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NORM-ONE PROJECTIONS IN BANACH SPACES

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Abstract. This is a survey of results about norm-one projections and 1-complemented subspaces in Köthe function spaces and Banach sequence spaces. The historical development of the theory is presented from the 1930s to the newest ideas. Proofs of the main results are outlined. Open problems are also discussed. Every effort has been made to include as complete a bibliography as possible.

CONTENTS

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1. Introduction

One of the main topics in the study of Banach spaces has been, since the inception of the field, the study of projections and complemented subspaces. Here by a projection we mean a bounded linear operator P satisfying $P^2 = P$, and by a complemented subspace we mean a range of a bounded linear projection P.

Banach [8] posed several problems about projections and complemented subspaces. Some of the most famous ones are:

Problem 1.1. *Does every complemented subspace of a space with a basis have a basis*?

Problem 1.2. *Does every complemented subspace of a space with an unconditional basis have an unconditional basis*?

Problem 1.3. *Does every Banach space X admit a nontrivial bounded linear projection*?

Here nontrivial means that $\dim P(X) = \infty$ and $\dim X/P(X) = \infty$; notice that the Hahn-Banach theorem guarantees that for every Banach space X and every subspace $Y \subset X$ with $\dim Y < \infty$, there exists a bounded linear projection with $P(X) = Y$.

These problems proved to be very elusive. In fact, Problem 1.2 is still open, even for subspaces of L_p, despite the intensive work of many people in the area.

Problems 1.1 and 1.3 were both answered negatively more than 50 years after they were posed. Problem 1.1 was solved by Szarek in 1987 [149] and Problem 1.3 was solved by Gowers and Maurey in 1993 [79].

The work, of more than 50 years, on Problems 1.1, 1.2 and 1.3 has led to significant developments in Banach space theory and also to many intriguing questions.

In this survey we want to concentrate on the developments of the theory of projections of norm one in Banach spaces. It seems that this theory, being of isometric rather than isomorphic nature, should be much less complicated than the theory of bounded projections of arbitrary norm.

Note, however, that any Banach space can be equivalently renormed so that the given bounded projection P has norm one on X. Thus, without loss of generality, one can rephrase Problems 1.1 and 1.2 to ask for the 1-complemented subspaces with the additional properties as stated there. Maybe this is the reason why the theory of norm-one projections still has many open problems.

One of the most interesting of them is the isometric version of Problem 1.2:

Problem 1.4. *Does every* 1*-complemented subspace of a space X with a* 1*-unconditional basis have an unconditional basis* (*with any constant C*)?

This problem has the affirmative answer if space X is over $\mathbb{C}$ [90] (and then constant $C = 1$), but it is open if space X is real; see Section 7 (it is known that $C = 1$ does not work in the real case). We will discuss also some other interesting open problems related to Problem 1.3; see Questions 7.3 and 7.4.

Norm-one projections are important because they are one of the most natural generalizations of the concept of orthogonal projections from Hilbert spaces to arbitrary Banach spaces.

Another very natural generalization of orthogonal projections are metric projections (also called proximity mappings, nearest point mappings; cf. Definition 2.10), which play a very important role in the theory of best approximation.

Metric projections are usually set-valued and one of the major research directions in the area is to determine when a metric projection admits a continuous or linear selection (see, e.g., [144, 53]). Not surprisingly contractive projections and metric projections as two natural generalizations of orthogonal projections are intrinsically related to each other and there are results about linear selections of metric projections using characterizations of norm-one projections as well as results giving characterizations of contractive projections using facts about metric projections (see Section 5.e).

It is striking that despite the intensive work of many authors on contractive projections (our bibliography includes over 120 items and probably is not complete), the full characterization of norm one projections is known only in L_p-spaces among nonatomic Köthe function spaces.

There are examples demonstrating that 1-complemented subspaces of general Banach spaces other than L_p cannot be described purely in isomorphic or isometric terms (in L_p a subspace X is 1-complemented if and only if X is isometrically isomorphic to an L_p-space, possibly on a different measure space). Thus it seems that the 1-complementability of a subspace Y in X in fact depends on the way that Y is embedded in X and that norm one projections are best described in terms of conditional expectation operators in both nonatomic Köthe function spaces and sequence Banach spaces; see Conjectures 5.26 and 7.9.

This survey is organized as follows. We start from recalling well-known facts about the abundance of contractive projections in Hilbert spaces and the characterizations of Hilbert spaces through existence of enough contractive projections (Section 3).

Next we divide the survey into two parts. Part I is about nonatomic function spaces and Part II is about sequence spaces. Each part starts with the section devoted to Lebesgue spaces L_p and ℓ_p, respectively (Sections 4, 6). Sections 5 and 7 are devoted to a variety of partial results valid in Köthe function spaces and in sequence spaces.

In the present survey we limit ourselves to the theory of contractive projections in Köthe function spaces and sequence spaces, which are not M-spaces. There exists a vast literature on contractive projections in spaces of continuous functions, vector-valued function spaces, noncommutative Banach spaces as well as nonlinear contractive projections, so we could not possibly cover everything here. We refer the interested reader to surveys [120, 65] of some of these topics; some additional references are also mentioned in the text below.

Throughout the survey we use standard notations as may be found, e.g., in [103, 104]. For the convenience of the reader we collect in Section 2 some of the most important definitions that we use.

2. Definitions

As indicated in the Introduction, throughout the paper we use standard definitions and notations as may be found, e.g., in [103, 104]. In this section, for the convenience of the reader, we collect some of the most important definitions that we use. Throughout the main body of the paper we will refer to these definitions whenever we use them.

Definition 2.1. [104, Definition 1.b.17]. Let (Ω, Σ, μ) be a complete σ-finite measure space. A Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable functions on Ω is called a *Köthe function space* if the following conditions hold:

(1) If $|f(w)| \leq |g(w)|$ a.e. on Ω, with f measurable and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

(2) For every $A \in \Sigma$ with $\mu(A) < \infty$, the characteristic function χ_A of A belongs to X.

Definition 2.2. We say that a Köthe function space X is *order-continuous* if whenever $f_n \in X$ with $f_n \downarrow 0$ a.e. then $\|f_n\|_X \downarrow 0$.

Definition 2.3. The Köthe dual X' of X is the Köthe space of all g such that $\int_\Omega |f||g| \, d\mu < \infty$ for every $f \in X$. X' is equipped with the norm

$$\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int_\Omega |f||g| d\mu.$$

Köthe dual X' can be regarded as a closed subspace of the dual X^* of X. If X is order-continuous, then $X' = X^*$.

Definition 2.4 [104, Definition 2.a.1]. Let (Ω, Σ, μ) be one of the measure spaces [0, 1], $[0, \infty)$ or $\{1, 2, \ldots\}$ (with the natural measure). A Köthe function space X on (Ω, Σ, μ) is said to be a *rearrangement invariant space* (*r.i. space*) if the following conditions hold:

(1) If τ is a measure-preserving automorphism of Ω onto itself and f is measurable function on Ω, then $f \in X$ if and only if $f \circ \tau \in X$ and $\|f\|_X = \|f \circ \tau\|_X$.

(2) X' is a norming subspace of X^*.

(3) If $A \in \Sigma$ and $\mu(A) = 1$, then $\|\chi_A\|_X = 1$.

R.i. spaces are also sometimes called *symmetric spaces*, especially when $\Omega = \{1, 2, \ldots\}$.

The most commonly used r.i. spaces, besides the Lebesgue spaces L_p and ℓ_p, are Orlicz and Lorentz spaces.

Definition 2.5 (see, e.g., [103, Definition 4.a.1]). An *Orlicz function* φ is a left-continuous, nondecreasing convex function $\varphi : [0,\infty) \to [0,\infty]$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. We will also additionally assume that $\varphi(1) = 1$.

Let (Ω, Σ, μ) be one of the measure spaces $\{1, 2, \ldots\}$, $[0,\ 1]$ or $[0,\ \infty)$ (with the natural measure). The Orlicz space $L_\varphi(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) measurable functions f on Ω so that

$$\int_\Omega \varphi\left(\frac{|f(t)|}{\lambda}\right) d\mu < \infty$$

for some $\lambda > 0$. The space L_φ is equipped with the norm

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f(t)|}{\lambda}\right) \le 1\right\}.$$

$\|\cdot\|_\varphi$ is called a *Luxemburg norm*.

Sometimes L_φ is considered with a different equivalent norm $\|\cdot\|_{\varphi,O}$ which can be described as follows:

$$\|x\|_{\varphi,O} = \inf_{\lambda>0} \frac{1}{\lambda}\left(1 + \int_\Omega \varphi(\lambda\|x_n\|)d\mu\right). \tag{1}$$

$\|\cdot\|_{\varphi,O}$ is called an *Orlicz norm* and (1) is called an *Amemiya formula for the Orlicz norm*.

Orlicz functions were introduced by Birnbaum and Orlicz [23] and Orlicz spaces were first considered by Orlicz [114, 115]. Since then they were extensively studied by many authors and became a source of many examples and counterexamples in Banach space theory. There are many monographs devoted to Orlicz spaces (see, e.g., [93, 131, 49]).

The other natural extensions of Lebesgue spaces L_p and ℓ_p are Lorentz spaces $L_{w,p}$ and $\ell_{w,p}$ introduced by Lorentz [106] in connection with some problems of harmonic analysis and interpolation theory. In this paper we do not use Lorentz function spaces, so we just recall the definition of Lorentz sequence spaces $\ell_{w,p}$.

Definition 2.6 (see, e.g., [103, Definition 4.e.1]). Let $1 \le p < \infty$ and let $w = \{w_n\}_{n=1}^\infty$ be a nonincreasing sequence of nonnegative numbers such that

$w_1 = 1$ and $\lim_{n\to\infty} w_n = 0$. The Banach space of all sequences of scalars $x = (x_1, x_2, \dots)$ for which

$$\|x\|_{w,p} = \sup_{\pi} \left(\sum_{n=1}^{\infty} \|a_{\pi(n)}\|^p w_n \right)^{\frac{1}{p}} < \infty,$$

where π ranges over all permutations of the integers, is called a *Lorentz sequence space* and is denoted by $\ell_{w,p}$ (or $d(w,p)$).

Note that $\|\cdot\|_{w,p}$ can also be computed as follows:

$$\|x\|_{w,p} = \left(\sum_{n=1}^{\infty} (x_n^*)^p w_n \right)^{\frac{1}{p}},$$

where $x = (x_1, x_2 \cdots) \in \ell_{w,p}$ and $\{x_n^*\}_{n=1}^{\infty}$ is a nonincreasing sequence obtained from $\{\|x_n\|\}_{n=1}^{\infty}$ by a suitable permutation of the integers. $\{x_n^*\}_{n=1}^{\infty}$ is called a *nonincreasing rearrangement of the sequence* $\{x_n\}_{n=1}^{\infty}$.

Definition 2.7. Let X be a Banach space. We define a *duality map* J from X into subsets of X^* by the condition that $f \in J(x) \subset X^*$ if and only if $\|f\|_{X^*} = \|x\|_X$ and $\langle f, x\rangle = \|x\|_X^2$.

If $J(x)$ contains exactly one functional, then element x is called *smooth* in X.

If every element $x \in X$ is smooth in X, then X is called *smooth*.

Definition 2.8. We say that the norm in the Banach lattice X is *strictly monotone* if $\|x+y\|_X > \|x\|_X$ whenever $x, y \geq 0$ and $y \neq 0$.

Definition 2.9. A Schauder basis $\{x_i\}_i$ for X is called *monotone* if $\sup_n \|P_n\| = 1$, where P_n are the natural projections associated to the basis, i.e.,

$$P_n\left(\sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=1}^{n} a_i x_i.$$

If $\sup_n \|I - P_n\| = 1$ (where I denotes the identity operator), the basis is called *reverse monotone*.

If for all scalars $(a_i)_i$ we have

$$\|\sum_{i=1}^{\infty} a_i x_i\|_X = \|\sum_{i=1}^{\infty} |a_i| x_i\|_X,$$

then the basis $\{x_i\}_i$ is called 1-*unconditional*.

Most commonly studied examples of spaces with 1-unconditional bases include ℓ_p, Orlicz and Lorentz spaces.

Definition 2.10 Let X be a real or complex normed linear space and M a subset of X. The *metric projection* onto M is the mapping $P_M : X \to 2^M$ which associates with each x in X its (possibly empty) set of nearest points in M, i.e.,

$$P_M(x) = \{m \in M : \|x - m\| = \inf\{\|x - y\| : y \in M\}\}$$

Other terms for metric projections used in the literature include *best approximation operator*, *nearest point map*, *Chebyshev map*, *proximity mapping*, *normal projection*, *projection of minimal distance*.

Some authors use the term metric projection for a particular selection of the set-valued mapping P_M (cf. Definition 2.12).

Definition 2.11. A subset M of a normed linear space X is called *proximinal* (resp. *Chebyshev*) if for each $x \in X$, $P_M(x)$ contains at least one (resp. exactly one) element of M.

Definition 2.12. Let $Q : X \to 2^Y$ be a set-valued map. A *selection* for Q is any map $q : X \to Y$ such that $q(x) \in Q(x)$ for all $x \in X$.

3. Hilbert Spaces

The first results about norm one projections were proven in the setting of Hilbert spaces.

It is well-known that Hilbert spaces contain many norm-one projections. Namely, orthogonal projections are contractive and for every subspace Y of a Hilbert space H there exists an orthogonal projection whose range is precisely Y. In fact this property characterizes Hilbert spaces as was proven by Kakutani [88] (see also [122]) in the case of real spaces, and by Bohnenblust [26] in the complex case (cf. [120]). This was later refined by James [86] and Papini [117] (see also [133]). We have:

Theorem 3.1 (cf. [3]). *For a Banach space X with* $\dim X \geq 3$*, the following statements are equivalent*:

(i) *X is isometrically isomorphic to a Hilbert space*,

(ii) *every* 2*-dimensional subspace of X is the range of a projection of norm* 1;

(iii) *every subspace of X is the range of a projection of norm* 1;

(iv) (*James* [86]) *every* 1*-codimensional subspace of X is the range of a projection of norm* 1;

(v) (*Papini* [117], *de Figueiredo*, *Karlovitz* [71] *for the case when* $\dim X < \infty$ *and* X *is strictly convex*) *for some* $2 \leq n \leq \dim X - 1$, *every n-dimensional subspace of* X *is the range of a projection of norm* 1,

(vi) (*Papini* [117]) *for some* $1 \leq n \leq \dim X - 2$, *every n-codimensional subspace of* X *is the range of a projection of norm* 1.

Note also that in Hilbert spaces every subspace is isometrically isomorphic to a Hilbert space and therefore the equivalence (i) $\leftrightarrow$ (iii) in Theorem 3.1 can be restated as follows:

Proposition 3.2. *Let* X *be a Hilbert space. Then* $Y \subset X$ *is contractively complemented if and only if* Y *is isometrically isomorphic to a Hilbert space.*

However, as we will see in the sequel, the statement in Proposition 3.2 does not characterize Hilbert spaces.

Part 1. Nonatomic Function Spaces

4. Lebesgue Function Spaces L_p

It follows from Theorem 3.1 that in spaces other than Hilbert space there are many subspaces which are not 1-complemented. But even before the results of Kakutani (1939) and Bohnenblust (1942), Murray [109] showed, answering a question of Banach [8], that if $1 < p < \infty$, $p \neq 2$, then there exist subspaces of L_p and of ℓ_p which are not complemented. This appears to be the first result indicating that not every subspace of L_p is 1-complemented.

4.a. **Conditional expectation operators.** In 1933, in the treatise on the foundations of probability [92], Kolmogorov introduced conditional expectation operators, which are very important examples of norm-one projections in L_p and other Banach spaces. Moreover conditional expectation operators play a crucial role in describing general contractive projections, so we start from recalling their definitions and basic properties.

Definition 4.1 [92, 58]. Let (Ω, Σ, μ) be a σ-finite measure space and $\Sigma_0 \subset \Sigma$ be a σ-subalgebra of Σ such that μ restricted to Σ_0 is σ-finite (i.e., so that Σ_0 does not have atoms of infinite measure). By the Radon-Nikodym Theorem, for every $f \in L_1(\Omega, \Sigma, \mu) + L_\infty(\Omega, \Sigma, \mu)$ there exists a unique, up to an equality a.e., Σ_0-measurable locally integrable function $\mathcal{E}^{\Sigma_0} f$ so that

$$\int_\Omega g \mathcal{E}^{\Sigma_0} f d\mu = \int_\Omega g f d\mu$$

for every bounded, integrable, Σ_0-measurable function g. The function $\mathcal{E}^{\Sigma_0} f$ is called the *conditional expectation of f with respect to* Σ_0.

