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NORM AND ORDER PROPERTIES OF BANACH LATTICES

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Abstract

The interrelationships between norm convergence and two forms of convergence defined in terms of order, namely order and relative uniform convergence are considered. The implications between conditions such as uniform convexity, uniform strictness, uniform monotonicity and others are proved. In particular it is shown that a σ-order continuous, σ-order complete Banach lattice is order continuous.

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1. Introduction

A number of authors have studied the connection between various notions of convergence defined on a Banach lattice, see for example Birkhoff (1967), Leader (1970), Lotz (1974) and Wirth (1975). In addition they characterize those Banach lattices in which there is a special connection between the norm and order such as the monotone convergence property. Another such property is uniform strictness, considered by Berens and Lorentz (1973) in work on Korovkin sets of positive contractions on L^p spaces. We collect the various results in this work and prove several additional ones. These are illustrated by using L^p and M spaces as examples.

2. Definitions

Let $(E, \leqslant, \|\cdot\|)$ be a real Banach lattice. We say E is *o-complete* (*σ-o-complete*) if every net (sequence) in E which has an upper bound has a supremum. E is said to be *order continuous* (*σ-order-continuous*) if every monotonically decreasing net (sequence) with infimum 0 is norm convergent to 0.† E is *monotone convergent* (Lotz (1974)) if each norm bounded increasing net in E^+ is norm convergent. The Banach lattice is *uniformly monotone* (Birkhoff (1967)) if for each $\varepsilon>0$ there exists $\delta>0$ such that f, $g\geqslant 0$, $\|f\|=1$, $\|f+g\|<1+\delta$ implies that $\|g\|<\varepsilon$. Whilst a *uniformly strict* Banach lattice is one in which for each $\varepsilon>0$ there exists $\delta>0$ such that $0\leqslant g\leqslant f$, $\|f\|=1$, $\|f-g\|\geqslant\varepsilon$ implies that $\|g\|\leqslant 1-\delta$. Finally E is called *uniformly convex* if for each $\varepsilon>0$ there exists a $\delta>0$ such that $\|f\|=\|g\|=1$, $\|f-g\|\geqslant\varepsilon$ implies that $\|\frac{1}{2}(f+g)\|\leqslant 1-\delta$.

We also consider two types of convergence in E defined in terms of the order. A net $\{f_\alpha\}_A$ is *order convergent* to f if there exists an increasing net $\{g_\beta\}_B$ and a decreasing net $\{h_\gamma\}_C$ such that given any $\beta\in B$, $\gamma\in C$ there exists $\alpha_0\in A$ such that $g_\beta\leqslant f_\alpha\leqslant h_\gamma$ for all $\alpha\geqslant\alpha_0$ and

$$\bigvee_B g_\beta = f = \bigwedge_C h_\gamma \quad \text{(Vulikh (1967)).}$$

If $\{f_n\}$ is an order convergent sequence then there exist sequences $\{g_n\}$ and $\{h_n\}$ such that $g_n\leqslant f_n\leqslant h_n$ for all n.

The net $\{f_\alpha\}_A$ is *relatively uniformly* (abbreviated *ru*) *convergent* to f if there exists $u>0$ such that given any $\varepsilon>0$ there exists $\alpha_0\in A$ with $|f_\alpha-f|<\varepsilon u$ for all $\alpha>\alpha_0$.

3. Main result

The following result details the implications between some of the conditions mentioned above.

PROPOSITION 1. *The following implications hold in the class of all Banach lattices:*

$$\begin{array}{c}
\textit{uniformly convex}\\
\Downarrow\\
\textit{uniformly monotone}\\
\Updownarrow\\
\textit{uniformly strict}\\
\Downarrow\\
\textit{monotone convergent}\\
\Downarrow\\
\sigma\textit{-o-continuous} \Leftarrow \textit{o-continuous} \Rightarrow \textit{o-complete} \Rightarrow \sigma\textit{-o-complete}\\
\Updownarrow\\
\sigma\textit{-o-continuous and } \sigma\textit{-o-complete}
\end{array}$$

No other implications hold between these conditions, except those resulting from the above.

