BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, January 1992

# NONUNIQUE TANGENT MAPS AT ISOLATED SINGULARITIES OF HARMONIC MAPS

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ABSTRACT. Shoen and Uhlenbeck showed that "tangent maps" can be defined at singular points of energy minimizing maps. Unfortunately these are not unique, even for generic boundary conditions. Examples are discussed which have isolated singularities with a continuum of distinct tangent maps.

Let $\Omega$ be a bounded domain in $R^m$ (or more generally a compact riemannian manifold with boundary) and let $N$ be a compact riemannian manifold. By the Nash embedding theorem, $N$ can be regarded as a submanifold of some euclidean space. The energy of a map $f\colon \Omega \to N$ is defined to be

$$E(f) = \int_\Omega |Df|^2.$$

(Here $f$ is allowed to be any measurable map from $\Omega$ to $R^d$ such that $f(x) \in N$ for almost every $x$ and such that the distributional first derivative of $f$ is square integrable.) The map $f$ is said to be energy minimizing if its energy is less than or equal to the energy of each other map having the same boundary values. It is fairly easy to prove that if $g\colon \Omega \to N$ has finite energy, then there is an energy minimizing map $f\colon \Omega \to N$ with the same boundary values as $g$. In [SU], Schoen and Uhlenbeck proved that if $f$ is energy minimizing, then $f$ is smooth except on a set $K \subset \Omega$ of Hausdorff dimension at most $m-3$.

Suppose $f$ is energy minimizing and that $x \in \Omega$ is a singularity of $f$. Schoen and Uhlenbeck also proved that for every sequence $r_i$ of positive numbers converging to zero, a subsequence of the maps

$$y \mapsto f(x + r_i y) \tag{1}$$

converges weakly to a map $f_\infty\colon R^m \to \Omega$ that is constant on rays through the origin. Such a map is called a *tangent map* to $f$ at $x$. Intuitively, $f_\infty$ is the result of looking at $f$ near $x$ through a microscope with infinite magnification. The map $f_\infty$ is simpler than $f$ because it is constant on rays, but one would like to think that it provides a good picture of $f$ near $x$. Note that $f_\infty$ would not give a very good picture of $f$ if there were more than one tangent map at $x$; that is, if a different subsequence of the maps (1) could converge to another limit map. Whether or not such pathological behavior is possible has been perhaps the most basic open question about singularities of energy minimizing maps.

There have been some positive results (ruling out pathological behavior). First, Leon Simon [S] showed that if $N$ is analytic and if $f_\infty$ has an isolated

Received by the editors April 3, 1991.
1991 *Mathematics Subject Classification*. Primary 49F22, 35J60.
The author was partially funded by NSF grants DMS85-53231 (PYI), DMS87-03537, and DMS9012-718.

discontinuity, then $f_\infty$ is unique (i.e., it is the only tangent map at $x$). Second, Gulliver and White [GW] showed that if $m = 3$ and $\dim(N) = 2$ (the lowest dimensions in which singularities are possible), then $f_\infty$ is unique whether or not $N$ is analytic.

This paper is an announcement of the first example of nonuniqueness:

**Theorem 1.** *There exists a $C^\infty$ 5-manifold $N$ and a nonempty open set $U$ of smooth maps $\phi : \partial B^4 \to N$ such that*

(1) *each $\phi \in U$ bounds one or more energy minimizing maps from $B^4$ to $N$, and*
(2) *if $f: B^4 \to N$ is an energy minimizing map with $f \mid \partial B^4 \in U$, then $f$ has an isolated singularity $x$ and a continuum of tangent maps at $x$. Each of the tangent maps is regular except at $0$.*

The proof is too long to give here; see [W4]. However, we can prove a simpler but nonetheless interesting result that has the same flavor. Let $f: \Omega \to N$ be a finite energy map that is smooth except at a finite set of discontinuities $\{p_i : i = 1, \dots, k\}$. We say that $f$ is *harmonic* if it satisfies the Euler-Lagrange partial differential equations for the energy functional. Such an $f$ is a critical point for energy, but it need not be a minimum. Nonetheless, the existence of tangent maps and the uniqueness results of Simon and Gulliver-White in fact hold for such harmonic maps. Thus it is interesting to note:

**Theorem 2.** *There is a $C^\infty$ 4-manifold $N$ and a harmonic map $f: B^3 \to N$ such that $f$ has an isolated singularity at $0$ and a continuum of distinct tangent maps at $0$.*

*Proof.* Let $N$ be the product $S^1 \times R \times S^2$ with the metric

$$dx^2 + dy^2 + (2 - V(x, y))dz^2$$

($V$ will be specified later). Note that this defines a complete metric on $N$ provided $V$ is everywhere less than $2$. Of course $N$ is not compact, but the image of the harmonic map we construct will be contained in a compact subset of $N$, so we could easily modify $N$ to make it compact.

