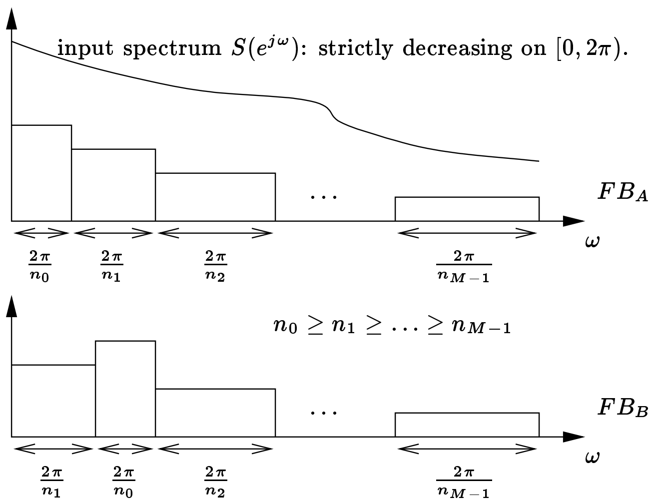

Fig. 5. Brickwall FB's used in proving Theorem 1.

Proof of Step 1: The sequential algorithm consists of maximizing the largest variance in the appropriate variance vector, then searching among all variance vectors having this maximum possible largest variance in order to maximize the second–largest variance, and so on. The entries of the equivalent uniform variance vector (8) are the nonuniform subband variances repeated appropriately many times. Hence running the algorithm on $\mathcal{E}^u$ is equivalent to sequentially maximizing the nonuniform FB subband variances. The ideal *compaction*(M) *filter*[12] produces the largest possible variance in a subband with decimator M, in any orthonormal FB. For a spectrum that is strictly decreasing on $[0, 2\pi)$ the compaction(M) filter is unique, supported on $[0, 2\pi/M]$, and has output variance that is strictly increasing in M. As n_0 is the largest decimator in the FB (i.e. the largest possible M), this proves that among all FB's in $\mathcal{N}^u$, FB_A maximizes the largest subband variance. (This maximum is the subband variance produced by the filter in FB_A corresponding to decimator n_0.) Now all FB's are orthonormal, and so their analysis filters $H_i(z)$ satisfy $[H_i(e^{j\omega})H_j^*(e^{j\omega})]\downarrow_{\gcd(n_i,n_j)}= 0$ for $i \neq j$. Hence, if an analysis filter in the FB, corresponding to decimator M, has an aliasfree(M) support, then this support does not overlap with the supports of any of the other analysis filters. Thus we can repeat the same argument for the second–largest variance, and so on. This shows that FB_A is the unique output of the sequential compaction algorithm.

Proof of Step 2: If a filter in an orthonormal FB is ideal, supported on $[c, d]$ and has constant magnitude on its support (like all the filters in FB_A, FB_B), then the corresponding subband variance is given by direct calculation as

$$\frac{f(d) - f(c)}{d - c}, \quad \text{where} \quad f(\omega) = \int_0^{\omega} S(e^{j\omega'})\, d\omega'. \tag{11}$$

The calculation uses the fact that the support length $d-c = 2\pi/n_i$ where n_i is the corresponding channel decimator, and that the constant passband magnitude of the filter is $\sqrt{n_i}$ (both these are due to orthonormality). Notice that the variance expression (11) is the slope of the chord on the graph of $y = f(\omega)$, connecting the end points with abscissae c, d. In our case, f is strictly concave on $[0, 2\pi)$, as its derivative is $S(e^{j\omega})$ which is strictly decreasing. For chords of concave functions, increasing either or both of the abscissae of their endpoints can never cause the slope to increase. As a result, for both the brickwall FB's FB_A and FB_B, the subband variance is nonincreasing as the