† We will henceforth abbreviate order-continuous to o-continuous and σ-order continuous to σ-o-continuous.

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PROOF. (i) Uniformly convex implies uniformly monotone. Suppose E is not uniformly monotone. Then there exists $0<\varepsilon_0<1$ such that for any $\delta>0$ there exist $f,g\in E^+$ with $\|f\|=1$, $\|f+g\|<1+\delta$ and $\|g\|\geqslant\varepsilon_0$. Suppose also that E is uniformly convex. Then there exists $\delta_0>0$ such that $\|h\|=\|k\|=1$, $\|h-k\|\geqslant\frac{1}{2}\varepsilon_0$ implies that $\|\frac{1}{2}(h+k)\|\leqslant 1-\delta_0$. Without loss of generality we may assume that $\delta_0<\frac{1}{4}\varepsilon_0$. Choose $f,g\geqslant 0$ satisfying $\|f\|=1$, $\|f+g\|\leqslant 1+\delta_0$ and $\|g\|\geqslant\varepsilon_0$. Let $h=f$, $k=(f+g)/\|f+g\|$. Hence

$$\begin{aligned}\|h-k\| &= \left\|\frac{f\|f+g\|-(f+g)}{\|f+g\|}\right\| \\ &\geqslant \left\|\frac{g}{\|f+g\|}\right\|-\left\|\frac{f(\|f+g\|-1)}{\|f+g\|}\right\| \\ &\geqslant \frac{\varepsilon_0}{1+\frac{1}{4}\varepsilon_0}-\delta_0 \\ &> \frac{\varepsilon_0}{2}.\end{aligned}$$

So

$$\left\|\frac{h+k}{2}\right\|\leqslant 1-\delta_0.$$

But

$$\begin{aligned}\left\|\frac{h+k}{2}\right\| &= \frac{\|f\|f+g\|+(f+g)\|}{2\|f+g\|} \\ &\geqslant \frac{\|f(\|f+g\|+1)\|}{2(1+\delta_0)}\geqslant\frac{1}{1+\delta_0}>1-\delta_0,\end{aligned}$$

which is a contradiction.

(ii) Uniformly monotone is equivalent to uniformly strict. Suppose that E is not uniformly monotone. So, as in (i), there exists $\varepsilon_0>0$ such that for any $\delta>0$ there exist $f,g\in E^+$ with $\|f\|=1$, $\|f+g\|\leqslant 1+\delta$ and $\|g\|\geqslant\varepsilon_0$. Suppose also that E is uniformly strict. Then there exists δ_0, $1>\delta_0>0$, such that $0\leqslant k\leqslant h$, $\|h\|=1$ and $\|h-k\|\geqslant\frac{1}{2}\varepsilon_0$ imply that $\|k\|<1-\delta_0$. Choose $f,g\geqslant 0$ such that $\|f\|=1$, $\|f+g\|<1+\delta_0$ and $\|g\|\geqslant\varepsilon_0$. Put

$$h=\frac{f+g}{\|f+g\|},\quad k=\frac{f}{\|f+g\|}.$$

Then $\|h\|=1$ and $0\leqslant k\leqslant h$. Also

$$\|h-k\|=\frac{\|g\|}{\|f+g\|}>\frac{\varepsilon_0}{1+\delta_0}>\frac{\varepsilon_0}{2}.$$

So $\|f\|/\|f+g\|=\|k\|<1-\delta_0$; hence $\|f\|<(1-\delta_0)(1+\delta_0)<1$, a contradiction.

If we now suppose that E is not uniformly strict then there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exists $h, k, 0 \leqslant k \leqslant h$, $\|h\| = 1$, $\|h-k\| \geqslant \varepsilon_0$ and $\|k\| > 1-\delta$. Let

$$f_1 = \frac{k}{\|k\|} \quad \text{and} \quad g_1 = \frac{(h-k)}{\|k\|}.$$

We can show that E is not uniformly monotone.

(iii) Uniformly monotone implies monotone convergent (Birkhoff (1967), Theorem 21, p. 371).