Basic properties of conditional expectations are as follows [92, 58]:

(1) $\mathcal{E}^{\Sigma_0}$ is a linear mapping,

(2) $\mathcal{E}^{\Sigma_0}$ is positive, i.e., $\mathcal{E}^{\Sigma_0}(f) \geq 0$ whenever $0 \leq f \in L_1(\Omega, \Sigma, \mu) + L_\infty(\Omega, \Sigma, \mu)$,

(3) $\mathcal{E}^{\Sigma_0}(f) = f$ if and only if $f \in L_1(\Omega, \Sigma_0, \mu) + L_\infty(\Omega, \Sigma_0, \mu)$; in particular, $\mathcal{E}^{\Sigma_0}(\mathbf{1}) = \mathbf{1}$ (here $\mathbf{1}$ denotes a function on Ω constantly equal to 1),

(4) $\|\mathcal{E}^{\Sigma_0} f\|_1 \leq \|f\|_1$ for all $f \in L_1(\Omega, \Sigma, \mu)$ (this follows easily since if $f \geq 0$ then $\|\mathcal{E}^{\Sigma_0} f\|_1 = \|f\|_1$ and by the positivity of $\mathcal{E}^{\Sigma_0}$ we have $\|\mathcal{E}^{\Sigma_0}(f)\| \leq \mathcal{E}^{\Sigma_0}(\|f\|)$ which gives the desired conclusion),

(5) (cf. [108]) if $p \geq 1$ then $\|\mathcal{E}^{\Sigma_0}(f)\|^p \leq \mathcal{E}^{\Sigma_0}(\|f\|^p)$ almost everywhere so $\|\mathcal{E}^{\Sigma_0}(f)\|_p \leq \|f\|_p$ for all $f \in L_p(\Omega, \Sigma, \mu)$,

(6) $\mathcal{E}^{\Sigma_0}$ satisfies the *averaging identity*, i.e.,

$$\mathcal{E}^{\Sigma_0}(f \cdot \mathcal{E}^{\Sigma_0}(g)) = \mathcal{E}^{\Sigma_0}(f) \cdot \mathcal{E}^{\Sigma_0}(g)$$

for all $f \in L_\infty(\Omega, \Sigma, \mu)$ and $g \in L_1(\Omega, \Sigma, \mu) + L_\infty(\Omega, \Sigma, \mu)$.

These properties can be summarized as follows:

Theorem 4.2 (for $p = 1$ implicit in [92], for $p > 1$ see [58, 108]). *If $p \geq 1$ and (Ω, Σ, μ) is a σ-finite measure space and Σ_0 is a σ-subalgebra of Σ, then the conditional expectation operator $\mathcal{E}^{\Sigma_0}$ is a positive contractive projection from $L_p(\Omega, \Sigma, \mu)$ onto $L_p(\Omega, \Sigma_0, \mu)$, that leaves the constants invariant.*

Several authors have investigated whether any subset of properties (1)-(6) characterizes conditional expectation operators. For the most recent results in this direction as well as a nice account of the literature, we refer the reader to [57].

In this survey we will concentrate on the papers that deal directly with the converse of Theorem 4.2.

First such results go back to the '50s [108, 7, 140, 138, 34]. These papers dealt with the converse of Theorem 4.2 under some additional assumptions on the contractive projection operator.

4.b. **Main characterizations.** In 1965, Douglas [62] gave a complete characterization of contractive projections on $L_1(\Omega, \Sigma, \mu)$ when (Σ, μ) is a finite measure space. In the following year, Ando [4] extended this characterization to $L_p(\Omega, \Sigma, \mu)$ on a finite measure space for arbitrary $p, 1 \leq p < \infty$, $p \neq 2$. They proved that every norm-one projection on $L_p(\Omega, \Sigma, \mu)$, $1 < p < \infty$, $p \neq 2$, which leaves constants invariant is a conditional expectation operator (when $p = 1$ this is true for most norm-one projections; for the precise statement, see the following Theorem 4.3). Moreover, if the norm-one projection P does not leave constants invariant, i.e., if $P\mathbf{1} = h$ is an arbitrary function in $L_p(\Omega, \Sigma, \mu)$, then h has a maximum support among functions from the range of P, the range of P is isometrically isomorphic to a weighted L_p-space (with weight h) and the projection P is a, so-called, weighted conditional expectation operator. The precise statement of their results is given below:

Theorem 4.3 [62, 4]. *Let $1 \le p < \infty, p \ne 2$ and let (Ω, Σ, μ) be a finite measure space. Then $P : L_p(\Omega, \Sigma, \mu) \longrightarrow L_p(\Omega, \Sigma, \mu)$ is a contractive projection if and only if there exists a σ-subalgebra $\Sigma_0 \subset \Sigma$ and a function $h \in L_p(\Omega, \Sigma, \mu)$ such that the support B of h is the maximum element of Σ_0 and for all $f \in L_p(\Omega, \Sigma, \mu)$, P is represented in the form*:

$$Pf = \frac{h}{\mathcal{E}^{\Sigma_0}(|h|^p)} \cdot \mathcal{E}^{\Sigma_0}(f \cdot \bar{h}^{p-1}) + Vf,$$

where, when $p \ne 1$, $V = 0$ and when $p = 1$, V is a contraction such that $V^2 = 0$, $P_B V = V$, $VP_B = 0$ and Vf/h is Σ_0-measurable (here P_B denotes the projection defined by $P_B f = f \cdot \chi_B$, where χ_B is the characteristic function of set B).

The method of Douglas and Ando depends on studying positive projections and projections whose range is a sublattice. Subsequently, as we will discuss in Section 5, many of the key steps leading to Theorem 4.3 were generalized to other spaces. Thus we will present these key steps here:

Proposition 4.4 (related to results of [7, 34, 108]). *Suppose that $Y \subset L_1$ is a closed sublattice of $L_1(\Omega, \Sigma, \mu)$. Then there exists a σ-subalgebra $\Sigma_0 \subset \Sigma$ and a weight function k such that $Y = k \cdot L_1(\Omega, \Sigma_0, \mu_{\Sigma_0})$. Moreover, the pair (Σ_0, k) is unique. (The measure μ_{Σ_0} is the restriction of μ to Σ_0.)*

Proposition 4.5. *If P is a positive contractive projection on $L_1(\Omega, \Sigma, \mu)$, then $\mathcal{R}(P)$ is a closed sublattice of $L_1(\Omega, \Sigma, \mu)$. (Here $\mathcal{R}(P)$ denotes the range of P.)*

This follows from the fact that if $f \in \mathcal{R}(P)$ then we have

$$f^+ \ge f \Longrightarrow P(f^+) \ge P(f) = f$$

and therefore $P(f^+) \ge f^+ \ge 0$.

Thus

$$\|P(f^+) - f^+\|_1 = \int_\Omega (P(f^+) - f^+) d\mu = \|P(f^+)\|_1 - \|f^+\|_1 \le 0.$$

Hence $\|P(f^+)-f^+\|_1 = 0$, i.e., $P(f^+) = f^+$ so $f \in \mathcal{R}(P)$ implies $f^+ \in \mathcal{R}(P)$ and the conclusion of Proposition 4.5 follows easily.

Theorem 4.6. *For a linear operator $P : L_1(\Omega, \Sigma, \mu) \to L_1(\Omega, \Sigma, \mu)$, the following are equivalent*:

(a) *P is a contractive projection with $P(\mathbf{1}) = \mathbf{1}$;*

(b) *P is a conditional expectation operator, i.e., there exists a unique σ-subalgebra $\Sigma_0 \subset \Sigma$ such that $P = \mathcal{E}^{\Sigma_0}$.*

A very elegant and short proof of this theorem was given recently by Abramovich, Aliprantis and Burkinshaw [1], so we will not reproduce it here. The main idea is to first show that every contraction T in L_1 with $T(\mathbf{1}) = \mathbf{1}$ is positive (first proven by Ando [4]) and then use Propositions 4.5 and 4.4 to see that $\mathcal{R}(P) = L_1(\Omega, \Sigma_0, \mu_{\Sigma_0})$ for some σ-subalgebra $\Sigma_0 \subset \Sigma$ (with $\Omega \in \Sigma_0$). In particular, we see that $P\chi_A = \chi_A$ for all $A \in \Sigma_0$, and Theorem 4.6 quickly follows (see [1]).

Next Douglas notices that in L_1 there exist contractive projections of a certain "irregular" form, which is impossible in L_p with $p > 1$; namely, we have:

Lemma 4.7. *Let $\Sigma_0 \subset \Sigma$ be a σ-subalgebra with a maximal element $B \subsetneq \Omega$. Let $V : L_1(\Omega, \Sigma, \mu) \to L_1(\Omega, \Sigma, \mu)$ be a contractive linear operator such that $V^2 = 0$, $P_B V = V$ and $V P_B = 0$. Then the operator $P = \mathcal{E}^{\Sigma_0} + V$ is a contractive linear projection from $L_1(\Omega, \Sigma, \mu)$ onto $L_1(B, \Sigma_0, \mu_{\Sigma_0})$.*

Sketch of Proof. It is not difficult to check that $P^2 = P$ and

$$\begin{aligned} Pf &= P(f\chi_B + f\chi_{B^c}) \\ &= \mathcal{E}^{\Sigma_0}(f\chi_B) + V(f\chi_B) + \mathcal{E}^{\Sigma_0}(f\chi_{B^c}) + V(f\chi_{B^c}) \\ &= \mathcal{E}^{\Sigma_0}(f\chi_B) + 0 + 0 + V(f\chi_{B^c}). \end{aligned}$$

Thus

$$\begin{aligned} \|Pf\|_1 &\le \|\mathcal{E}^{\Sigma_0}(f\chi_B)\|_1 + \|V(f\chi_{B^c})\|_1 \\ &\le \|f\chi_B\|_1 + \|f\chi_{B^c}\|_1 = \|f\|_1. \end{aligned}$$

So P is contractive. ∎

Moreover, Ando showed:

Theorem 4.8. *A contractive projection P in L_p $(1 < p < \infty,\ p \neq 2)$ with $P\mathbf{1} = \mathbf{1}$ is contractive with respect to the L_1-norm.*

The final step in the proof of Theorem 4.3 is to drop the assumption that $P\mathbf{1} = \mathbf{1}$ in Theorem 4.6. (So far any attempts to generalize this final step to spaces other than L_p have failed; see the discussion in Section 5).

Theorem 4.3 can be restated as follows:

Theorem 4.9 [4, Corollary 1]. *A contractive projection P in L_p is isometrically equivalent to a conditional expectation* (*with respect to a measure*), *if* $1 < p < \infty$, *or if* $p = 1$ *and* $PP_B = P$, *where* B *is the maximum support of elements of the range of* P.

Also as a corollary one immediately obtains an analogue of Proposition 3.2. Namely, we have:

Theorem 4.10 (Douglas, Ando, see [19, Theorem 4.1] for general measures).

(A) *For a subspace* Y *of* $L_p(\Omega, \Sigma, \mu)$, *the following statements are equivalent*:

- (A1) Y *is the range of a positive contractive projection*,
- (A2) Y *is the closed sublattice of* $L_p(\Omega, \Sigma, \mu)$,
- (A3) *there exists a positive isometrical isomorphism from* Y *onto some* L_p (B, Σ_0, ν).

(B) *For a subspace* Y *of* $L_p(\Omega, \Sigma, \mu)$, *the following statements are equivalent*:

- (B1) Y *is the range of a contractive projection*,
- (B2) *there exists a function* ϕ *on* Ω, *with* $\|\phi\| = 1$ *a.e.*, *such that* $\phi \cdot Y$ *is the closed sublattice of* $L_p(\Omega, \Sigma, \mu)$,
- (B3) Y *is isometrically isomorphic to some* $L_p(B, \Sigma_0, \nu)$.

4.c. **Characterizations of L_p through contractive projections.** In 1969, Ando [5] showed that Theorem 4.10 does characterize L_p among Banach lattices of dimension bigger or equal than 3. He proved:

Theorem 4.11. *Let* X *be a Banach lattice with* $\dim X \geq 3$. *Then* X *is order isometric to* $L_p(\mu)$, *for some* $1 \leq p < \infty$ *and measure* μ, *or to* $c_0(\Gamma)$, *for some index set* Γ, *if and only if there is a contractive positive projection from* X *onto any closed sublattice of it.*

For expositions of this theorem, see, e.g., [95], [94, §16], [104, Section 1.b]. Theorem 4.11 has been strengthened in a series of articles by Calvert and Fitzpatrick (1986-1991) which will be discussed in Section 6. Here we just mention one of their theorems which directly applies to nonatomic function spaces.

As Calvert and Fitzpatrick indicate, in the following statement, they had in mind A being a set of characteristic functions of measurable sets:

Theorem 4.12 [42, Theorem 4.4]. *Let* X *be a Banach lattice and suppose* $A = \{e_i\}_{i \in I}$ *is a set of positive elements of* X *such that* $e, f \in A$ *implies* $(e - f)^+ \in A$

and $(e-f)^+ \wedge f = 0$. *Suppose* X *is the closed span of* A, *and* $\dim(\mathrm{span} A) \geq 3$. *Suppose that any two-dimensional sublattice of* X *containing any* e_i, $i \in I \backslash \{i_0\}$ (*where* $i_0 \in I$ *is a fixed arbitrary element of* I) *is the range of a contractive projection. Then* X *is an* L_p*-space* $(1 \leq p < \infty)$ *or an* M*-space.*

4.d. **Extensions of the main characterizations to general measure spaces.** Earlier, in 1955, Grothendieck [80] showed part of Theorem 4.10, namely, he showed that when $p=1$ and (Ω, Σ, μ) is a general measure space then a range of a contractive projection in $L_1(\Omega, \Sigma, \mu)$ is isometrically isomorphic to another L_1-space.

In 1970, Wulbert [151] extended Theorem 4.3 to arbitrary measure spaces under the additional assumption that projection P is contractive both in L_p-norm and L_∞-norm. At the same time, Tzafriri [150] extended Theorem 4.10 to L_p-spaces on arbitrary measure spaces.

Dinculeanu and M. M. Rao studied generalizations of Theorems 4.2 and 4.3 to $L_p(\Omega, \Sigma, \mu)$, where Σ is not assumed to be a σ-algebra but only a δ-ring, i.e., Σ is closed under finite unions, differences and countable intersections and μ is a finitely additive general measure (not necessarily finite). Dinculeanu [54] showed that Theorem 4.2 still holds in this more general setting, and Dinculeanu and Rao [55] obtained analogues of Theorem 4.3 under additional assumptions that projection P besides being contractive is also positive, or the range of P equals $L^p(\Lambda)$ for some sub-δ-ring $\Lambda \subset \Sigma$, or P satisfies the averaging identity (6).

In 1974, Bernau and Lacey [19] (see also [95, 94]) gave new unified presentation of the proof of Theorem 4.3 and Theorem 4.10 generalized to arbitrary measure spaces. When $1 < p < \infty$, their approach does not rely on the reduction to the case of L_1, but instead they use duality arguments depending on the smoothness of L_p and $L_q = (L_p)^*$. Their proof uses the following two key lemmas:

Lemma 4.13 [4, Lemma 1; 19, Lemma 2.2]. *Suppose* $1 < p < \infty$ *and let* P *be a contractive projection on* $L_p(\Omega, \Sigma, \mu)$. *Then* $f \in \mathcal{R}(P)$ *if and only if* $J(f) \in \mathcal{R}(P^*)$ (*here* J *denotes the duality map in* L_p, *see Definition* 2.7).

Sketch of Proof. Let $f \in \mathcal{R}(P)$. Notice that we have

$$\begin{aligned}
\|f\|_p^2 &= \langle f, J(f)\rangle = \langle Pf, J(f)\rangle \\
&= \langle f, P^*(J(f)\rangle \leq \|f\|_p \cdot \|P^*(J(f))\|_q \\
&\leq \|f\|_p \cdot \|J(f)\|_q = \|f\|_p^2.
\end{aligned}$$

Since L_p, $1 < p < \infty$, is smooth, we conclude that

$$P^*(J(f)) = J(f),$$

i.e.,

$$J(f) \in \mathcal{R}(P^*).$$

Since $L_q = (L_p)^*$ is also smooth, we similarly obtain that $J(f) \in \mathcal{R}(P^*)$ implies $J(J(f)) = f \in \mathcal{R}(P^{**}) = \mathcal{R}(P)$.

Notice that this proof is valid not only in L_p, but in any smooth, reflexive Banach space X with a smooth dual. ■

Lemma 4.14 [19, Lemma 2.3(i)]. *Suppose* $1 < p < \infty$, $p \neq 2$. *Let* P *be a contractive projection on* $L_p(\Omega, \Sigma, \mu)$. *If* $f, g \in \mathcal{R}(P)$, *then* $|f|\text{sgn}\, g \in \mathcal{R}(P)$.

Idea of Proof. [19, Lemma 2.3(i)]. The proof of this important lemma is somewhat technical. It involves an inductive procedure as follows:

Set $k_0 = g$. Then, by Lemma 4.13, $J(f), J(f+\lambda k_0) \in R(P^*)$ for every $\lambda \in \mathbb{R}$. It is then shown that

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(J(f + \lambda k_0) - J(f))$$

exists a.e. and is in L_q.