Note that the orthogonal map $O(3)$ acts on $B^3$ and on $N$ (on $N$ by $\rho: (x, y, z) \mapsto (x, y, \rho z)$). We simplify the harmonic map equations by looking for solutions that are $O(3)$-equivariant. It is not hard to see that every equivariant map is of the form:

$$p \mapsto (v^1(|p|), v^2(|p|), \pm p/|p|) \in S^1 \times R \times S^2. \tag{2}$$

Here $v: (0, 1] \to S^1 \times R$. It is convenient to introduce a change of variable. Let $t = \log r$ and $u(t) = v(e^t)$, so $u: (-\infty, 0] \to S^1 \times R$. Then the energy of the map (2) is

$$\int_{t=-\infty}^{0} (|\dot{u}|^2 + (2 - V(u)))e^t \, dt.$$

(If the domain were $k$-dimensional, then $e^t$ would be $e^{(k-2)t}$.) The associated Euler-Lagrange equation is

$$\ddot{u} + \dot{u} + \nabla V(u) = 0. \tag{3}$$

Thus equivariant harmonic maps are equivalent to solutions of the ordinary differential equation (3). This equation has a physical interpretation: it is the equation of motion of a unit mass moving in $S^1 \times R$ subject to a potential $V$ and a viscous force. Thus the physical energy $E(u, t) = \frac{1}{2}|\dot{u}|^2 + V(u)$ is monotonically decreasing. To see this mathematically, multiply (3) by $\dot{u}$:

$$\frac{d}{dt}\left(\frac{1}{2}\dot{u}^2 + V(u)\right) = -\dot{u}^2. \tag{4}$$

Now we choose $V$ to be

$$V(x, y) = -\exp\left(-\frac{1}{y^2}\right)\sin\left(x + \frac{1}{y}\right).$$

**Lemma.** (1) *Let $u$ be a solution of* (3) *with $E(u, \cdot)$ constant. Then $u(t) \equiv p$ for some $p \in S^1 \times [0]$.*

(2) *Let $u$ be a solution of* (3) *such that the physical energy $E(u, 0)$ at time 0 is negative. Then the solution exists for all $t \in [0, \infty)$ and becomes unbounded as $t \to \infty$.*

*Proof.* If $E(u, \cdot)$ is constant, then $\dot{u} \equiv 0$, since otherwise the particle would be dissipating physical energy to viscosity (see (4)). Thus $u$ is a constant $p$, so (3) implies that $\nabla V(p) = 0$. But $\nabla V(x, y) = 0$ if and only if $y = 0$. This proves (1).

Now suppose that $u$ is a solution of (3) with $E(u, 0) < 0$. By elementary ODE theory, the solution exists for all positive times unless the particle moves infinitely far in a finite time. But $\frac{1}{2}|\dot{u}(t)|^2 + V(u(t)) \le E(u, 0) < 0$, so $\frac{1}{2}|\dot{u}(t)|^2 < -V(u(t)) \le \sup(-V) = 1$. Thus the solution exists for $t \in [0, \infty)$.

Suppose $u(t)$ remains in a bounded region of $S^1 \times R$. Then the set of pairs $(u(n), \dot{u}(n))$ is bounded, so a subsequence $(u(n_i), \dot{u}(n_i))$ converges. It follows (from the smooth dependence of ODE solutions on initial conditions) that the solutions $u_i(t) = u(n_i + t)$ converge smoothly to a solution $v(t)$ of (3). Now

$$E(v, t) = \lim_{i\to\infty} E(u, n_i + t) = \lim_{t\to\infty} E(u, t) \le E(u, 0) < 0 \tag{5}$$

(where $\lim_{t\to\infty} E(u, t)$ exists because $E(u, \cdot)$ is monotonic). Thus $E(v, \cdot)$ is constant, so by (1) of the lemma $v(t) \equiv p \in S^1 \times [0]$. But then $E(v, t) = V(p) = 0$, contradicting (5). This proves (2). □

Now let $u_n \colon [0, \infty) \to S^1 \times R$ be the solution to (3) with initial position $u_n(0) = (\frac{1}{2}\pi, \frac{1}{2\pi n})$ and initial velocity $\dot{u}_n(0) = 0$. Note that the initial physical energy is negative:

$$\frac{1}{2}|\dot{u}_n(0)|^2 + V(u_n(0)) = 0 + V(\frac{1}{2}\pi, \frac{1}{2\pi n}) < 0.$$

Thus by the lemma, there is a first time $t_n > 0$ at which $u_n(t_n) \in S^1 \times \{-1, 1\}$. If $u_n(t)$ were ever in $S^1 \times [0]$, then $E(u_n, t) \ge V(u_n(t)) = 0$, which is impossible. Thus $u_n(t_n) \in S^1 \times [1]$ and $u_n(t) \in S^1 \times (0, 1)$ for $t \in (0, t_n)$.

Note that $u_n(0) \to (\frac{1}{2}\pi, 0)$ and $\dot{u}_n(0) \equiv 0$, so the $u_n$ converge to a solution $w$ with $w(0) = (\frac{1}{2}\pi, 0)$ and $\dot{w}(0) = 0$. By uniqueness of solutions to ODEs, $w(t) \equiv (\frac{1}{2}\pi, 0)$. Thus $\lim_{n\to\infty} u_n(t) = (\frac{1}{2}\pi, 0)$, so $t_n \to \infty$ since $u_n(t_n) = (x_n, 1)$.