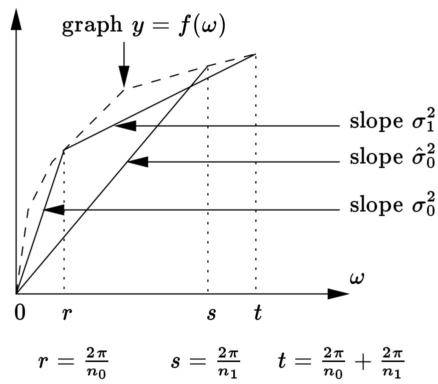

Fig. 6. Subband variances as chord slopes.

corresponding filter band-edges increase from 0 to 2π. Thus the largest and next–largest subband variances of FB_A are σ_0^2 and σ_1^2, corresponding to decimators n_0 and n_1 respectively; while the largest subband variance of FB_B is $\hat{\sigma}_0^2$ which corresponds to decimator n_1. All these variances are identifiable as slopes of chords on the graph $y = f(\omega)$ as in Fig. 6. Now we define $L = \text{lcm}\{n_i\}$ and $p_i = L/n_i$, and let $\mathbf{v}_A, \mathbf{v}_B$ be the equivalent uniform subband variance vectors of FB_A, FB_B respectively. We use the relation (8) between the subband variances of a nonuniform FB and its equivalent uniform FB. This shows that the p_1 largest variances in $\mathbf{v}_B$ have value $\hat{\sigma}_0^2$. On the other hand, of the p_1 largest variances in $\mathbf{v}_A$, there are p_0 variances of value σ_0^2 and $p_1 - p_0$ variances of value σ_1^2. (Recall that $n_0 > n_1$, so $p_1 - p_0 > 0$.) Now referring to the definition (7) of majorization, we will have proved that the $\mathbf{v}_A$ does not majorize $\mathbf{v}_B$ if we show that

$$p_0\sigma_0^2 + (p_1 - p_0)\sigma_1^2 < p_1\hat{\sigma}_0^2 \tag{12}$$

To do this, we substitute the chord slopes from Fig. 6 for the variances. Using $p_k = L/n_k$ and deleting a factor of $L/(2\pi)$, we see that proving the above equation reduces to proving that

$$\frac{n_1}{n_0}f(r) + (1 - \frac{n_1}{n_0})f(t) < f(s) \tag{13}$$

(where r, s, t are as in Fig. 6). This is true by strict concavity of f, since $0 < \frac{n_1}{n_0} < 1$ and $s = \frac{n_1}{n_0}r + (1 - \frac{n_1}{n_0})t$.

Generalizing the proof. If the input spectrum is strictly increasing rather than decreasing on $[0, 2\pi)$ then very similar arguments hold ($f(\omega)$ is now convex). Alternatively we may observe that reflecting the frequency band $\omega \in [0, 2\pi)$ about $\omega = \pi$ gives a decreasing spectrum, and so a separate proof is really unnecessary. Similar comments hold if the interval of strict monotonicity is $[a, a + 2\pi)$ for some $a \neq 0$. If after arranging the decimators as $n_0 \geq n_1 \geq \ldots \geq n_{M-1}$ it happens that $n_0 = n_1$, then we find the smallest i for which $n_i \neq n_{i+1}$, and have n_i, n_{i+1} play the roles played by n_0, n_1 respectively in the above proof. (There is such an i because the FB's are not uniform.)

5. CONCLUSION

We have presented two different definitions of nonuniform principal component filter banks, generalizing the uniform ones. We have studied their optimality, showing that they minimize many concave objectives just like the uniform ones, though the form of the objective here must be somewhat more restricted in order to ensure PCFB optimality. We have shown that there are no nonuniform PCFB's for the unconstrained nonuniform FB classes if the input spectrum is strictly monotone (over an interval of length 2π). From this result we conclude that nonuniform PCFB existence is much more delicate than uniform PCFB existence, as it can be destroyed by small perturbations of the input spectra.

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