Set

$$g_0 = \lim_{\lambda \to 0} \frac{1}{\lambda}(J(f + \lambda k_0) - J(f)),$$
$$k_1 = J^{-1}(g_0).$$

Then $g_0 \in R(P^*)$ and, by Lemma 4.13, $k_1 \in \mathcal{R}(P)$. Define inductively

$$k_{n+1} = J^{-1}(\lim_{x \to 0} \frac{1}{\lambda}(J(f + \lambda k_n) - J(f))).$$

Then Bernau and Lacey show that $\lim_{n\to\infty} k_n$ exists in L_p and that

$$\lim_{n\to\infty} k_n = |f|\text{sgn} g.$$

Since $k_n \in \mathcal{R}(P)$ for each n, the lemma is proven. ■

By Lemma 4.14, it is not difficult to deduce:

Lemma 4.15 [19, Lemma 3.1]. *Suppose* $1 < p < \infty$, $p \neq 2$ *and let* P *be a contractive projection on* $L_p(\Omega, \Sigma, \mu)$. *Define* $\Sigma_0 = \{\text{supp}\, f : f \in \mathcal{R}(P)\} \subset \Sigma$. *Then* Σ_0 *is a* σ*-subalgebra of* Σ.

In the next step of the proof of Theorem 4.3, Bernau and Lacey show that there exists a function h in $\mathcal{R}(P)$ with maximal support and that $\mathcal{R}(P)$ is isometrically isomorphic to $L_p(\text{supp}\, h, \Sigma_0, |h|^p \mu_{\Sigma_0})$, which leads them to the final conclusion.

4.e. **Results coming from the approximation theory and the nonlinear approach**. Contractive linear projections in L_p were also studied from the point of view of nonlinear analysis. This comes from the fact that the existence of a contractive linear projection onto a subspace is intrinsically related to the metric projection onto a complementary subspace. In Section 5.e below we discuss this relationship in greater detail. Here we just note the fact that was used (and observed) by de Figueiredo and Karlovitz [71]:

Proposition 4.16. *Let X be a normed space and P be a linear projection on X with* $\operatorname{codim}\mathcal{R}(P) = 1$. *Then P has norm one if and only if $I - P$ is a metric projection onto* $\operatorname{Ker}(P)$, *i.e., for each $x \in X$ and $y \in \operatorname{Ker}(P)$, $\|x - (I - P)x\| \leq \|x - y\|$ (see Definition* 2.10).

Thus, if P is a norm-one projection, then there exists a linear selection of a metric projection onto $Ker(P)$.

De Figueiredo and Karlovitz used this to obtain the following:

Proposition 4.17 [71, Proposition 3] (see also [15, Proposition 1]). *Let $1 < p < \infty$, $p \neq 2$, and (Ω, Σ, μ) be a σ-finite nonatomic measure space. Then no subspace of $L_p(\Omega, \Sigma, \mu)$ of codimension one is the range of a linear projection of norm one in $L_p(\Omega, \Sigma, \mu)$.*

This proposition, of course, also follows quickly from Theorem 4.3 (also for $p = 1$) but the proof in [71] is significantly simpler. The proof in [15] is also an application of Proposition 4.16. In fact, Beauzamy and Maurey obtained a somewhat stronger result not limited to nonatomic measures:

Theorem 4.18 [15, Proposition 1] ($cf.\ also$ [52, Theorem 5.2]). *Let $1 < p < \infty$, $p \neq 2$, and let $f \in L_q(\Omega, \Sigma, \mu)$, $f \neq 0$ $((1/p) + (1/q) = 1)$. Then the hyperplane $f^{-1}(0)$ is the range of a norm-one linear projection in $L_p(\Omega, \Sigma, \mu)$ if and only if f is of the form $f = \alpha \chi_A + \beta \chi_B$, where A and B are atoms of μ and α, β are scalars.*

Proposition 4.17 for $p = 1$ was proven using the methods of L-summands [82, Corollary IV.1.15].

Contractive projections in L_p were used to construct monotone and unconditional bases in L_p. Such constructions depend on the study of increasing sequences of contractive projections (we say that a sequence of projections $\{P_j\}_{j=1}^{\infty}$ is increasing if $P_i P_j = P_{\min\{i,j\}}$ for all $i, j \in \mathbb{N}$); see [61, 121, 35, 64] and the survey [63] for the discussion of different unconditionality properties for contractive projections. We are not aware of any generalizations of these constructions to spaces other than L_p, maybe because contractive projections are not well understood in spaces other than L_p (see Section 5).

4.f. **Bibliographical remarks.** There are many papers that deal with different properties of contractive projections in Lebesgue spaces L_p in the situation when the projection satisfies some additional conditions. We do not have the space here to describe all the relevant results, we just list the papers that the author of this survey is aware of: [37, 36, 18, 20, 17, 147, 148, 125].

5. Nonatomic Köthe Function Spaces

To this day, there are no classes of Köthe function spaces other than L_p where the form of contractive projections is fully characterized. In this section we present the history of different partial results describing different properties of norm-one projections and 1-complemented subspaces of different Köthe function spaces.

5.a. **Results related to the ergodic theory.** The first general results about the existence of norm-one projections are the consequence of the classical Mean Ergodic Theorem (see [66, Section VIII.5]. This theorem goes back to 1930s; see the excellent bibliographical notes in [66, Section VIII.10, p. 728]). We have:

Theorem 5.1. *Let X be a reflexive Banach space and T be a norm-one operator on X. Then the operators*

$$A_{T,n} = \frac{1}{n+1}\sum_{j=0}^{n} T^j$$

converge strongly to a norm-one projection onto the space $F_T = \{x \in X : T(x) = x\}$.

Also Lorch [105] studied monotone sequences of projections $\{P_n\}_{n=1}^{\infty}$ whose norm has a common bound (i.e., $\exists K \in \mathbb{R}$ such that $\|P_n\| \le K$ for all $n \in \mathbb{N}$).

Definition 5.2. We say that a sequence of projections $\{P_n\}_{n=1}^{\infty}$ is *increasing* (resp. *decreasing*) if for all $n, m \in \mathbb{N}$, $P_nP_m = P_{\min(n,m)}$ (resp., $P_nP_m = P_{\max(n,m)}$).

Lorch showed in particular that:

Theorem 5.3. *Suppose that X is a reflexive Banach space. Let $\{P_n\}_{n=1}^{\infty}$ be a monotone sequence of contractive projections. Then*

(a) *if the sequence $\{P_n\}_{n=1}^{\infty}$ is increasing, then $Y = \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(P_n)}$ is 1-complemented in X;*

(b) *if the sequence $\{P_n\}_{n=1}^{\infty}$ is decreasing, then $Y = \bigcap_{n=1}^{\infty} \mathcal{R}(P_n)$ is 1-complemented in X.*

(Recall that $\mathcal{R}(P)$ denotes the range of the projection P.)

The study of contractive projections in connection with the ergodic theory is still very active. It includes in particular a study of Korovkin sets and Korovkin approximations; see, e.g., [18, 152, 46, 134] and their references ([46, Section VI] contains a short overview of related results).

5.b. **Other general results.** The next result illustrates that 1-complemented subspaces can be very rare and unstable (see also Theorem 6.2 and Section 7.a). Lindenstrauss [101] showed the following striking fact:

Theorem 5.4. *There exist Banach spaces $Z \supset X$ with $\dim Z/X = 2$ such that*

(1) *for every $\varepsilon > 0$, there is a projection with norm $\leq 1+\varepsilon$ from Z onto X;*
(2) *for every Y with $Z \supsetneq Y \supsetneq X$, there is a projection of norm one from Y onto X;*
(3) *there is no projection of norm one from Z onto X.*

Thus there does not exist any limiting process that can be applied to obtain 1-complemented subspaces out of $(1+\varepsilon)$-complemented subspaces.

Results similar to Theorem 5.3 were considered in the context of nonlinear projections and with emphasis on what kind of sequences of (both linear and nonlinear) contractive projections and products of contractive projections converge to a contractive projection, see [81] for a study of products of projections in Hilbert space, and [33, 133, 68] for the initial results on products of contractive projections in reflexive Banach spaces (see also the survey [69]). The literature on this subject continues to grow but the discussion of the results in this direction is beyond the scope of the present survey.

The next result that we want to mention here is the following fact observed by Cohen and Sullivan in 1970 [51]:

Theorem 5.5. *A subspace of a smooth space X can be the range of at most one projection of norm one.*

The question of uniqueness has been subsequently studied in spaces which are not necessarily smooth; see [110, 111, 112, 113].

5.c. **Characterizations using the duality map.** In 1975, Calvert proved a very important characterization of 1-complemented subspaces of a reflexive Banach space X with X and X^* strictly convex. He showed:

Theorem 5.6 [38]. *Suppose that X is a strictly convex reflexive Banach space with strictly convex dual X^*. Let $J : X \to X^*$ be the duality map (i.e., $\|Jx\| =$*

$\|x\|, \langle Jx, x\rangle = \|x\|^2$; *see Definition* 2.7). *Then a closed linear subspace M of X is the range of a linear contractive projection if and only if $J(M)$ is a linear subspace of X^*.*

Implication "$\Longrightarrow$" in Theorem 5.6 is fairly immediate. Indeed, this really is the statement of Lemma 4.13, which says that if $M = \mathcal{R}(P)$, where P is a contractive projection, then $J(M) = \mathcal{R}(P^*)$ and thus $J(M)$ is a linear subspace of X^*. It is easy to check that the proof of Lemma 4.13 is valid not only in L_p $(1 < p < \infty)$, but in any smooth reflexive Banach space with smooth dual (cf. also [51, Theorem 8]).

Implication "$\Longleftarrow$" follows from the following generalization of Proposition 4.16 (cf. Section 5.e):

Proposition 5.7 [38; 15, Lemma 8]. *Suppose that X is reflexive, smooth and strictly convex. Let P be a projection on X. Then $I - P$ has norm one if and only if P is the metric projection* (*cf. Definition* 2.10).

Theorem 5.6 proved extremely useful in describing the form of 1-complemented subspaces in various Banach spaces. We already saw that the proof of Bernau and Lacey in L_p was based on one part of this theorem (the key Lemma 4.13) and it will be used frequently in the results described in Section 7. In fact, Theorem 5.6 is so important that it was proved again (with different methods) by Arazy and Friedman as a starting point for their study of contractive projections in $\mathcal{C}_p$ [6].

In 1977, Calvert extended Theorem 5.6 to general Banach spaces without the assumption of reflexivity or strict convexity. He proved:

Theorem 5.8 [39]. *Let X be a Banach space over $\mathbb{C}$ or $\mathbb{R}$. Let M be a closed linear subspace of X. M is the range of a contractive projection if and only if there exists a weak*-closed linear subspace L of X^* with $M \subset J^{-1}(L)$ and $L \subset \overline{J(M)}^{\|\cdot\|_{X^*}}$.*

This appears to be the most general characterization of contractively complemented subspaces known.

5.d. **Characterizations obtained from the nonlinear approach.** Contractive projections were also studied in the nonlinear theory of Banach spaces. We do not have the necessary space here to describe all the interesting results concerning nonlinear contractive projections (see, e.g., the survey [116] and the excellent exposition in [141]) but we do want to mention a couple of results concerning linear projections which were proven as a "side bonus" of the nonlinear approach.

We start from the collection of facts when the existence of a nonlinear contractive projection from X onto a linear subspace $Y \subset X$ implies the existence of a linear contractive projection from X onto Y (these facts are listed in [141, Proposition 2.1] and are described as “folklore” or “should be folklore”).

Proposition 5.9. *Let X be a Banach space and Y be a closed linear subspace of X.*

(a) *If* $\mathrm{codim}Y = 1$ *and there exists a nonlinear contractive projection from X onto Y, then Y is 1-complemented (linearly) in X.*

(b) *If X is smooth and if $P : X \longrightarrow Y$ is a contractive retraction onto Y, then P is a linear contractive projection.*

We also have the following deep fact:

Theorem 5.10 [100]. *Let X be a Banach space and Y be a closed linear subspace of X such that there exists a nonlinear contractive projection from X onto Y. If Y is a conjugate space then Y is 1-complemented $(linearly)$ in X.*

Beauzamy [13] introduced the following notions:

Definition 5.11. Let M be a subset of a Banach space X. A point $x \in X$ is called *minimal with respect to M* if for all $y \in X \setminus \{x\}$ there exists at least one $m \in M$ with $\|m - y\| > \|m - x\|$.

A set of all points minimal with respect to M is denoted $\min(M)$. Clearly $M \subset \min(M)$. A set M is called *optimal* if $M = \min(M)$.

Beauzamy and Maurey proved that this notion is closely related to contractive projections.

Theorem 5.12 ([15; see also [14] for the “$\Longrightarrow$” direction). *Suppose that X is reflexive, strictly convex and smooth. Then the closed subspace $Y \subset X$ is optimal if and only if Y is the range of a linear norm-one projection on X.*

The proof uses Proposition 5.7.

Theorem 5.12 has been extended by Godini, who weakened the conditions on X, but considered a slightly more restrictive definition of minimal sets.

Definition 5.13. Let X be a real normed linear space and Y a linear subspace of X. To each nonempty subset $M \subset Y$ we assign a subset $M_{Y,X} \subset X$ defined as follows:

$M_{Y,X} = \{x \in X :$ for all $y \in Y \setminus \{x\}$ there exists $m \in M$ such that $\|y - m\| > \|x - m\|\}$.

Thus the set $\min(M)$ defined by Beauzamy and Maurey equals $M_{X,X}$.

Godini proved that

Theorem 5.14 [78, Theorem 2]. *Let X be a normed linear space and Y a closed linear subspace of X. A necessary, and if every element of Y is smooth in X also sufficient, condition for the existence of a norm-one linear projection P of X onto Y is that $Y_{Y,X} = Y$. If every element of Y is smooth in X, then there exists at most one norm one linear projection of X onto Y.*

Godini used this result to characterize spaces X which are 1-complemented in X^{**}, provided every element of X is smooth in X^{**}.

5.e. **Relations with metric projections.** As we mentioned a few times above there is a fruitful line of investigating contractive linear projections in connection with metric projections. Both norm-one projections and metric projections (see Definition 2.10) are natural generalizations of orthogonal projections from Hilbert spaces to general Banach spaces. Thus, not surprisingly, there is an intrinsic connection between them. We have the following very clear but important fact:

Proposition 5.15. *Let X be a normed linear space and M a linear subspace of X. Then for all $x \in X$ and any $y \in P_M(x)$,*

$$\|x - y\| \leq \|x\|$$

and

$$\|y\| \leq 2\|x\|.$$

In this proposition we do not assume that $P_M(x)$ has a linear selection (see Definition 2.12). The earliest explicit reference that we could find for Proposition 5.15 is [142] (cf. [144, Theorem 4.1]). However, different (usually weaker) versions of Proposition 5.15 were observed on many occasions by different authors; see Propositions 4.16, 5.7 above and the excellent exposition of this topic with full references in [144] (cf. also [53]). For us, the most important is the following corollary of Proposition 5.15.

Proposition 5.16. *Let X be a normed linear space. Let P be a linear projection on X. Then $\|P\| = 1$ if and only if $I - P$ is a selection of a metric projection onto* Ker$P \subset X$. (*here I denotes the identity operator on X*).

In particular, if $\|P\| = 1$ then there exists a linear selection of a metric projection onto Ker$P$.

We include the simple proof below:

Proof (cf., e.g., [71]). Suppose that P is a linear projection on X with $\|P\| = 1$. Then for each $x \in X$ and $y \in$ Ker$P$,

$$\|x - (I - P)x\| = \|Px\| = \|P(x - y)\| \leq \|x - y\|.$$

Thus Ker$P$ is a proximinal subspace of X (cf. Definition 2.10) and $I - P$ is a selection of a metric projection onto Ker$P$.

Conversely, if $I - P$ is a selection of a metric projection onto Ker$P$ then for each $x \in X$,

$$\|Px\| = \|x - (I - P)x\| \leq \|x - 0\| = \|x\|.$$

Hence $\|P\| = 1$.

The proof is finished by noting that if P is linear so is $I - P$. ■

As we illustrated above, Proposition 5.16 is a very important tool in obtaining characterizations of contractive projections (Proposition 4.17, Theorems 4.18, 5.6, 5.12, 5.14, 5.38, 5.39, 5.40). Proposition 5.16 was also used to obtain characterizations of subspaces of L_p which admit a linear selection of a metric projection through the analysis of known results about norm-one projections in L_p-spaces [97, 146].

5.f. **Relations with a notion of orthogonality.** Contractive projections may be treated as a generalization of orthogonal projections from Hilbert spaces to general Banach spaces. This has been explored by Papini [117], Faulkner, Huneycutt [70], Campbell, Faulkner, Sine [46] and Kinnunen [91], who considered the following extension of orthogonality to general Banach spaces:

Definition 5.17 [3, §4]. Let X be a real Banach space and $x, y \in X$. We say that x is *orthogonal to* y in the sense of Birkhoff-James (or simply x *is* BJ-*orthogonal to* y), denoted by $x \perp y$, if

$$\|x\| \leq \|x + \lambda y\|$$

for all $\lambda \in \mathbb{R}$.

The notion of BJ-orthogonality was introduced by Birkhoff [22] and developed by James [84, 85, 86]; cf. also [3].

In general, $x \perp y$ does not imply $y \perp x$. Thus we introduce two notions of orthogonal projections:

Definition 5.18. A projection P on X is called *left-orthogonal* (resp. *right-orthogonal*) if for each $x \in X$,

$$Px \perp (x - Px)$$

$$\text{(resp. } (x - Px) \perp Px).$$

Papini [117] obtained characterizations of Hilbert spaces among general Banach spaces using these notions (see Theorem 3.1).