(iv) Monotone convergent implies o-continuous. If E is monotone convergent and $\{x_\alpha\}_A$ is decreasing with infimum 0 then $\{x_{\alpha_0} - x_\alpha\}_{\alpha \geqslant \alpha_0}$ is norm convergent for any fixed $\alpha_0 \in A$.

(v) o-continuous implies o-complete. See, for example, Lacey (1974), p. 20.

(vi) σ-o-complete together with σ-o-continuous implies o-continuous.
Lotz (1974), Proposition 1.6, p. 120, proves that if E is o-complete and not o-continuous then it contains a closed sublattice order isomorphic to l^∞. In fact his proof remains valid if we replace o-complete by σ-o-complete. If E is σ-o-continuous then it cannot possess a closed sublattice order isomorphic to l^∞. Hence if E is σ-o-complete and σ-o-continuous it is o-continuous.

4. Counterexamples

(i) $L^1(0,1)$ is uniformly monotone but not uniformly convex. See Proposition 2 below.

(ii) $\mathbf{R}^2$ with the sup norm is monotone convergent but not uniformly monotone. Take $f = (\frac{1}{3}\delta, 1)$, $g = (1+\frac{1}{3}\delta, 0)$. Then $\|f+g\| < 1+\delta$ but $\|g\| > 1$.

(iii) c_0 is o-continuous but not monotone convergent.

(iv) l^∞ is o-complete but not σ-o-continuous.

(v) If X is the one-point compactification of an uncountable discrete space then $C(X)$ is σ-o-continuous but not σ-o-complete (Nagel (1973), p. 14).

(vi) Let X be an uncountable set and select $x_\infty \in X$. Choose all subsets of X not containing x_∞ and all subsets containing x_∞ with countable complement as the basis for a topology on X. Then the space $C_b(X)$ of continuous bounded functions on X is σ-o-complete but not o-complete (Luxemburg and Zaanen (1971), p. 290).

5. L^p and M spaces

An L^p *space* $(1 \leqslant p < \infty)$ is a Banach lattice for which $\|f+g\|^p = \|f\|^p + \|g\|^p$ if $f \wedge g = 0$. Every L^p space may be identified with an $L^p(\mu)$ space for some measure μ. An *M-space* is a Banach lattice for which $\|f \vee g\| = \|f\| \vee \|g\|$ whenever $f, g \geqslant 0$.

PROPOSITION 2 (Clarkson, Lotz). *Every L^p space $(1<p<\infty)$ is uniformly convex. L^1 is uniformly monotone but not uniformly convex unless it is one dimensional. An M space is o-continuous if and only if it is $c_0(\Gamma)$ for some set Γ. An M space is monotone convergent (uniformly monotone) if and only if it is finite dimensional (one dimensional).*

PROOF. Clarkson (1936) proved that every $L^p(\mu)$ space $(1<p<\infty)$ is uniformly convex. L^1 is clearly uniformly monotone, since $\|f+g\| = \|f\|+\|g\|$ if $f, g \geqslant 0$ (Lacey (1974), p. 133). However, if the L^1 space is at least two dimensional then there exist $f, g > 0$ such that $\|f\| = \|g\| = 1$ and $f \wedge g = 0$. So

$$|f-g| = (f-g)\vee(-f+g) = 2f \vee 2g - (f+g) = f+g.$$

So $\|f-g\| = 2$, but $\|f+g\| = 2$.

Lotz (1974), Corollary 2.2, shows that an M space is o-continuous if and only if it is of the form $c_0(\Gamma)$ for some set Γ. Clearly $c_0(\Gamma)$ is monotone convergent if and only if Γ is finite. Finally, counterexample (ii) shows that if $c_0(\Gamma)$ is uniformly monotone then Γ is a singleton.

Berens and Lorentz (1973) state implicitly that a Lorentz space $\Lambda(\varphi, p)$ is uniformly convex if and only if it is uniformly monotone, provided $1<p<\infty$. It is well known, see for example Luxemburg and Zaanen (1971), pp. 287–288, that for X compact and Hausdorff, $C(X)$ is o-complete (σ-o-complete) if and only if X is extremally disconnected (the closure of every open F_σ-set is open). It is also easily shown that if X is compact, Hausdorff and first countable then $C(X)$ is σ-o-continuous if and only if X is finite.