Now as in the proof of the lemma, there is a sequence $n(i)$ such that the solutions $v_i(t) = u_{n(i)}(t_{n(i)} + t)$ convergesmoothly to a solution $v$ on $(-\infty, \infty)$.

Of course

$$v(0) \in S^1 \times [1],$$
$$v(t) \in S^1 \times [0, 1] \quad \text{for} \quad t < 0,$$

and

$$E(v, t) \leq 0 \quad \text{for all } t.$$

In fact $E(v, t)$ must be strictly negative for every $t$. For since $E(v, t)$ is a nonpositive and nonincreasing function of $t$, if it were $0$ for some $t = a$, then it would be $0$ for each $t \leq a$. But then by the lemma, $v(t) \equiv p \in S^1 \times [0]$ for all $t \leq a$. By unique continuation for ODE, $v(t) \equiv p \in S^1 \times [0]$ for all $t$. But $v(0) \in S^1 \times [1]$. This proves that $E(v, t)$ is strictly negative.

Now I claim that $v$ defines a harmonic map with a continuum of tangent maps at the origin. That is, I claim that $v(t)$ has a continuum of subsequential limits as $t \to -\infty$.

As in the proof of the lemma, every sequence of $t$'s tending to $-\infty$ has a subsequence $\tau_i$ such that the solutions $w_i(t) = v(t + \tau_i)$ converge to a solution $w(t)$. Of course

$$E(w, t) = \lim_{i \to \infty} E(v, t + \tau_i) = \lim_{t \to -\infty} E(v, t) \leq 0$$

(where $\lim_{t \to -\infty} E(v, t)$ exists because $E(v, \cdot)$ is monotonic). Thus $E(w, \cdot)$ is constant, so by the lemma $w(t) \equiv p$, where $p \in S^1 \times [0]$.

What we have shown is $\lim_{t \to -\infty} v^2(t) = 0$, where $v^2(t)$ is the second component of $v(t) = (v^1(t), v^2(t)) \in S^1 \times R$.

Now the set $Z = \{p \in S^1 \times R : V(p) = 0\}$ consists of $S^1 \times [0]$ together with a collection of curves that wind around the cylinder infinitely many times as they approach $S^1 \times [0]$. Since $V(v(t)) \leq E(v, t) < 0$, $v(t)$ is never in $Z$. Thus $v(t)$ must also wind around the cylinder infinitely many times as $t \to -\infty$. This proves Theorem 2.

(To make this last argument more formal, note from the definition of $V$ that for each $x \in S^1$ and each $\varepsilon > 0$, the set $Z \cup ([x] \times (-\varepsilon, \varepsilon))$ divides $S^1 \times R$ into infinitely many connected components, the closure of each of which is disjoint from $S^1 \times [0]$. Since $v(t)$ approaches $S^1 \times [0]$ as $t \to -\infty$, the particle must cross the set $Z \cup ([x] \times (-\varepsilon, \varepsilon))$. Since it never crosses $Z$, it must cross $[x] \times (-\varepsilon, \varepsilon)$. As this holds for every $x$ and $\varepsilon$, each $(x, 0) \in S^1 \times [0]$ is a subsequential limit of $v(t)$.) □

## Remarks

Exactly the same construction provides examples of harmonic maps from $B^m$ to $N = S^1 \times R \times S^{m-1}$ (metrized as above) with a continuum of tangent maps at an isolated singularity. The only difference is that the viscosity (i.e., the coefficient in front of $\dot{u}$ in (3)) is $m - 2$ instead of $1$.

In all those examples, the dimension of the target manifold is one more than the dimension of the domain. But we can also prove that there is a harmonic map $f$ from $B^4$ to the 4-manifold $N$ of Theorem 2 such that $f$ has a continuum of tangent maps at an isolated singularity. The proof is the same as the proof of Theorem 2, except that we consider maps of the form

$$f: p \to (f^1(|p|), f^2(|p|), h(p/|p|)),$$

where $h: S^3 \to S^2$ is the Hopf fibration.

## Open questions

**1.** Must tangent maps be unique if the target manifold $N$ is 2-dimensional? The answer is "yes" if the domain is 3-dimensional [GW].

**2.** Must tangent maps be unique for generic metrics on the target manifold $N$?

**3.** If $T$ is a minimal variety in a riemannian manifold $N$, then at each singular point $x \in T$ there are one or more tangent cones (i.e., subsequential limits of images of $T$ under dilations about $x$). Can there be more than one? See [AA; T1,2; W1–3], and Simon [S] for results in special cases. Simon [S] proved that if a tangent cone has multiplicity one and has an isolated singularity, then it is unique. Unlike his analogous result for harmonic maps, this does not require that the metric on $N$ be analytic.

The construction in this paper does not seem to have any analogue in the case of minimal varieties.

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