In real Banach spaces we have the following:

Theorem 5.19 [70, 91]. *Let X be a real Banach space and M a complemented linear subspace of X. Let P be a linear projection from X onto M. Then*:

(a) $\|P\| = 1$ *if and only if P is left-orthogonal.*

(b) *P is (a selection of) a metric projection if and only if P is right-orthogonal.*

(c) *If $\|P\| = 1$, then $M \perp \mathrm{Ker}P$.*

(d) *If P is (a selection of) a metric projection then* $\mathrm{Ker}P \perp M$.

(e) *If $M \perp \mathrm{Ker}P$ and $X = M \oplus \mathrm{Ker}P$ then $\|P\| = 1$.*

(f) *If $\mathrm{Ker}P \perp M$ and $X = M \oplus \mathrm{Ker}P$, then P is (a selection of) a metric projection.*

In particular, as a corollary Kinnunen obtained another proof of Proposition 5.16. Also, as an application of Theorem 5.19, he obtained a characterization of norm-one projections of a finite rank. For this we need the following definition:

Definition 5.20 [143]. Let $\{x_n\}$ be a basis of a Banach space X. Then the *sequence of coefficient functionals $\{f_n\}$ associated to the basis $\{x_n\}$* is defined by

$$f_j(x_k) = \delta_{jk}$$

for all j, k, where δ_{jk} denotes the Kronecker delta.

A basis $\{x_n\}$ is called *normal* if $\|x_n\|_X = \|f_n\|_{X^*} = 1$ for all n.

By [143, Theorem II.2.1], a basis $\{x_n\}$ is normal if and only if $\|x_n\| = 1$ and $x_n \perp \overline{\mathrm{span}}\{x_1, \cdots, x_{n-1}, x_{n+1}, \cdots\}$ for each n.

Further, by [143, Theorem II.2.2] every finite-dimensional Banach space has a normal basis. Kinnunen proved the following characterization of norm-one projections in terms of normal bases:

Theorem 5.21 [91, Theorem 5.3]. *Let X be a real Banach space, M be a 1-complemented linear subspace of X with $\dim M = n < \infty$, and $P : X \overset{onto}{\longrightarrow} M$ be a norm-one projection. Then P is of the form*:

$$Px = \sum_{k=1}^{n} f_k(x) u_k$$

for all $x \in X$, where $\{u_k\}_{k=1}^{n}$ is a normal basis of M and $\{f_k\}_{k=1}^{n} \subset X^*$ are norm-one functionals such that $f_k(u_i) = \delta_{ki}$ for all $k, i = 1, \cdots, n$.

The problem of describing the basis structure of 1-complemented subspaces in general real Banach spaces is still open; see Section 7 for the discussion of known results in both real and complex Banach spaces.

5.g. **Relations with isometries.** Here we consider the following question:

Question 5.22. *Let X be a Banach space and $T \overset{into}{\longrightarrow} X$ be an isometry. Is the range of T, $Y = T(X)$, 1-complemented in X?*

This question has an affirmative answer in Hilbert spaces (see Proposition 3.2) and in L_p, $1 \le p < \infty$ (see Theorem 4.11(B3)).

Question 5.22 for reflexive Banach spaces was posed by Faulkner and Huneycutt [70]. It is known that in $C[0,1]$ a range of an isometry does not have to be even complemented [56].

Question 5.22 was considered by Campbell, Faulkner and Sine [46], who proved:

Theorem 5.23. *Let X be a reflexive Banach space and T be an isometry on X. If the range of T, $Y = T(X)$, is 1-complemented in X, then T is a Wold isometry.*

Here "Wold isometry" is defined as follows:

Definition 5.24. Let T be an injective linear map on a Banach space X. Then T is called a *unilateral shift* provided there exists a subspace L of X for which

$$X = \bigoplus_{n=0}^{\infty} T^n(L).$$

An isometry T on X is called a *Wold isometry* provided $X = M_\infty \oplus N_\infty$, where $M_\infty = \bigcap_{n=1}^{\infty} T^n(X)$ and $N_\infty = \sum_{n=0}^{\infty} \oplus T^n(L)$, where L is a complement for the range of T, $T(X)$, in X.

Then $T|N_\infty$ is a shift and $T|M_\infty$ is a surjective isometry (sometimes referred to as a unitary operator).

This definition was introduced in [70] and used to study extensions to reflexive Banach spaces of the Wold Decomposition Theorem, which says that every isometry on a Hilbert space is the direct sum of a unitary operator and copies of the unilateral shift.

Campbell, Faulkner and Sine [46] also gave an example of a $C(K)$ space and a Wold isometry T on $C(K)$ such that the range of T is 2-complemented but not 1-complemented in $C(K)$ (in this example, the range of T is even finite-codimensional in $C(K)$).

It is not known whether every isometry in a reflexive Banach space is a Wold isometry and thus Question 5.22 is still open (see also Remark in Section 6.c).

5.h. **Characterizations in terms of the conditional expectation operators.** Next came a series of results which studied the form of a contractive projection in terms of the conditional expectation operators. We have the following generalization of Theorem 4.2 from L_p to general rearrangement invariant spaces.

Theorem 5.25 [104, Theorem 2.a.4]. *Let X be a rearrangement invariant function space* (*see Definition* 2.4) *on the interval I, where $I = [0,1]$ or $I = [0,\infty)$. Then, for every σ-subalgebra Σ_0 of measurable subsets of I so that the Lebesgue measure restricted to Σ_0 is σ-finite, the conditional expectation operator $\mathcal{E}^{\Sigma_0}$ is a projection of norm one from X onto the subspace X_{Σ_0} of X consisting of all Σ_0-measurable functions in X.*

As Lindenstrauss and Tzafriri pointed out, Theorem 5.25 is a consequence of Theorem 4.2 and general interpolation theorem (although they do give a direct proof of it), and thus it is really valid in any interpolation space between L_1 and L_∞.

Remark. The statement of Theorem 5.25 appears explicitly in [67, Section 11.2]; we do not know whether or not this is the first reference for it.

Similarly as in the case of L_p, a lot of effort has been put into proving the converse of Theorem 5.25, i.e., into proving the following conjecture, which would generalize Theorem 4.3 for L_p:

Conjecture 5.26. *Let X be a rearrangement invariant function space on the interval $[0,1]$* (*so X contains constant functions*). *Suppose that P is a contractive projection on X with $P(\mathbf{1}) = \mathbf{1}$. Then there exists a σ-algebra Σ_0 of measurable subsets of $[0,1]$ so that P is the conditional expectation operator $\mathcal{E}^{\Sigma_0}$.*

In fact, many extend Conjecture 5.26 to any function space X for which Theorem 5.25 is valid, i.e., to spaces X with $\mathbf{1} \in X$ and where conditional expectation operators are norm-one projections (Bru and Heinich [30] call spaces X with this property *invariant under conditioning*).

Duplissey [67, Theorem II.2.1] showed that arbitrary conditional expectation operators are contractive in a Köthe space X if and only if all conditional expectation operators with finite-dimensional range are contractive on X. Duplissey also studied Conjecture 5.26 but with the additional assumption that P is contractive in L_∞-norm as well as in norm of X. He proved:

Theorem 5.27 [67, Theorem II.5.5]. *Let X be a strictly monotone Köthe function space on a σ-finite measure space (Ω, Σ, μ). Then the following are equivalent*:

(a) *If* $\mu(\Omega)$ *is not finite, then for each* $A \in \Sigma$ *such that* $0 \neq \chi_A \in X$ *and* $\mu(A) < \infty$, *the element* χ_A *is smooth in* X *and*

$$J(\chi_A) = \frac{\|\chi_A\|_X}{\mu(A)} \cdot \chi_A.$$

If $\mu(\Omega)$ *is finite, then* $\mathbf{1}$ *is a smooth point of* X *and* $J(\mathbf{1}) = (\|\mathbf{1}\|_X/\mu(\Omega)) \cdot \mathbf{1}$.

(b) *Each positive contractive projection on* X *such that* $\|Pf\|_\infty \leq \|f\|_\infty$ *for all* $f \in X$ *is a conditional expectation operator.*

First developments in the study of Conjecture 5.26 without any additional assumptions on a contractive projection are due to Bru and Heinich, [30, 31] and Bru, Heinich and Lootgieter [32]. We will outline here the results in [31] which contain and expand on earlier work [30, 32]. The authors start from generalizing the crucial Lemma 4.14, which was used in the proof of Bernau and Lacey in L_p. They prove:

Proposition 5.28 [31, Proposition 8] (cf. [30, Proposition 7]. *Let* X *be an order continuous Köthe function space such that conditional expectation operators are contractive on* X. *Assume that the norm of* X *is twice differentiable at* $\mathbf{1}$ *and* $\|\mathbf{1}\|_X = 1$. *If* $X \subset X^*$, *then there exists a constant* $k \geq 0$ *such that for all* $f \in X$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(J(\mathbf{1} + \varepsilon f) - \mathbf{1}) = k(f - (\int_I f(t)dt) \cdot \mathbf{1}),$$

where the limit is taken in norm of X^*, *and recall that* $J : X \to X^*$ *denotes the duality map* (*see Definition* 2.7).

The statement of Proposition 5.28 captures the most essential element of the proof of Lemma 4.14, but its proof is much less technical; it makes an elegant use of differentiability of $\|\cdot\|_X$ and $J(\cdot)$ at $\mathbf{1}$.

To generalize the exact statement of Lemma 4.14, Bru and Heinich introduce the following definition:

Definition 5.29. Let X be an order continuous, smooth Köthe function space with $\mathbf{1} \in X$ and $\|\mathbf{1}\|_X = 1$. Then X is called *D-concave* (Bru and Heinich use D to denote the duality map which in this survey is denoted by J, following most of English-language literature; so it would be natural for us to use "J-concave") if

(i) $X^* \subset X$, the inclusion map is continuous and

$$\lim_{c \to \infty} \sup\{\|f \cdot \chi_{\{|f|>c\}}\| : f \in X^*, \|f\|_{X^*} = 1\} = 0$$

(i.e., the unit ball of X^* is X-equiintegrable),

(ii) $J(J(f)) = f \iff f = \text{sgn}(f)/\|\text{sgn}(f)\|_{X^*}$.

X is called *D-convex* if X^* is D-concave.

Notice that $L_p[0,1],\ 1 < p < 2$, are D-concave and $L_p[0,1],\ 2 < p < \infty$, are D-convex.

Now we are ready to state the generalization of Lemma 4.14.

Proposition 5.30 (cf. [31, Theorem 11]). *Let X be a D-concave Köthe function space, and let P be a contractive projection on X with $P(\mathbf{1}) = \mathbf{1}$. Then if $f \in \mathcal{R}(P)$ then* $\text{sgn}(f) \in \mathcal{R}(P)$.

Then using the analogue of Lemma 4.15, Bru and Heinich obtain the generalization of Theorem 4.3 for constant-preserving contractive projections:

Theorem 5.31 [31, Theorem 13 and its Corollary]. *Suppose that X is a D-concave Köthe function space such that the norm of X^* is twice differentiable at $\mathbf{1}$, or X is a D-convex Köthe function space such that the norm of X is twice differentiable at $\mathbf{1}$. Let P be a contractive projection on X with $P(\mathbf{1}) = \mathbf{1}$. Then P is a conditional expectation operator.*

As a corollary, they obtain a characterization of constant preserving contractive projections on special Orlicz spaces:

Theorem 5.32 [31, Proposition 28]. *Let φ be an Orlicz function which is twice differentiable on $\mathbb{R}_+$ and such that φ'' is either strictly increasing to infinity or φ'' is strictly decreasing to* 0. *Then every contractive projection P with $P(\mathbf{1}) = \mathbf{1}$ on the Orlicz function space with Luxemburg norm L_φ, or with Orlicz norm $L_{\varphi,O}$ is a conditional expectation operator.*

They also obtain the same conclusion under the assumption that the Orlicz function φ has a continuous strictly increasing derivative φ' so that φ' is of the concave type (i.e. there exist constants $\gamma,\ t_0 > 0$ so that for all $\lambda,\ 0 < \lambda \leq 1$, and all $t \geq t_0$, $\varphi'(\lambda x)/(\lambda x) \geq \gamma(\varphi'(x)/x)$; for all $t,\ 0 < t < 1$, $\varphi'(t) > t$; for all $t > 1$, $\varphi'(t) < t$; φ' is differentiable at $t = 1$ and $\lim\limits_{t\to\infty} \varphi'(t) = \infty$) or the inverse $(\varphi')^{-1}$ is of the concave type [30, Theorem 4], [32, Application].

One might say that the restrictions on Köthe function space X in Theorem 5.31 and on the Orlicz function in Theorem 5.32 are somewhat severe; however, these results are the most general results known about the form of contractive projection P with $P(\mathbf{1}) = \mathbf{1}$. Nothing, outside of L_p, is known about contractive projections which do not satisfy $P(\mathbf{1}) = \mathbf{1}$. In particular, it is not known which functions in X can be the image of $\mathbf{1}$.

Next development in the study of contractive projections in terms of conditional expectation operators is due to P. Dodds, Huijmans and de Pagter [57], who obtained very general results under the assumption that the contractive projection P is positive or that the range of P is a sublattice. They extended Theorem 4.6 to very general lattices, but under the restriction that P is positive.

Theorem 5.33 [57, Proposition 4.7]. *Let X be a Köthe function space on a finite measure space (Ω, Σ, μ) and let $P : X \to X$ be a linear map. Then the following statements are equivalent*:

(a) *There exists a σ-algebra $\Sigma_0 \subset \Sigma$ such that P is the conditional expectation operator $\mathcal{E}^{\Sigma_0}$.*

(b) *P is a positive order continuous projection with $P(\mathbf{1}) = \mathbf{1}$ and $P'(\mathbf{1}) = \mathbf{1}$ (here $P' : X' \to X'$ denotes the dual mapping to P defined on the Köthe dual X').*

Corollary 5.34 [57, Corollary 4.9]. *Let X be a Köthe function space such that the norm on X is smooth at $\mathbf{1}$ and $\|\mathbf{1}\|_X \cdot \|\mathbf{1}\|_{X'} = \mu(\Omega)$. If P is a positive order continuous contractive projection with $P(\mathbf{1}) = \mathbf{1}$, then there exists a σ-subalgebra $\Sigma_0 \subset \Sigma$ such that $P = \mathcal{E}^{\Sigma_0}$.*

This corollary is a significant extension of Theorem 5.27.

Next P. Dodds, Huijmans and de Pagter obtain characterizations of contractive projections onto a sublattice.

Theorem 5.35 [57, Corollary 4.14]. *Let X be a Köthe function space with an order continuous norm and let P be a contractive projection in X such that $\mathcal{R}(P)$ is a sublattice. If $\mathcal{R}(P)$ contains some strictly positive functions and if the norm X is smooth at all such strictly positive functions*, *then P is a weighted conditional expectation operator*, *i.e.*, *there exists a σ-subalgebra $\Sigma_0 \subset \Sigma$, $0 \leq w \in X'$ and $0 < k \in L_1(\Omega, \Sigma, \mu)$ with*

$$\mathcal{E}^{\Sigma_0}(wk) = \mathcal{E}^{\Sigma_0}(k) = \mathbf{1}$$

such that

$$Pf = k\mathcal{E}^{\Sigma_0}(wf)$$

for all $f \in X$.

Theorem 5.36. *Let X be a Köthe function space with order continuous norm such that the norm is smooth at $\mathbf{1}$ and $\|\mathbf{1}\|_X\|\mathbf{1}\|_{X'} = \mu(\Omega)$. If P is a contractive projection in X such that $\mathcal{R}(P)$ is a sublattice and $\mathbf{1} \in \mathcal{R}(P)$, then P is a* *conditional expectation operator, i.e., there exists a σ-subalgebra $\Sigma_0 \subset \Sigma$ so that $P = \mathcal{E}^{\Sigma_0}$.*

The paper [57] contains also many very interesting results about what conditions on positive operators T (which are not necessarily projections nor contractive) assure that they will be conditional expectation operators, but these results will not be summarized here. We finish the account of the work in [57] with results which relate when a contractive projection P on a Banach lattice X is positive and when the range of P is a sublattice of X.

