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Norm Bounds for Ehrhart Polynomial Roots

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Abstract M. Beck et al. found that the roots of the Ehrhart polynomial of a d-dimensional lattice polytope are bounded above in norm by $1 + (d + 1)!$. We provide an improved bound which is quadratic in d and applies to a larger family of polynomials.

Keywords Lattice polytopes · Polynomial roots · Ehrhart theory

Let P be a convex polytope in R^n with vertices in Z^n and affine span of dimension d; we refer to such polytopes as *lattice polytopes* and to elements of Z^n as *lattice points*. A remarkable theorem due to Ehrhart [5] is that the number of lattice points in the tth dilate of P, for non-negative integers t, is given by a polynomial in t of degree d called the *Ehrhart polynomial* of P. We denote this polynomial by $L_P(t)$, and let $\mathrm{Ehr}_P(x) = \sum_{t\geq 0} L_P(t)x^t$ denote its associated rational generating function. For more information regarding Ehrhart theory, see [2].

In [1] it was shown that for a lattice polytope P of dimension d, the roots of $L_P(t)$ are bounded above in norm by $1 + (d + 1)!$. However, the authors suggested that a bound that is polynomial in d should exist and questioned whether this is a property of Ehrhart polynomials in particular or of a broader class of polynomials (see Remark 4.4 on p. 26 of [1]). Our answer is the following:

Theorem 1 *If f is a nonzero polynomial of degree d with real-valued, non-negative coefficients when expressed with respect to the polynomial basis*

$$B_d := \left\{ \binom{t+d-j}{d} : 0 \leq j \leq d \right\},$$

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then all the roots of f *lie inside the disc with center* $-1/2$ *and radius* $d(d-\frac{1}{2})$.

The link between this situation and Ehrhart polynomials is that for a polynomial f of degree d over the complex numbers, there always exist complex values h_j so that

$$\frac{\sum_{j=0}^{d} h_j x^j}{(1-x)^{d+1}} = \sum_{t\geq 0} f(t)x^t.$$

As a result, f can be expressed as

$$f(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}.$$

This is easily seen by expanding the rational function as a formal power series. We then apply the following theorem, originally due to Stanley:

Theorem 2 (See [7] and [2]) *If* P *is a* d*-dimensional lattice polytope with*

$$\mathrm{Ehr}_P(x) = \frac{\sum_{j=0}^{d} h_j x^j}{(1-x)^{d+1}},$$

then the h_j *are non-negative integers.*

Thus, our result applies to Ehrhart polynomials and more generally to Hilbert polynomials of certain Cohen–Macaulay modules (see Corollary 4.1.10 of [3]).

Proof of Theorem 1 Let d be a positive integer, let $D_d := \{z\colon |z+\frac{1}{2}| \leq d(d-\frac{1}{2})\}$, and let f be as given in the theorem. It is enough to show that for any complex number z not in D_d there exists an open half-plane with zero on the boundary containing $B_d(z) := \{\binom{z+d-j}{d}\colon\ 0 \leq j \leq d\}$, since this implies that $f(z)$ is a nontrivial, non-negative linear combination of elements in a common open half-plane and is hence nonzero.

Each element of $B_d(z)$ is given by the product of $1/d!$ and d consecutive members of $M := \{(z+d), (z+d-1), \ldots, (z-d+2), (z-d+1)\}$. The elements of M are contained in a disk $D(z)$ of diameter $2d-1$ centered at $z+\frac{1}{2}$. We claim that if $|z+\frac{1}{2}| > d(d-\frac{1}{2})$, which holds for $z \notin D_d$, then the angular width of $D(z)$ is less than $\frac{\pi}{l}d$. To see this, consider one of the lines through the origin tangent to $D(z)$. The triangle formed by the origin, the point of tangency, and $z+\frac{1}{2}$ is a right triangle with hypotenuse of length $|z+\frac{1}{2}|$ and a side of length $d-\frac{1}{2}$ opposite the interior angle formed at the origin. Hence, the interior angle at the origin is $\sin^{-1}(d-\frac{1}{2}/|z+\frac{1}{2}|)$, and thus the total angular width of $D(z)$ is $2\sin^{-1}(d-\frac{1}{2}/|z+\frac{1}{2}|)$. Finally, we see that

$$2\sin^{-1}\left(\frac{d-\frac{1}{2}}{|z+\frac{1}{2}|}\right) < 2\sin^{-1}\left(\frac{d-\frac{1}{2}}{d(d-\frac{1}{2})}\right) = 2\sin^{-1}\left(\frac{1}{d}\right) < \frac{\pi}{d}.$$

Therefore, the elements of M all lie in a cone in the plane with apex the origin and angle width less than π/d. Thus, the angular difference between $(z+d-j)\cdots(z-j+1)$ and $(z+d-j-1)\cdots(z-j)$ is less than π/d for any j, $0 \le j < d$. Hence, $B_d(z)$ lies in an open half-plane and our proof is complete. □

All the polynomials in B_d have roots contained in $\{-d, -d+1, \ldots, d-1\}$. For $1 \le j \le d$, the number of polynomials in B_d with $-j$ as a root is equal to the number with $-1+j$ as a root. Thus, the location of the center of the disc in our theorem should not come as a surprise since the roots of the elements of B_d are highly symmetric with respect to the point $-1/2$. The line $x = -1/2$ also plays a prominent role for Ehrhart polynomials of cross-polytopes, as shown in [4] and [6].

It is interesting that our result only depends on f having a "nice" representation with respect to B_d. In our situation, the reason that B_d is better than the standard monomial basis is that each of the polynomials in B_d is of full degree d, and hence each such polynomial has d roots. By adapting our method, one can obtain root bounds for any polynomial in the non-negative real span of any basis for degree d polynomials containing only polynomials of degree d having positive real leading coefficients and known roots.

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