PROBLEM. Characterize σ-o-continuous, o-complete and σ-o-complete M spaces.

6. Relative uniform convergence

In general, relative uniform convergence implies both norm and order convergence and there are no other implications. However, Birkhoff (1967) does show that if a sequence is norm convergent then each subsequence contains an ru convergent subsequence.

PROPOSITION 3. *The following implications hold for nets in a Banach lattice E.*

(i) (*norm convergence* $\Rightarrow$ *ru convergence*) $\Leftrightarrow$ *E has a strong unit* $\Leftrightarrow$ (*E is norm equivalent to* $C(X)$, *X compact Hausdorff*) $\Leftrightarrow$ (*norm convergence* $\Rightarrow$ *order convergence*).

(ii) (*order convergence* $\Rightarrow$ *ru convergence*) $\Leftrightarrow$ *E is o-continuous.*

(iii) (*norm convergence* $\Leftrightarrow$ *order convergence*) $\Leftrightarrow$ *E is* $\mathbf{R}^n$.

For sequences we have

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(iv) (*norm convergence* $\Rightarrow$ *ru convergence*) $\Leftrightarrow$ *E is norm equivalent to an M space* $\Leftrightarrow$ (*norm convergence* $\Rightarrow$ *order convergence*).

(v) (*order convergence* $\Rightarrow$ *ru convergence*) $\Leftrightarrow$ *E is σ-o-continuous.*

PROOF. (i) Take the unit ball in E as the net indexed so that $\alpha<\beta$ if $\|\alpha\|>\|\beta\|$. If E is ru convergent then E possesses a strong unit e. In that case let

$$\|x\|_e = \inf\{\lambda\colon |x|\leqslant \lambda e\}.$$

Then $(E,\|\cdot\|_e)$ is an M-space with unity and is hence isomorphic to $C(X)$ for some compact Hausdorff space X. Also $\|\cdot\|_e$ is equivalent to the original norm on E.

Conversely if norm convergence implies order convergence then again taking the unit ball as above we have an upper bound e for the unit ball. Clearly e is a strong unit.

(ii) If E is o-continuous and $\{f_\alpha\}_A$ is monotonically decreasing with $\wedge_A f_\alpha = 0$ then clearly there exist $\alpha_n \in A$ such that $|f_{\alpha_n}| < 1/n^3$ for all $n \in \mathrm{N}$. So if $u = \sum_{n=1}^{\infty} n f_{\alpha_n}$ then $|f_\alpha| \leqslant u/n$ provided $\alpha \geqslant \alpha_n$.

(iii) By Proposition 2 $C(X)$ is o-continuous if and only if X is finite.

(iv) See Wirth (1975), Proposition 2.

(v) As for (ii) above.

References

H. Berens and G. G. Lorentz (1973), 'Korovkin theorems for Banach lattices', in *Approximation theory* (editor G. G. Lorentz) (Academic Press, New York).

G. Birkhoff (1967), *Lattice theory* (American Mathematical Society, Providence, Rhode Island), pp. 366–373.

J. A. Clarkson (1936), 'Uniform convex spaces', *Trans. Amer. Math. Soc.* **40**, 396–414.

H. E. Lacey (1974), *The isometric theory of classical Banach spaces* (Springer-Verlag, New York).

S. Leader (1970), 'Sequential convergence in lattice groups', in *Problems in analysis* (editor R. C. Gunning) (University Press, Princeton, New Jersey), pp. 273–290.

W. A. J. Luxemburg and A. C. Zaanen (1971), *Riesz spaces* 1 (North Holland, Amsterdam).

H. P. Lotz (1974), 'Minimal and reflexive Banach lattices', *Math. Ann.* **209**, 117–126.

R. J. Nagel (1973), 'Ordnungsstetigkeit in Banachverbanden', *Manuscripta Mathematica* **9**, 9–27.

B. Z. Vulikh (1967), *Introduction to the theory of partially ordered spaces* (Wolters-Noordhoff, Groningen), p. 67.

A. Wirth (1975), 'Relatively uniform Banach lattices', *Proc. Amer. Math. Soc.* **52**, 178–180.

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