Theorem 5.37. *Let X be a Banach lattice and P be a contractive projection on X.*

(a) [57, Lemma 4.5; 67, Theorem II.3.2(i)]. *If X is strictly monotone and P is positive, then $\mathcal{R}(P)$ is a sublattice.*

(b) [57, Remark after Lemma 4.5]. *There exists X not strictly monotone (e.g., $X = \ell^3_\infty$) and P positive with $\mathcal{R}(P)$ not a sublattice.*

(c) [57, Proposition 4.10]. *If X is smooth and $\mathcal{R}(P)$ is a sublattice, then P is positive.*

(d) [57, Example 4.11]. *There exists X nonsmooth ($\dim X = 3$, ball of X is a dodecahedron, thus X is also non-strictly monotone, but it is symmetric) and non-positive P with $\mathcal{R}(P)$ a sublattice.*

(e) [57, Proposition 4.13]. *If X is order continuous and $\mathcal{R}(P)$ is a sublattice such that $\mathcal{R}(P)$ contains some strictly positive functions from X and the norm of X is smooth at all strictly positive functions in $\mathcal{R}(P)$, then P is positive.*

(f) [57, Proposition 4.15]. *If X is order continuous and $\mathcal{R}(P)$ is a sublattice such that there exists a strictly positive function $w \in \mathcal{R}(P)$ so that the norm of X is smooth at w and $J(w)$ is strictly positive then P is positive (this is satisfied, for example, if $\mathbf{1} \in \mathcal{R}(P)$ and the norm of X is smooth at $\mathbf{1}$).*

5.i. **Nonexistence of 1-complemented subspaces of finite codimension.** We finish this section with results about nonexistence of contractive projections onto subspaces of finite codimension in a Köthe function space X, which extend Proposition 4.17. We have:

Theorem 5.38 [89, Theorem 4.3] (cf. also [127, Theorem 2]). *Suppose that X is a separable, real, order-continuous Köthe function space on (Ω, Σ, μ), where μ is nonatomic and finite. Then the hyperplane $M = f^{-1}(0), (f \in X^*)$ is 1-complemented if and only if there exists a nonnegative measurable function w with*

$\operatorname{supp} w = B = \operatorname{supp} f$, *so that for any* $x \in X$ *with* $\operatorname{supp} x \subset B$,

$$\|x\|_X = \left(\int |x|^2 w d\mu\right)^{\frac{1}{2}},$$

i.e., *there are no* 1*-complemented hyperplanes in* X *unless* X *contains a band isometric to* L_2.

Around the same time a similar result was obtained by Franchetti and Semenov for rearrangement-invariant function spaces, but without the restriction of separability:

Theorem 5.39 [76, Theorem 1]. *Let* X *be a real rearrangement-invariant function space on* (Ω, Σ, μ), *where* μ *is nonatomic and* $\mu(\Omega) = 1$. *Denote by* S *a rank-one projection*

$$Sx = \left(\int_\Omega x(s) d\mu(s)\right) \mathbf{1}.$$

Then $\|I - S\| = 1$ *if and only if* X *is isometric to* $L_2(\Omega, \Sigma, \mu)$; *i.e.*, *the hyperplane* $f^{-1}(0)$, *where* $f(x) = \int_\Omega x(s) d\mu(s)$ *for all* $x \in X$, *is* 1*-complemented in* X *if and only if* $X = L_2(\Omega, \Sigma, \mu)$.

Next the author of this survey extended Theorem 5.38 to subspaces of any finite codimension:

Theorem 5.40 [127, Theorem 4]. *Suppose* μ *is nonatomic and* X *is a real separable rearrangement-invariant space on* $[0,1]$ *not isometric to* L_2. *Then there are no* 1*-complemented subspaces of any finite codimension in* X.

The proofs of Theorems 5.38, 5.39 and 5.40 all use the classical Liapunoff Theorem (see, e.g., [139]) and facts related to Proposition 5.16. Moreover, in [89, 127] the following fact is used:

Proposition 5.41 (cf. [89, 136]). *Let* X *be a real Banach space and* P *be a projection in* X. *Then* $\|I - P\| = 1$ (*where* I *denotes the identity operator*) *if and only if for all* $x \in X$ *there exists* $x^* \in X^*$ *with* $x^* \in J(x)$ *and* $\langle x^*, Px\rangle \geq 0$. ($X$ *does not have to be smooth*, *see Definition* 2.7).

The proof of this fact uses the theory of numerical ranges [27, 28] and it relates contractive projections to accretive operators in real Banach spaces. Proposition 5.41 is very useful in characterizing contractive projections in Banach spaces with 1-unconditional bases; see Section 7.

Remark. A careful reader may have noticed that, despite our effort to have as complete a bibliography as possible, there are several papers concerning contractive projections in function spaces by M. M. Rao that have not been referenced here. The reason for this omission is that, regrettably, many of the results in those papers are not valid in the full generality as stated there, but the methods are in fact limited only to the L_p-case; see [19, 83, 107].

Part 2. Sequence Spaces

6. Lebesgue Sequence Spaces ℓ_p

In this section, we discuss the development of the study of contractive projections in the case of the Lebesgue sequence spaces ℓ_p.

6.a. **General results.** The first result about 1-complemented subspaces of ℓ_p is due to Bohnenblust who considered finite-dimensional spaces ℓ_p^n:

Theorem 6.1 [25, Theorem 3.2]. *A subspace S of an n-dimensional space ℓ_p^n is 1-complemented in ℓ_p^n if and only if S is spanned by disjointly supported vectors.*

The method of the proof of Theorem 6.1 is technically very complicated; it involved conditions of Plücker Grassmann coordinates of the subspace S (we will not present the definition here). However the proof, in addition to Theorem 6.1, gives also a characterization of 1-complemented subspaces of n-dimensional subspaces $S \subset \ell_p^n$. As a corollary Bohnenblust showed that there exist subspaces of ℓ_p^n which do not have any 1-complemented subspaces (see also Section 7).

Theorem 6.2 [25, Theorem 3.3]. *Let $1 < p < \infty$, p not an integer and let $l \in \mathbb{N}$ be such that $2(2l-3) < n$. Then there exist l-dimensional subspaces S_l of ℓ_p^n such that only S_l and subspaces of dimension one are 1-complemented in S_l.*

The case of infinite-dimensional ℓ_p is simpler than the case of general L_p-spaces; however the original proofs of Douglas and Ando do not cover it (as they work only on finite measure spaces). Subsequent generalizations by Tzafriri and Bernau, Lacey do not consider this case separately. The simple proof specifically for ℓ_p is included in [103, Theorem 2.a.4]. We quote the statement of this theorem below because it illustrates the geometric properties of 1-complemented subspaces which we will try to transfer to other sequence spaces.

Theorem 6.3 [103]. *Let $1 \leq p < \infty$, $p \neq 2$, and $F \subset \ell_p$ be a closed linear subspace of ℓ_p. Then the following conditions are equivalent*:

(1) *F is 1-complemented in ℓ_p,*

(2) *F is isometric to $\ell_p^{\dim F}$,*

(3) *there exist vectors $\{u_j\}_{j=1}^{\dim F}$ of norm one and the form*

$$u_j = \sum_{k \in S_j} \lambda_k e_k,$$

with $S_j \subseteq \mathbb{N}$, $S_j \cap S_i = \emptyset$ for $j \neq i$, and such that $F = \overline{\text{span}}\{u_j\}_{j=1}^{\dim F}$ (here $\{e_k\}_{k \in \mathbb{N}}$ denotes the usual basis of ℓ_p).

Moreover, if these conditions are satisfied then the norm-one projection $P : \ell_p \overset{onto}{\longrightarrow} F$ is given by

$$Px = \sum_{j=1}^{\dim F} u_j^*(x) u_j,$$

where $\{u_j^*\}_{j=1}^{\dim F} \subset X^*$ satisfy $\|u_j^*\| = u_j^*(u_j) = 1$ (i.e., $u_j^* = J(u_j)$).

Sketch of Proof. The proof of Theorem 6.3 (1) $\Longrightarrow$ (3) presented in [103] follows the essential steps of the proof of Bernau and Lacey for general L_p, i.e., it also rests on the key Lemma 4.14. However the finishing step to reach the final conclusion is now much simpler than in L_p.

Indeed, let $\Sigma_0 = \{\operatorname{supp} f : f \in F = \mathcal{R}(P)\} \subset \mathcal{P}(\mathbb{N})$. Then Lemma 4.14 implies that if $A, B \in \Sigma_0$ then $A \cap B \in \Sigma_0$. Thus, for each $i_0 \in \mathbb{N}$ such that i_0 belongs to the support of Pf, for some $f \in \ell_p$, there is a set $A_{i_0} \in \Sigma_0$ which is minimal in Σ_0 and such that $i_0 \in A_{i_0}$. Let $A_{i_0} = \operatorname{supp} f_{i_0}$, where $f_{i_0} \in \mathcal{R}(P)$, and consider a subspace of $F = \mathcal{R}(P)$ consisting of all functions g so that $\operatorname{supp} g \subset A_{i_0} = \operatorname{supp} f_{i_0}$. We claim that this subspace of $\mathcal{R}(P)$ is one-dimensional. Indeed, if it was not one dimensional then there would exist $g \in F = \mathcal{R}(P)$ linearly independent with f_{i_0} and such that $\operatorname{supp} g \subset A_{i_0}$. For any $i \in A_{i_0} \setminus \{i_0\}$, we now can find a linear combination of g and f_{i_0} so that $(ag + bf_{i_0})(i) = 0$. Since $ag + bf_{i_0} \in F = \mathcal{R}(P)$, we get $\operatorname{supp}(ag + bf_{i_0}) \in F = \mathcal{R}(P)$ and $\operatorname{supp}(ag + bf_{i_0}) \subseteq A_{i_0}$, which contradicts the minimality of A_{i_0}. Now let $u_{i_0} \in F$ be the unique vector with $\operatorname{supp} u_{i_0} = A_{i_0}$ and $\|u_{i_0}\|_p = 1$. It is easy to see that (3) holds.

The final statement of Theorem 6.3 about the form of the projection P follows from the uniqueness of this projection. ■

The precursor of Theorem 6.3 was proved in 1960 by Pełczyński [118], who showed that in ℓ_p $(1 < p < \infty)$ the subspaces that are isometric to ℓ_p are 1-complemented in ℓ_p.

Theorem 6.3 explicitly relates the one-complementability of the subspace F with the property that F is spanned by disjointly supported vectors. In the next section we will analyze such a relation in other sequence spaces.

6.b. **1-complemented subspaces of finite codimension.** Contractive projections in ℓ_p were also investigated from the approximation theory point of view, as part of the study of minimal projections.

Definition 6.4. A projection $P : X \overset{\text{onto}}{\longrightarrow} Y$ is called *minimal* if $\|P\| = \inf\{\|Q\| : Q : X \overset{\text{onto}}{\longrightarrow} Y,\, Q \text{ projection}\}$.

Results on minimal projections appeared in the 1930s, in connection with geometry of Banach spaces, and they have many applications in numerical analysis and approximation theory; see the survey [50] for the early results, and [113] for the book length presentation of more modern developments. Here we will just give a brief account of results on contractive projections (which clearly are always minimal). We start with a result of Blatter and Cheney, who studied minimal projections onto hyperplanes (i.e., subspaces of codimension 1) in ℓ_1 and c_0 [24]. In particular, they proved:

Theorem 6.5 [24, Theorem 3]. *Let $0 \neq f \in \ell_\infty$. The hyperplane $Y = f^{-1}(0) \subset \ell_1$ is a range of a norm-one projection in ℓ_1 if and only if at most two coordinates of f are different from 0. The norm-one projection onto Y is unique if and only if exactly two coordinates of f are different from 0.*

Precisely the same characterization is valid for all $p,\ 1 \leq p < \infty,\ p \neq 2$, as shown by Beauzamy and Maurey [15]; see Theorem 4.18; exept when $p > 1$, all norm-one projections are unique, see Theorem 5.5.

These results have been generalized by Baronti and Papini to subspaces of arbitrary finite codimension and to arbitrary $p, 1 \leq p < \infty$. They proved:

Theorem 6.6 [10, Theorem 3.4; 12, Theorem 5.5]. *Let Y be a subspace of ℓ_p $(1 \leq p < \infty,\ p \neq 2)$ of finite codimension* $\operatorname{codim} Y = n \in \mathbb{N}$. *Then Y is 1-complemented in ℓ_p if and only if Y is the intersection of n 1-complemented hyperplanes, i.e., if and only if there exist functionals $f_1, \cdots, f_n \in (\ell_p)^*$ such that for each $j \leq n$ at most two coordinates of f_j are different from 0 and $Y = \bigcap_{j=1}^{n} f_j^{-1}(0)$.*

The proof of this theorem depends on Theorem 5.6 and is slightly simpler than the proof of Theorem 6.3 since it is restricted to subspaces of finite codimension. Notice that the descriptions given in Theorems 6.3 and 6.6 are equivalent. This is an intuitively straightforward fact but since we have not seen it in the literature we present the full proof below. Unfortunately, the proof is somewhat techinical. We have:

Proposition 6.7. *Let X be a Banach space with basis $\{e_n\}_{n\in\mathbb{N}}$ and Y be a subspace of X with* codim $= n$. *Then the following conditions are equivalent*:

(i) *there exist vectors $\{u_j\}_{j\in\mathbb{N}}$ of the form $u_j = \sum_{k\in S_j} u_{jk}e_k$ with $S_j \subseteq \mathbb{N}$, $S_j \cap S_i = \emptyset$ for $j \neq i$ and such that $Y = \overline{\text{span}}\{u_j\}_{j\in\mathbb{N}}$.*

(ii) *there exist functionals $f_1, \cdots, f_n \in X^*$ such that*

$$f_j = \sum_{k\in F_j} f_{jk}e_k^*$$

with $\text{card}(F_j) \leq 2$ *for each $j = 1, \cdots, n$ and such that $Y = \bigcap_{j=1}^{n} f_j^{-1}(0)$.*

Proof. (i) $\Longrightarrow$ (ii): It is easy to see that since $S_j's$ are mutually disjoint,

$$\text{codim}Y = \sum_{j=1}^{\infty}(\text{card}(S_j) - 1).$$

Since codim$Y = n$, all vectors $\{u_j\}_{j\in\mathbb{N}}$, except at most n of them, have a singleton support.

After reordering of $\{u_j\}_{j\in\mathbb{N}}$ if necessary, let $m \leq n$ be such that $\text{card}(S_j) \geq 2$ for $j \leq m$ and $\text{card}(S_j) = 1$ for $j > m$.

For each $j \leq m$, select $\varphi(j) \in S_j$, and for each $k \in S_j\backslash\{\varphi(j)\}$, set

$$f_k = u_{jk}e_{\varphi(j)}^* - u_{j\varphi(j)}e_k^*.$$

Then

$$\bigcap_{k\in S_j\backslash\{\varphi(j)\}} f_k^{-1}(0) = \overline{\text{span}}(\{u_j\} \cup \{e_i\}_{i\notin S_j}). \tag{2}$$

Indeed, for each $k \in S_j\backslash\{\varphi(j)\}$,

$$f_k^{-1}(0) = \overline{\text{span}}(\{e_i\}_{i\neq k,\varphi(j)} \cup \{u_{j\varphi(j)}e_{\varphi(j)} + u_{jk}e_k\}),$$

so $u_j \in f_k^{-1}(0)$ and $\{e_i\}_{i\notin S_j} \subset f_k^{-1}(0)$ for each $k \in S_j\backslash\{\varphi(j)\}$. By the equality of codimensions, we obtain (2).

Further,

$$\begin{aligned} Y &= \overline{\text{span}}\{u_j\}_{j\in\mathbb{N}} \\ &= \overline{\text{span}}(\{u_j\}_{j=1}^m \cup \{e_i : i \notin \bigcup_{j=1}^m S_j\}) \\ &= \bigcap_{j=1}^m \overline{\text{span}}(\{u_j\} \cup \{e_i : i \notin S_j\}) \\ &= \bigcap_{j=1}^m \bigcap_{k\in S_j\backslash\{\varphi(j)\}} f_k^{-1}(0) \end{aligned}$$

and (ii) is proven.

(ii) $\Longrightarrow$ (i) : Suppose that there exist $k > 1$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\mathrm{card}(\bigcup_{\nu=1}^{k} F_{i_\nu}) \leq k$. Since functionals $\{f_{i_\nu}\}_{\nu=1}^{k}$ are linearly independent, the matrix of their coefficients $(f_{i_\nu,j})_{\nu=1, j\in\cup_{\nu=1}^{k} F_{i_\nu}}^{k}$ has rank k. Thus $\mathrm{card}(\bigcup_{\nu=1}^{k} F_{i_\nu}) \geq k$.

If $\mathrm{card}(\bigcup_{\nu=1}^{k} F_{i_\nu}) = k$, then, again by the linear independence of the functionals $\{f_{i_\nu}\}_{\nu=1}^{k}$, there exists an isomorphism which transforms the $k \times k$ matrix of the coefficients of $\{f_{i_\nu}\}_{\nu=1}^{k}$ into an identity matrix (of size $k \times k$), that is, there exist functionals $\{g_{i_\nu}\}_{\nu=1}^{k}$ so that for all $\nu = 1, \cdots, k$ we have $\mathrm{card}(\mathrm{supp}\, g_{i_\nu}) = 1$ and

$$\bigcap_{\nu=1}^{k} g_{i_\nu}^{-1}(0) = \bigcap_{\nu=1}^{k} f_{i_\nu}^{-1}(0).$$

Thus without reducing the generality, we will assume that sets $\{F_j\}_{j=1}^{n}$ satisfy the following property:

(3) For each subset $S \subset \{1, \cdots, n\}$ with $\mathrm{card}(S) > 1$, $\mathrm{card}(\bigcup_{i\in S} F_i) > \mathrm{card}(S)$.

After reordering of $\{F_j\}_{j=1}^{n}$, if necessary, let m, $0 \leq m \leq n$, be such that

$$\mathrm{card}(F_j) = 2 \text{ for all } j \leq m,$$
$$\mathrm{card}(F_j) = 1 \text{ for all } j > m.$$

Then $(\bigcup_{j=1}^{m} F_j) \cap (\bigcup_{j=m+1}^{n} F_j) = \emptyset$ and

$$\bigcap_{j=m+1}^{n} f_j^{-1}(0) = \overline{\mathrm{span}}\{e_k : k \notin \bigcup_{j=m+1}^{n} F_j\}. \tag{4}$$

We introduce a relation $\sim$ on a set of indices $\{1, \cdots, m\}$ as follows: $i \sim j$ if there exists $t \leq m$ and $k_1, \cdots, k_t \in \{1, \cdots, m\}$ such that $i = k_1$, $j = k_t$ and $F_{k_s} \cap F_{k_{s+1}} \neq \emptyset$ for all $1 \leq s < t$.

Clearly, $\sim$ is an equivalence relation. Let $S \subset \{1, 2, \cdots, m\}$ be a class of equivalence of $\sim$, and set $A_S = \bigcup_{j\in S} F_j$.

Notice that

$$\mathrm{card}(A_S) = \mathrm{card}(S) + 1. \tag{5}$$

Indeed, since S is a class of equivalence of $\sim$, there exists a bijection $\sigma : \{1, \cdots, \mathrm{card} S\} \to S$ such that $F_{\sigma(i)} \cap F_{\sigma(i+1)} \neq \emptyset$ for each $i < \mathrm{card}(S)$. Thus for $k < \mathrm{card}(S)$, we have

$$\begin{aligned} \mathrm{card}(\bigcup_{i=1}^{k+1} F_{\sigma(i)}) &\leq \mathrm{card}(\bigcup_{i=1}^{k} F_{\sigma(i)}) + 1 \\ &\leq \mathrm{card}(F_{\sigma(1)}) + k = 2 + k. \end{aligned}$$

Thus

$$\mathrm{card}(A_S) = \mathrm{card}(\bigcup_{i=1}^{\mathrm{card}(S)} F_{\sigma(i)}) \leq \mathrm{card}(S) + 1.$$

On the other hand, by (3), if $\mathrm{card}(S) > 1$, then

$$\mathrm{card}(A_S) > \mathrm{card}(S).$$

Thus $\mathrm{card}(A_S) = \mathrm{card}(S) + 1$ if $\mathrm{card}(S) > 1$.

If $\mathrm{card}(S) = 1$, then $S = \{s\}$ for some $s \in \{1, \ldots, m\}$ and $A_S = F_s$, so $\mathrm{card}(A_S) = \mathrm{card}(F_s) = 2$, and (5) holds.

Now we are ready to show that for each class of abstraction of $\sim$, S, there exists a vector $u_S \in X$ with $\mathrm{supp}\, u_S \subset A_S$ such that

$$\bigcap_{j \in S} f_j^{-1}(0) = \overline{\mathrm{span}}(u_S \cup \{e_k : k \notin A_S\}). \tag{6}$$

To see this, notice that, clearly, if $k \notin A_S$ then $e_k \in \bigcap_{j \in S} f_j^{-1}(0)$. Next let us consider u with $\mathrm{supp}\,(u) \subset A_S$, say,

$$u = \sum_{i \in A_S} u_i e_i.$$

Then $u \in \bigcap_{j \in S} f_j^{-1}(0)$ if and only if $\{u_i\}_{i \in A_S}$ is a solution of a system of linear homogenous equations ($f_j(u) = 0$, $j \in S$), where the number of equations is $\mathrm{card}(S)$ and the number of variables is $\mathrm{card}(A_S)$. But by (5), $\mathrm{card}(A_S) = \mathrm{card}(S) + 1$ and the functionals $\{f_j\}_{j \in S}$ are linearly independent, so this system has exactly one solution, which we will denote by u_S and (6) holds. Combining (6) with (4), we obtain

$$\begin{aligned} \bigcap_{j=1}^{n} f_j^{-1}(0) = \overline{\mathrm{span}}\{\{e_k : k \notin \bigcup_{j=1}^{n} F_j\} \cup \{u_S : S \subset \{1, \cdots, m\}, \\ S \text{ class of abstraction of } \sim\}, \end{aligned}$$

so (i) is proven. ∎

6.c. **Characterizations of ℓ_p through 1-complemented subspaces.** We finish this section with results that generalize Ando's characterization of L_p-spaces among Banach lattices (Theorem 4.11) which were obtained by Calvert and Fitzpatrick in a series of papers [41, 42, 43, 44, 40, 45, 72, 73, 74]. The first result shows that Theorem 6.6 characterizes ℓ_p and c_0:

Theorem 6.8 [41, Theorem 1]. *Let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis for a Banach lattice X with $e_i \wedge e_j = 0$ if $i \neq j$. Suppose each hyperplane F which contains all but two of the basis vectors (i.e., $F = f^{-1}(0)$ for some functional f with at most two coordinates different from 0) and which is a sublattice is the range of a contractive projection. Then $X = \ell_p$ $(1 \leq p < \infty)$ or $X = c_0$.*

Next they obtained a generalization of Ando's theorem about 1-complementability of two-dimensional sublattices [5], [94, Theorem 16.4]:

Theorem 6.9 [42, Theorem 4.2] (cf. also [43, Theorem 2]). *Let X be a Banach lattice and $A = \{e_i\}_{i\in I}$ be a set of elements of X with $e_i \wedge e_j = 0$ for $i \neq j$ and such that* span$\{e_i\}_{i\in I}$ *is dense in X. Let $i_0 \in I$ be any fixed element of I. Suppose that any two-dimensional sublattice of X which contains some e_i, $i \in I\backslash\{i_0\}$, is the range of a contractive projection. Then X is linearly isometric and lattice isomorphic to $\ell_p(I)$ $(p \in [1, \infty])$ or $c_0(I)$.*

Calvert and Fitzpatrick also show that in the assumptions of the above theorem one cannot exclude two elements of the index set I, i.e., there exists a Banach lattice $X \neq \ell_p(I)$, $c_0(I)$ and i_0, $i_1 \in I$, $i_0 \neq i_1$, such that every 2-dimensional sublattice of X containing some e_i, $i \in I\backslash\{i_0, i_1\}$, is 1-complemented [42, Example 4.].

Further, Calvert and Fitzpatrick obtained a result analogous to Theorem 6.9 where the assumption about X being a lattice is replaced by the assumption of the existence of enough smooth points in X.

Theorem 6.10 [44, Theorem C]. *Let X be a real Banach space with $\dim(X) \geq 3$. Let $\{e_i\}_{i\in I}$ be a linearly independent set of smooth points in X with $\overline{\text{span}}\{e_i\}_{i\in I} = X$. Suppose that every two-dimensional subspace of X intersecting $\{e_i\}_{i\in I}$ is the range of a nonexpansive projection. Then X is isometrically isomorphic to $\ell_p(I)$, $1 \leq p \leq \infty$ or $c_0(I)$.*

Finally in [45], Calvert and Fitzpatrick studied extensions of Theorems 6.9 and 6.10 for spaces which are not lattices and without the assumption of the existence of a linearly dense set of smooth points in X (like in Theorem 6.10). They showed that there exist 3-dimensional real Banach spaces X, $X \neq \ell_p(3)$, which have two linearly independent elements e_1, e_2 such that every 2-dimensional subspace of X containing either e_1 or e_2 is 1-complemented in X. Moreover, they gave a complete characterization of 3-dimensional real Banach spaces X with the above property. In particular, as a corollary of [45] one obtains the following:

Corollary 6.11. *Let X be a real symmetric Banach space with $\dim(X) = 3$ such that X contains two linearly independent elements e_1, e_2 such that every 2-dimensional subspace of X containing either e_1 or e_2 is 1-complemented in X. Then X is isometrically isomorphic to $\ell_p(3)$, $1 \leq p \leq \infty$.*

We do not know whether this corollary can be extended for symmetric spaces X with $\dim X > 3$.

Remark. It seems that the property like in Theorem 6.3(2), i.e.,

$$\begin{array}{l} an\ infinite\text{-}dimensional\ subspace\ Y \subset X\ is\ 1\text{-}complemented \\ in\ X\ if\ and\ only\ if\ Y\ is\ isometrically\ isomorphic\ to\ X, \end{array} \tag{7}$$

does not characterize ℓ_p. Indeed, Gowers and Maurey [79] constructed spaces which have very few complemented subspaces, and we believe that these spaces can be slightly modified so that they would satisfy property (7) vacuously (cf. also Sections 7.a and 5.g).

6.d. **Bibliographical remarks.** Other papers containing results about contractive projections in ℓ_p include [91, 11, 9]. Contractive projections in ℓ_p for $0 < p < 1$ were studied in [123, 124].

7. Sequence Banach Spaces

7.a. **General nonexistence results.** The situation about the existence of contractive projections in Banach spaces with bases is somewhat ambivalent. That is, one may say that there is an abundance of contractive projections since every infinite-dimensional Banach space with a basis has a conditional basis, which can be renormed so that the projections associated with this conditional basis are all contractive [143, p. 250]. Moreover, Lindenstrauss proved the following property of nonseparable reflexive Banach spaces illustrating the richness of 1-complemented subspaces:

Theorem 7.1 [99]. *Let X be any reflexive Banach space. For any separable subspace $Y \subset X$ there exists a separable subspace $Z \subset X$ such that $Y \subset Z$ and Z is 1-complemented in X.*

On the other hand, we mentioned above a striking example of Lindenstrauss illustrating the claim that norm-one projections are very rare; see Theorem 5.4. Also, Bohnenblust showed that there are subspaces S of ℓ_p^n which do not have any nontrivial 1-complemented subspaces (Theorem 6.2 above). Moreover, Bosznay and Garay [29] showed:

Theorem 7.2. *Let X be a finite-dimensional real Banach space. Then for any $\varepsilon > 0$, there exists a norm $|||\cdot|||$ on X such that*

$$(1-\varepsilon)\|x\|_X \leq |||x||| \leq (1+\varepsilon)\|x\|_X$$

and such that $(X, |||\cdot|||)$ does not have any nontrivial 1-complemented subspaces.

Here, we say that a 1-complemented subspace F of X is nontrivial if $F \subsetneq X$ and $\dim F > 1$ (clearly, by the Hahn-Banach theorem all 1-dimensional subspaces are 1-complemented in all Banach spaces).

Thus, in the case of finite dimensional real spaces, the property of having 1-complemented subspaces is highly unstable. It seems clear that there should exist infinite-dimensional Banach spaces with no nontrivial 1-complemented subspaces (we believe that an appropriate modification of a hereditarily indecomposable space constructed in [79] would do the job). However the following questions are open:

Question 7.3. *Let $(X, \|\cdot\|)$ be a Banach space. Does there exist an equivalent renorming $(X, |||\cdot|||)$ of X so that $(X, |||\cdot|||)$ has no nontrivial 1-complemented subspaces?*

Theorem 7.2 says that when X is finite-dimensional then the answer to this question is positive in the very strong sense that one may require the new norm $|||\cdot|||$ to be arbitrarily close to the original norm $\|\cdot\|$.

Question 7.4. *Let X be a Banach space. Does there exist a subspace $Y \subset X$ so that Y does not have nontrivial 1-complemented subspaces?*

As mentioned above, by a result of Bohnenblust, Question 7.4 has a positive answer if $X = \ell_p^n$, $p \in (1, \infty) \setminus \mathbb{Z}$, $n \geq 7$, and therefore also if $X = L_p$ or $X = \ell_p$. We suspect that the answer is positive for all Banach spaces which are not isometric to a Hilbert space.

7.b. **More negative results concerning the inheritance of the isomorphic structure.** Here we will concentrate on the search of characterizations of 1-complemented subspaces and norm one-projections in classical types of spaces (with their usual norms) in the spirit of Theorem 6.3. However we have to start from the following two negative results:

Theorem 7.5 [119]. *Let X be any finite-dimensional Banach space* $(\dim X = n)$. *Then*

(a) *there exists a Banach space Y with $\dim Y = n^2$ such that Y has a basis with basis constant $\leq 1 + n^{-1/2}$ and X is 1-complemented in Y.*

(b) *there exists a Banach space Y such that Y has monotone basis* (*i.e.*, *basis constant equals* 1; *cf. Definition* 2.9) *and X is 1-complemented in Y* (*here* $\dim Y$ *could be infinite*).

Theorem 7.6 [102; 103, Theorem 3.b.1]. *Every space X with a 1-unconditional basis is 1-complemented in some symmetric space X.*

These results are very negative because they show that 1-complemented subspaces do not have to inherit the isomorphic structure of the space. In particular, it follows from Theorem 7.6 that there is no hope of extending Proposition 3.2 to general symmetric spaces (Proposition 3.2 does have an analogue in ℓ_p, Theorem 6.3(2)).

Moreover, very shortly after Theorem 7.6 it was established that in fact the isomorphic structure of X is not inherited by 1-complemented subspaces even in the most natural symmetric spaces, i.e., in Orlicz spaces and Lorentz spaces (cf. Definitions 2.5 and 2.6). We have:

Theorem 7.7 [98, 2]. *There exists a class of Orlicz spaces ℓ_φ* [98] *and a class of Lorentz spaces $\ell_{w,p}$* [2] *such that if X belongs to either of these classes then X has an infinite-dimensional 1-complemented subspace Y which is not isomorphic with X.*

Thus it seems that there is no hope of giving any sort of characterization of 1-complemented subspaces in terms of isomorphisms. However the analogue of Theorem 4.2 still holds:

Theorem 7.8. *Let X be a symmetric sequence space with basis $\{e_i\}_{i\in I}$ $(I \subseteq \mathbb{N})$. Let $\{f_j\}_{j\in J}$ be a block basis with constant coefficients of any permutation of $\{e_i\}_{i\in I}$, i.e., there exists a permutation σ of I, an increasing sequence $\{p_j\} \subseteq \mathbb{N}$ and scalars $\{\theta_i\}_i$ with $|\theta_i| = 1$ for all $i \in I$ such that*

$$f_j = \sum_{k=p_j+1}^{p_{j+1}} \theta_k e_{\sigma(k)}.$$

Then $Y = \overline{\text{span}}\{f_j\}_{j\in J} \subset X$ is a 1-complemented in X and the averaging projection $P : X \overset{onto}{\longrightarrow} Y$ is a norm-one projection.

In fact, all examples presented in the proofs of Theorems 7.6 and 7.7 were of the form described in Theorem 7.8. We postulate that Theorem 7.8 is the right form of the description of 1-complemented subspaces, i.e., that Conjecture 5.26 should be valid also in Banach spaces with bases.

Conjecture 7.9. *Let X be a strictly monotone Banach space with a 1-unconditional basis (sufficiently different from a Hilbert space). Then every norm-one projection in X is a weighted conditional expectation operator and every 1-complemented subspace of X is spanned by mutually disjoint elements.*

Below we present some results supporting this conjecture (see Theorem 7.23 and Corollary 7.24), but first we return to the chronological order of discoveries.

7.c. **Some spaces whose 1-complemented subspaces do inherit the basis.** The next developments in the study of 1-complemented subspaces of sequence space dealt with isometric versions of Banach's Problems 1.1 and 1.2. In this section we will discuss the isometric analogue of Problems 1.1, i.e.:

Problem 7.10. *Does every 1-complemented subspace of a space with a monotone basis have a monotone basis*? (*cf. Definition* 2.9)

As we mentioned above, Pełczyński [119] (see Theorem 7.5(b)) showed that the answer to Problem 7.10 is negative. However, Dor [60] proved that the answer is yes if we consider only finite-dimensional spaces:

Theorem 7.11. *Let X be a finite-dimensional Banach space with a monotone basis, and let Y be 1-complemented in X. Then Y has a monotone basis.*

Chronologically, the next development related to describing bases in 1-complemented subspaces is due to Kinnunen [91] which we described in Section 5.f.

The next development concerning Problem 7.10 is due to Rosenthal [136], who proved:

Theorem 7.12. *Let Y be a reflexive Banach space which is isometric to a contractively complemented subspace of a Banach space X with reverse monotone basis. Then Y has a reverse monotone basis.*

Here, a basis (b_j) for a Banach space X is said to be *reverse monotone* if $\|I - P_j\| = 1$ for all j, where

$$P_j b = \sum_{n=1}^{j} c_n b_n \quad \text{if} \quad b = \sum_{n=1}^{\infty} c_n b_n.$$

Clearly, if X is finite-dimensional, then X has a reverse monotone basis if and only if X has a monotone basis (simply reverse the order of the monotone basis $(b_n)_{n=1}^m$ to $(b_n)_{n=m}^1$).

Thus Theorem 7.12 generalizes Theorem 7.11 and also it illustrates that in infinite dimensions the concept of a reverse monotone basis is very different from a monotone basis (cf. Theorem 7.5(b)).

In [136], Rosenthal also studied the following concept:

Definition 7.13. A Banach space X has enough contractive projections (ECP) provided every 1-complemented nonzero subspace Y of X contains a contractively complemented subspace Z of codimension one in Y.

Property (ECP) is clearly inherited by 1-complemented subspaces.

This property seems to be modeled on the definition of the reverse monotone basis $(b_n)_{n\in\mathbb{N}}$ where we require that every subspace of the form $Y_M = \overline{\text{span}}\{b_n\}_{n\geq M}$ $(M \in \mathbb{N})$ has a 1-complemented subspace $Y_{M+1} = \overline{\text{span}}\{b_n\}_{n\geq M+1}$ of codimension one in Y_M.

Rosenthal proved:

Theorem 7.14 [136, Theorem 1.8]. *Every Banach space X with reverse monotone basis has enough contractive projections (ECP).*

The proof of Theorem 7.14 is not trivial. It uses the theory of numerical ranges (see [27, 28]) and Proposition 5.41.

On the other hand, Rosenthal suggests that there may exist reflexive separable spaces with ECP but with no basis or even without finite-dimensional decomposition (FDD). This question is still open.

Rosenthal proved the following characterization of property (ECP):

Theorem 7.15 [136, Theorem 2.1]. *A reflexive Banach space X has (ECP) if and only if X has a reverse monotone transfinite basis.*

Here transfinite basis is the concept that generalizes bases by dropping the assumption of countability (it was introduced by Bessaga [21]; see also [59]). It is defined as follows:

Definition 7.16 [145, Definition 17.7]. Let $\eta > 0$ be an ordinal. A transfinite sequence of elements $(b_\alpha)_{\alpha<\eta}$ in a Banach space X is called a *transfinite basis* (of length η) of X if for every $x \in X$ there exists a unique transfinite sequence of scalars $(x_\alpha)_{\alpha<\eta}$ such that $\sum_{\alpha<\eta} x_\alpha b_\alpha$ converges to x.

For most recent developments related to further generalizations of Problem 7.10, see Section 7.f.

7.d. **Preservation of the 1-unconditional basis in the complex case.** Let us now concentrate on the isometric version of Problem 1.2, i.e.:

Problem 7.17. *Does every* 1*-complemented subspace of a space with a* 1*-unconditional basis have a* 1*-unconditional basis*?

Curiously, the situation is different depending whether we consider Banach spaces over $\mathbb{C}$ or over $\mathbb{R}$.

The first result concerns complex spaces and was proven implicitly by Kalton and Wood [90] and explicitly explained in [75, 135]. We have

Theorem 7.18. *Let X be a complex Banach space with* 1*-unconditional basis and let Y be a* 1*-complemented subspace of X. Then Y has a* 1*-unconditional basis.*

The analogous result is false in real Banach spaces [96, 16]. Thus the answer to Problem 7.17 is negative in the real case.

The proof of Theorem 7.18 is based on the theory of numerical ranges in Banach spaces (see [27, 28]). We outline here the main elements of the proof.

Definition 7.19 [27]. Let X be a Banach space and T be a linear operator on X. We say that operator T is *hermitian* if the numerical range of T is contained in $\mathbb{R}$, i.e.,

$$\overline{\mathrm{conv}}\{\langle f, Tx\rangle : x \in X, f \in J(x)\} \subseteq \mathbb{R}.$$

Definition 7.20 [90]. An element $x_0 \in X$ is called *hermitian* if there exists a hermitian rank-1 projection from X onto $\mathrm{span}\{x_0\}$, i.e., equivalently, if for all $x \in X$, $f \in J(x)$, $f_0 \in J(x_0)$, we have

$$\langle f, x_0\rangle \cdot \langle f_0, x\rangle \in \mathbb{R}.$$

The set of all hermitian elements of X is denoted $H(X)$

Let $\{H_\lambda : \lambda \in \Lambda\}$ be the collection of maximal linear subspaces of $H(X)$. Then H_λ, $\lambda \in \Lambda$, are called *Hilbert components* of X.

A Hilbert component H_λ is called *nontrivial* if $\dim H_\lambda > 1$.

Kalton and Wood showed that Hilbert components are well-defined and they obtained the following characterization of hermitian elements in X:

Theorem 7.21 [90, Theorem 6.5]. *Let X be a complex Banach space with a normalized* 1*-unconditional basis $\{e_i\}_{i\in I}$. Then $x_0 \in X$ is hermitian if and only if*

(i) $\|y\|_X = \|y\|_2$ *for all $y \in X$ with* $\mathrm{supp}\, y \subset \mathrm{supp}\, x_0$, *and*

(ii) *for all* $y, z \in X$ *with* $\operatorname{supp} y \cup \operatorname{supp} z \subset \operatorname{supp} x_0$ *and for all* $v \in X$ *with* $\operatorname{supp} v \cap \operatorname{supp} x_0 = \emptyset$, *if* $\|y\|_X = \|z\|_X$ *then* $\|y+v\|_X = \|z+v\|_X$.

In (i) above, when we say $\|y\|_2$ we mean the ℓ_2-norm of y with respect to the given 1-unconditional basis of X, i.e., if $y = \sum_{i\in I} y_i e_i$ then $\|y\|_2 = (\sum_{i\in I} |y_i|^2)^{1/2}$.

In particular, it is easy to see

Corollary 7.22. *If X is a complex Banach space with a normalized* 1*-unconditional basis* $\{e_i\}_{i\in I}$ *and* x_0 *is hermitian in* X, *then*

$$S_{x_0} = \overline{\operatorname{span}}\{e_i : i \in \operatorname{supp} x_0\} \subset X$$

is a 1*-complemented subspace of* X *and* S_{x_0} *is isometric to a Hilbert space.*

Clearly, every element $\{e_i\}_{i\in I}$ of the 1-unconditional basis is hermitian, and Kalton and Wood proved that if P is a projection of norm one then also the elements $\{Pe_i\}_{i\in I}$ are hermitian in $\mathcal{R}(P)$ (see [75, Lemma 3]). This fact, together with Theorem 7.21, leads to Theorem 7.18.

A refinement of this technique allowed the author of this survey to obtain a strengthening of Theorem 7.18. Namely, we get a geometric description of the 1-unconditional basis of the 1-complemented subspace:

Theorem 7.23 [128, Corollary 3.2]. *Suppose that* X *is a complex strictly monotone Banach space with a* 1*-unconditional basis* $\{e_i\}_i$ *and let* P *be a projection of norm one in* X. *Suppose that* $Y = P(X) \subset X$ *has no nontrivial Hilbert components. Then there exist disjointly supported elements* $\{y_i\}_{i=1}^m \subset Y$ $(m = \dim Y \leq \infty)$ *which span* Y. *Moreover, for all* $x \in X$,

$$Px = \sum_{i=1}^m y_i^*(x) y_i,$$

where $\{y_i^*\}_{i=1}^m \subset X^*$ *satisfy* $\|y_i^*\| = \|y_i\|_X = y_i^*(y_i) = 1$ *for all* $i = 1, \dots, m$.

The statement in Theorem 7.23 exactly parallels and extends the characterization of 1-complemented subspaces of ℓ_p given in Theorem 6.3(3).

Remark. The assumption of X being strictly monotone cannot be omitted (Blatter and Cheney [24] showed examples of 1-complemented hyperplanes in ℓ_∞^3 that are not spanned by disjointly supported vectors). Also the assumption that Y does not have nontrivial Hilbert components cannot be dropped (see examples in [128]); the property of not having nontrivial Hilbert components is not necessarily inherited by 1-complemented subspaces.

As noted above (Theorem 7.7) one cannot hope to give any sort of isomorphic description of 1-comp- lemented subspaces of X and we believe that the geometric characterization of the position of Y in X as the span of a family of disjointly supported vectors is best possible. In fact, the examples of 1-complemented subspaces Y nonisomorphic to the whole space X which were described in [2, 98] all satisfied the conclusion of Theorem 7.23 (in both complex and real cases); even more, they in fact were spanned by disjoint elements with finite supports and constant coefficients, i.e., were of the form

$$Y = \overline{\text{span}}\{v_j\}_{j\in\mathbb{N}}$$

where $v_j = \sum_{\nu\in S_j} e_\nu,\ S_j \cap S_k = \emptyset$ whenever $j \neq k$. So Y satisfies the conditions of Theorem 5.25 and the norm-one projection onto Y is simply a conditional expectation operator.

Thus it appears that it is not that the class of contractive projections is richer in spaces other than ℓ_p, but rather that the isomorphic structure of subspaces of X is much more varied. In fact, Theorem 7.23 can be rephrased as follows:

Corollary 7.24. *Suppose that X is a complex Banach space with 1-unconditional basis and X does not contain a 1-complemented copy of a 2-dimensional Hilbert space ℓ_2^2. Then every norm one projection in X is a weighted conditional expectation operator.*

Proof. Theorem 7.23 and Corollary 7.22. ■

Remark. Recall that Calvert and Fitzpatrick showed that if all weighted conditional expectation operators are contractive projections in X, then X has to be isometric to ℓ_p or c_0 (see Section 6).

7.e. **The real case.** One can immediately see that when X is a real Banach space then all operators and all elements are hermitian, so above methods cannot be applied. And, in fact, Theorem 7.18 fails in the real case.

As mentioned above, Lewis [96] and Benyamini, Flinn, Lewis [16] showed examples of real Banach spaces without 1-unconditional bases which are 1-complemented in a real space with a 1-unconditional basis:

Proposition 7.25 [16, Propositions 1 and 2]. *The space $E_n = \{(x_i)_{i=1}^n \in \ell_\infty^n : \sum_{i=1}^n x_i = 0\}$ is 1-complemented in a space with 1-unconditional basis and if $n \geq 5$, then E_n does not admit a 1-unconditional basis.*

The proof of Proposition 7.25 is based on the following interesting observation:

Proposition 7.26 (stated as "well-known" in [16]). *A real Banach space X embeds as a norm-one complemented subspace of some Banach space Z with 1-unconditional basis if and only if the identity I on X can be written as*

$$I = \sum_i f_i \otimes x_i$$

for some $x_i \in X, f_i \in X^*$ with

$$\| \sum_i \varepsilon_i f_i \otimes x_i \| \leq 1$$

for all $\varepsilon_i = \pm 1$.

In Proposition 7.25, spaces E_n are not uniformly convex, but uniformly convex examples are also possible.

It is easy to see that spaces E_n do have a 2-unconditional basis. The following version of Problem 1.2 is open:

Problem 7.27. *Let X be a real Banach space with a 1-unconditional basis and let Y be 1-complemented in X. Does Y have to have an unconditional basis?*

After the negative examples of Lewis and Benyamini, Flinn, Lewis, there were few attempts to characterize 1-complemented subspaces of special real Banach spaces. The first development is due to Rosenthal [137] who considered spaces which are isometric to the direct sum of Hilbert spaces of dimension at least two via a one-unconditional basis according to the following definition:

Definition 7.28 [137]. Let Γ be a nonempty set and $(X_\alpha)_{\alpha \in \Gamma}$ be a family of nonzero Banach spaces. A Banach space X is said to be a *functional unconditional sum of the X_α's* if there exists a normalized 1-unconditional basis $\underline{b} = (b_\alpha)_{\alpha \in \Gamma}$ for some Banach space B so that X is (linearly isometric to) $(\sum_\Gamma \oplus X_\alpha)_{\underline{b}}$, i.e.,

$$X = \{x = (x_\alpha)_{\alpha \in \Gamma} \in \prod_{\alpha \in \Gamma} X_\alpha : \sum_{\alpha \in \Gamma} \|x_\alpha\|_{X_\alpha} b_\alpha \in B\}$$

and we define the norm on X by:

$$\|x\|_X = \| \sum_{\alpha \in \Gamma} \|x_\alpha b_\alpha\|_B.$$

In case X is real and each X_α is a real Hilbert space of dimension at least two, we call X a *Functional Hilbertian Sum* (FHS).

It is clear that each complex Banach space X with a 1-unconditional basis $(e_i)_{i\in I}$ is isometrically isomorphic over reals to $X_{\mathbb{R}}(\ell_2^2)$, $X_{\mathbb{R}} = \{\sum_{i\in I} a_i e_i \in X$: where $(a_i)_{i\in I} \subset \mathbb{R}\}$ i.e., X can be considered over reals as an FHS, where $B = X_{\mathbb{R}}$, $\underline{b} = (b_i)_{i\in I} := (e_i)_{i\in I}$, and each $(X_i)_{i\in I}$ is a two-dimensional Hilbert space ℓ_2^2. This similarity to the complex case allows to transfer some of the techniques related to hermitian operators and hermitian elements. We will use the following notion:

Definition 7.29. Let B be a complex or real Banach space. We say that operator $T : B \longrightarrow B$ is *skew-Hermitian* if $\mathrm{Re} f(Tb) = 0$ for all $b \in B$ and $f \in J(b) \subset B^*$.

Using skew-Hermitian operators, Rosenthal gave a sufficient condition for a 1-complemented subspace of a real Banach space with 1-unconditional basis to have a 1-unconditional basis:

Theorem 7.30 [137, Theorem 3.15]. *Let X be a real Banach space with a 1-unconditional basis and Y be a 1-complemented subspace of X. Suppose for all $y \in Y$, there is a skew-Hermitian operator T on X such that $Ty \in Y$ and $T^2 y = -y$. Then Y is FHS, so Y has a 1-unconditional basis.*

As a corollary, Rosenthal obtains Theorem 7.18. The method of Rosenthal is very similar in spirit to the method of Kalton and Wood [90], but Rosenthal's analysis of FHS spaces also involves deep considerations of Lie algebras of Banach space X (the Lie algebra of X consists of skew-Hermitian operators on X).

Next, Rosenthal considers orthogonal projections on X which are defined as follows:

Definition 7.31. Let X be a real or complex Banach space, and Y, Z be subspaces of X. Z is said to be an *orthogonal complement of* Y if $Y + Z = X$ and if for all $y \in Y, z \in Z$ and all scalars α, β with $\|\alpha\| = \|\beta\| = 1$, $\|y+z\| = \|\alpha y+\beta z\|$.

The projection P with range Y and kernel Z is called *the orthogonal projection onto Y*. Note that $\|P\| = \|I - P\| = 1$. We say that Y is *orthogonally complemented in X* if Y has an orthogonal complement.

It is easy to see that any space with 1-unconditional basis is orthogonally complemented in some FHS space. Rosenthal proved the converse:

Theorem 7.32 [137, Theorem 3.18]. *Let X be an FHS space and Y be an orthogonally complemented subspace of X. Then Y has a 1-unconditional basis.*

Theorems 7.30 and 7.32 are the most general partial answers to Problem 7.27 known today.

It is natural to ask whether the structures analogous to but weaker than 1-unconditional bases are preserved by 1-complemented subspaces. Results of this kind will be presented in Section 7.f below.

Next, results on 1-complemented subspaces of real Banach spaces with 1-unconditional bases are due to the author of this survey. In [129] we considered an extension of Theorem 6.6 from ℓ_p to a larger class of Banach spaces.

Definition 7.33 (cf. [104, Definition 1.d.3]). We say that a Banach lattice X is *one-p-convex* (resp. *one-p-concave*) if for every $n \in \mathbb{N}$ and every choice of elements $\{x_i\}_{i=1}^n$ in X we have

$$\|(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}\| \leq (\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < \infty,$$

or, respectively,

$$\|(\sum_{i=1}^{n} |x_i|^q)^{\frac{1}{q}}\| \geq (\sum_{i=1}^{n} \|x_i\|^q)^{\frac{1}{q}} \quad \text{if} \quad 1 \leq q < \infty.$$

This is the usual notion of p-convexity (resp. q-concavity) but with the additional requirement that the constant is equal to 1. Clearly, spaces ℓ_p are both one-p-convex and one-p-concave.

We have the following extension of Theorem 6.6.

Theorem 7.34 [129, Theorem 1]. *Let X be a strictly monotone Banach space with a 1-unconditional basis. Suppose that*

(a) *X is a one-p-convex, $2 < p < \infty$, or*

(b) *X is a one-q-concave, $1 < q < 2$, and smooth at each basis vector.*

Then any 1-complemented subspace F of codimension n in X contains all but at most $2n$ basis vectors of X, i.e., there exist functionals $f_n, \cdots, f_n \in X^*$ such that $\mathrm{card}(\bigcup_{j=1}^{n} \mathrm{supp}\, f_j) \leq 2n$ *and* $F = \bigcap_{j=1}^{n} f_j^{-1}(0)$.

Notice that in a general Banach space X it does not have to be true that every 1-complemented subspace $Y \subset X$ with $\mathrm{codim} Y < \infty$ is an intersection of 1-complemented hyperplanes in X (as is the case when $X = \ell_p$, by Theorem 6.6). Indeed, Bohnenblust's finite-dimensional examples from Theorem 6.2 do not have 1-complemented hyperplanes, while every 1-dimensional subspace is of finite codimension and, of course, 1-complemented (moreover these spaces are one-p-convex when $2 < p < \infty$); see also Theorem 7.2.

We wish to note here that the condition in the conclusion of Theorem 6.6 does not characterize ℓ_p among infinite dimensional Banach spaces with 1-unconditional basis (cf. [128, Theorem 5.1]). However, we believe that the following conjecture is true:

Conjecture 7.35.

(a) *Let X be a finite-dimensional Banach space such that every* 1*-dimensional subspace of X can be represented as an intersection of* 1*-complemented hyperplanes. Then X is isometric to ℓ_p^n for some $p, 1 \leq p \leq \infty$, $n = \dim X$.*

(b) *There exists a finite-dimensional Banach space X, $\dim X = n$, which is not isometric to ℓ_p^n for any $p, 1 \leq p \leq \infty$, and such that every* 1*-complemented subspace Y of X with* $\dim Y \geq 2$ *can be represented as an intersection of* 1*-complemented hyperplanes.*

As a corollary of Theorems 7.34 and 5.38, we obtain the extension of Theorem 4.18:

Corollary 7.36 [129, Corollary 3]. *Suppose that X is a separable strictly monotone real function space on (Ω, μ), where μ is finite on the nonatomic part of Ω. Suppose that X is either one-p-convex for some $2 < p < \infty$, or one-q-concave for some $1 < q < 2$ and smooth at χ_A for every atom A of μ. Let the hyperplane $F = f^{-1}(0)$ be* 1*-complemented in X. Then there exist $\alpha, \beta \in \mathbb{R}$ and A, B-atoms of μ so that $f = \alpha\chi_A + \beta\chi_B$.*

Theorem 7.34 does not guarantee that the 1-complemented subspaces of finite codimension in X are spanned by disjointly supported vectors, however we believe that this should be the case in “most” spaces. In [130], we proved that this is the case for most 1-complemented subspaces of finite codimension in Orlicz sequence spaces and 1-complemented hyperplanes in Lorentz sequence spaces.

We need the following notation:

Definition 7.37. We say that the Orlicz function φ is *similar* to t^p for some $p \in [1, \infty)$ if there exist $C, t_0 > 0$ so that $\varphi(t) = Ct^p$ for all $t < t_0$.

We say that φ is *equivalent* to t^p for some $p \in [1, \infty)$ if there exist $C_1, C_2, t_0 > 0$ so that $C_1, t^p \leq \varphi(t) \leq C_2 t^p$ for all $t < t_0$.

We obtained the following:

Theorem 7.38 [130, Theorem 7]. *Let φ be an Orlicz function such that φ is not similar to t^2 and $\varphi(t) > 0$ for all $t > 0$. Consider the Orlicz space ℓ_φ with either Luxemburg or Orlicz norm. Let $F \subset \ell_\varphi$ be a subspace of codimension n with* $\dim F > 1$*. If F contains at least one basis vector and F is* 1*-complemented in ℓ_φ, then F can be represented as $F = \bigcap_{j=1}^{n} f_j^{-1}(0)$, where* $\mathrm{card}(\mathrm{supp}\, f_j) \leq 2$ *for all $j = 1, \ldots, n$, i.e., F is spanned by disjointly supported vectors.*

Moreover, if φ is not equivalent to t^p for any $p \in [1, \infty)$, then $|f_{ji}|$ is either 1 *or* 0 *for all i, j, i.e., F is a span of a block basis with constant coefficients.*

If φ is equivalent but not similar to t^p for some $p \in [1, \infty)$, then there exists $\gamma \geq 1$ such that $\{|f_{ji}| : j = 1, \ldots, n, i \leq \dim \ell_\varphi\} \subset \{\gamma^m : m \in \mathbb{Z}\} \cup \{0\}$.

In [130], we were unable to eliminate the condition that F has to contain at least one basis vector of ℓ_φ for Theorem 7.38 to hold, even though we believe that this condition is not necessary. It follows from Theorem 7.34 that this condition will be automatically satisfied if $\dim(\ell_\varphi)$ is large enough and ℓ_φ is one-p-convex for some $2 < p < \infty$ or one-q-concave for some $1 < q < 2$. Very recently, we also obtained a new result which eliminates this condition (as well as the requirement that F is of finite codimension) provided that Orlicz function φ is smooth enough [126]. The other conditions in the assumptions of Theorem 7.38 are all necessary.

In [130], we also considered characterizations of 1-complemented hyperplanes in Lorentz sequence spaces. We obtained the following:

Theorem 7.39 [130, Theorem 3]. *Let $\ell_{w,p}$ be a real Lorentz sequence space with $1 < p < \infty$ and $w_2 > 0$. Suppose that $Y = f^{-1}(0)$ is 1-complemented in $\ell_{w,p}$ and $\mathrm{card}(\mathrm{supp}\, f) \geq n > 2$. Then $p = 2$ and $1 = w_1 = w_2 = \cdots = w_n$.*

Theorem 7.40 [130, Corollary 6]. *Let $\ell_{w,p}$ be a real Lorentz sequence space with $1 < p < \infty$ and $w_k > 0$ for all k, i.e., $\ell_{w,p}$ is strictly monotone. Suppose that $Y = f^{-1}(0)$ is* 1*-complemented in $\ell_{w,p}$ and $\mathrm{card}(\mathrm{supp}\, f) = 2$, i.e., $f = f_i e_i^k + f_j e_j^k$ for some $i \neq j$. Then $|f_i| = |f_j|$ or $\ell_{w,p} = \ell_p$, i.e., $w_k = 1$ for all k.*

The main tool in the proofs of Theorems 7.38, 7.39 and 7.40 is Proposition 5.41, which allows to transfer to the real space setting techniques analogous to hermitian elements in complex spaces (see [89]).

Finally, we want to mention two results valid in complex and real Orlicz and Lorentz sequence spaces, which characterize 1-complemented subspaces among the subspaces spanned by a family of mutually disjoint vectors.

Theorem 7.41 [128, Theorem 6.1]. *Let ℓ_φ be a real or complex Orlicz space and let $\{x_i\}_{i \in I}$ $(I \subseteq \mathbb{N},\ \mathrm{card}(I) > 1)$ be mutually disjoint elements in ℓ_φ with $\|x_i\|_\varphi = 1$ for all $i \in I$. Suppose that $X = \overline{\mathrm{span}}\{x_i\}_{i \in I} \subset \ell_\varphi$ is* 1*-compelemented in ℓ_φ. Then one of the three possibilities holds:*

(1) *for each $i \in I$, $\mathrm{card}(\mathrm{supp}\, x_i) < \infty$ and $|x_{ij}| = |x_{ik}|$ for all $j, k \in \mathrm{supp}\, x_i$; or*

(2) *there exists p, $1 \leq p \leq \infty$, such that $\varphi(t) = Ct^p$ for all $t \leq \sup\{\|x_i\|_\infty : i \in I\}$; or*

(3) *there exists* p, $1 \leq p \leq \infty$, *and constants* $C_1, C_2, \gamma \geq 0$ *such that* $C_2 t^p \leq \varphi(t) \leq C_1 t^p$ *for all* $t \leq \sup\{\|x_i\|_\infty : i \in I\}$ *and such that for all* $i \in I$ *and* $j \in \operatorname{supp} x_i$,

$$|x_{ij}| \in \{\gamma^k \cdot \|x_i\|_\infty : k \in \mathbb{Z}\}.$$

Theorem 7.42 [128, Theorem 6.3]. *Let* $\ell_{w,p}$ *with* $1 < p < \infty$ *be a real or complex Lorentz sequence space and let* $\{x_i\}_{i\in I}(I \subseteq \mathbb{N}, \operatorname{card}(I) > 1)$ *be mutually disjoint elements in* $\ell_{w,p}$ *such that* $X = \overline{\operatorname{span}}\{x_i\}_{i\in I}$ *is* 1*-complemented in* $\ell_{w,p}$. *Suppose*, *moreover*, *that* $w_\nu \neq 0$ *for all* $\nu \leq S \overset{def}{=} \sum_{i\in I} \operatorname{card}(\operatorname{supp} x_i)$ $(\leq \infty)$. *Then*:

(1) $w_\nu = 1$ *for all* $\nu \leq S$, *or*

(2) $|x_{ik}| = |x_{il}|$ *for all* $i \in I$ *and all* $k, l \in \operatorname{supp} x_i$.

Theorems 7.41 and 7.42 show that in Orlicz and Lorentz sequence spaces which are sufficiently different from ℓ_p, a subspace spanned by a block basis $\{f_j\}_j$ is 1-complemented if and only if all elements of the block basis $\{f_j\}_j$ have constant coefficients, i.e., if they satisfy Theorem 7.8, about the most obvious form of 1-complemented subspaces in symmetric spaces. We believe that in fact Theorem 7.8 provides not only a sufficient condition but also a necessary condition for a form of a 1-complemented subspaces in sequence spaces sufficiently different from ℓ_p.

7.f. **Preservation of approximation properties by norm-one projections.** In this final section, we mention briefly problems analogous to Problems 1.1 and 1.2, but concerned with structures which are analogous to, but weaker than unconditional bases or bases, namely, finite dimensional decompositions and approximation properties.

The approximation property appeared already in Banach's book [8] and it plays a fundamental role in the structure theory of Banach spaces; see [103] and the recent [47] for the exposition of the development and open problems in the theory. Here we just recall the definitions essential for the statement of the results below.

Definition 7.43 [48, 77] (see also [103, Sections 1.e and 1.g] and [47]). Let X be a separable Banach space. We say that X has the *approximation property* (AP for short) if there is a net of finite-rank operators T_α so that $T_\alpha x \longrightarrow x$ for $x \in X$, uniformly on compact sets. We say that X has the *bounded approximation property* (BAP for short) if this net can be replaced by a sequence T_n; alternatively X has BAP if there is a sequence of finite-rank operators T_n so that $T_n x \longrightarrow x$ for $x \in X$ and $\sup_n \|T_n\| < \infty$. A sequence T_n with these properties is called an *approximating sequence*. If X has an approximating sequence T_n with $\lim_{n\to\infty} \|T_n\| = 1$ then we say that X has the *metric approximation property* (MAP for short). If X has an approximating sequence T_n with $\lim_{n\to\infty} \|I - 2T_n\| = 1$ then we say that X has the *unconditional metric approximation property* (UMAP for short) (here I denotes the identity operator on X). If X has an approximating sequence T_n with $T_mT_n = T_{\min(m,n)}$ then we say that X has a *finite-dimensional decomposition* (FDD for short). Let $1 \le \lambda < \infty$. An FDD is called λ-*unconditional* if $\| \sum_{n=1}^{\infty} \theta_n(T_{n+1} - T_n)\| \le \lambda$ for every choice of signs $(\theta_n)_{n=1}^{\infty}$.

As mentioned above, we refer the reader to [47] for the analysis of these interesting properties. Here we just want to note that, clearly, spaces with bases have AP and FDD, spaces with monotone bases have MAP and spaces with 1-unconditional bases have UMAP. It is a very nontrivial fact that the reverse implications do not hold; see [103, 47].

It is very natural to consider the following analogue of Problems 1.1 and 1.2:

Problem 7.44. *Let X be a separable Banach space with one of the approximation properties: AP, BAP, MAP, UMAP or FDD or a basis. Suppose that Y is complemented in X. Which of the approximation properties must Y satisfy?*

We note here that given any pair of Banach spaces X, Y such that $Y \subset X$ and Y is complemented in X, it is possible to introduce a new equivalent norm on X so that Y is 1-complemented in X with the new norm. Thus Problem 7.44 can be restated for 1-complemented subspaces Y of X without losing the isomorphic nature of the problem. We refer the reader to [47] for an interesting account of what is known about Problem 7.44. Here we just quote one sample result related to this problem:

Theorem 7.45 [47, Theorem 3.13] (*due to Pełczyński* [119] *and Johnson, Rosenthal, Zippin* [87]). *A separable Banach space Y has BAP if and only if Y is isomorphic to a complemented (equivalently, 1-complemented) subspace of a space with a basis.*

Isometric versions of Problem 7.44 will necessarily deal with isometric versions of approximation properties, i.e., MAP, UMAP, 1-unconditional FDD, monotone basis, 1-unconditional basis, similarly as Problems 7.10 and 7.17.

Clearly, 1-complemented subspaces of spaces with MAP have MAP. Also, Godefroy and Kalton [77] observed that 1-complemented subspaces of spaces with UMAP have UMAP. Moreover, they proved:

Theorem 7.46 [77, Corollary IV.4]. *Let X be a separable Banach space. Then X has UMAP if and only if for every $\varepsilon > 0$, X is isometric to a 1-complemented subspace of a space V_ε with a $(1+\varepsilon)$-unconditional FDD.*

As far as we know, all other isometric versions of Problem 7.44 are open; see also Section 7.c.

We have for example:

Problem 7.47. *Does every* 1*-complemented subspace of a space with a* 1*-unconditional FDD have a* 1*-unconditional FDD* (*or any unconditional FDD*)?

But note that Read [132] gave examples of spaces without FDD which are complemented (equivalently, 1-complemented) in a space with an FDD, or even in a space with a basis (cf. [47]).

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References

1. Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, An elementary proof of Douglas' theorem on contractive projections on L_1-spaces, *J. Math. Anal. Appl.* **177** (1993), 641-644.
2. Z. Altshuler, P. G. Casazza and B. L. Lin, On symmetric basic sequences in Lorentz sequence spaces, *Israel J. Math.* **15** (1973), 140-155.
3. D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser Verlag, Basel, 1986.
4. T. Ando, Contractive projections in L_p-spaces, *Pacific J. Math.* **17** (1966), 391-405.
5. T. Ando, Banachverbände und positive Projektionen, *Math. Z.* **109** (1969), 121-130.
6. J. Arazy and Y. Friedman, Contractive projections in $\mathcal{C}_p$, *Mem. Amer. Math. Soc.* **459** (1992).
7. R. R. Bahadur, Measurable subspaces and subalgebras, *Proc. Amer. Math. Soc.* **6** (1955), 565-570.
8. S. Banach, *Theorie des Operations Lineaires*, Monografie Mat. 1, Polish Scientific Publishers, Warsaw, 1932.
9. M. Baronti, Strictly contractive projections on classical sequence spaces, *Portugal. Math.* **47** (1990), 131-138.
10. M. Baronti and P. L. Papini, Norm-one projections onto subspaces of ℓ_p, *Ann. Mat. Pura Appl.* **152** (1988), 53-61.
11. M. Baronti and P. L. Papini, Bicontractive projections in sequence spaces and a few related kinds of maps, *Comment. Math. Univ. Carolin.* **30** (1989), 665-673.

12. M. Baronti and P. L. Papini, Norm-one projections onto subspaces of finite codimension in ℓ_1 and c_0, *Period. Math. Hungar.* **22** (1991), 161-174.

13. B. Beauzamy, Points minimaux dans les espaces de Banach, *C. R. Acad. Sci. Paris Sér. A-B* **280:Aii** (1975), A717-A720.

14. B. Beauzamy, Points minimaux dans les espaces de Banach, in: *Séminaire Maurey-Schwartz 1974-1975: Espaces L^p, Applications Radonifiantes et Géométrie des Espaces de Banach*, Exp. No. XVIII et XIX, Centre Math., École Polytech., Paris, 1975, p.34.

15. B. Beauzamy and B. Maurey, Points minimaux et ensembles optimaux dans les espaces de Banach, *J. Funct. Anal.* **24** (1977), 107-139.

16. Y. Benyamini, P. Flinn and D. R. Lewis, A space without 1-unconditional basis which is 1-complemented in a space with a 1-unconditional basis, in: *Texas Functional Analysis Seminar 1983-1984 (Austin, Tex.)*, Univ. Texas Press, Austin, TX, 1984, pp. 143-149.

17. Y. Benyamini and P. K. Lin, An operator on L^p without best compact approximation, *Israel J. Math.* **51** (1985), 298-304.

18. S. J. Bernau, Theorems of Korovkin type for L^p-spaces, *Pacific J. Math. 53* (1974), 11-19.

19. S. J. Bernau and H. E. Lacey, The range of a contractive projection on an L_p-space, *Pacific J. Math.* **53** (1974), 21-41.

20. S. J. Bernau and H. E. Lacey, Bicontractive projections and reordering of L_p-spaces, *Pacific J. Math.* **69** (1977), 291-302.

21. C. Bessaga, Topological equivalence of unseparable reflexive Banach spaces. Ordinal resolutions of identity and monotone bases, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **15** (1967), 397-399.

22. G. Birkhoff, Orthogonality in linear metric spaces, *Duke Math. J.* **1** (1935), 169-172.

23. Z. Birnbaum and W. Orlicz, Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen, *Studia Math.* **3** (1931), 1-67.

24. J. Blatter and E. W. Cheney, Minimal projections on hyperplanes in sequence spaces, *Ann. Mat. Pura Appl.* **101** (1974), 215-227.

25. F. Bohnenblust, Subspaces of $l_{p,n}$ spaces, *Amer. J. Math.* **63** (1941), 64-72.

26. P. Bohnenblust, A characterization of complex Hilbert spaces, *Portugal. Math.* **3** (1942), 103-109.

27. F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Mathematical Society Lecture Note Series, 2, Cambridge University Press, London, 1971.

28. F. F. Bonsall and J. Duncan, *Numerical Ranges. II*, London Mathematical Society Lecture Notes Series, No. 10, Cambridge University Press, New York, 1973.

29. A. P. Bosznay and B. M. Garay, On norms of projections, *Acta Sci. Math.* (Szeged), **50** (1986), 87-92.

30. B. Bru and H. Heinich, Dérivations et contractions dans les espaces de Köthe, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 131-134.

31. B. Bru and H. Heinich, Applications de dualité dans les espaces de Köthe, *Studia Math.* **93** (1989), 41-69.

32. B. Bru, H. Heinich and J.-C. Lootgieter, Itération des applications de dualité dans les espaces d'Orlicz, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 745-747.

33. R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.* **3** (1977), 459-470.

34. H. D. Brunk, On an extension of the concept of conditional expectation, *Proc. Amer. Math. Soc.* **14** (1963), 298-304.

35. D. L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, *Ann. Probab.* **12** (1984), 647-702.

36. C. Byrne, Jr. and F. E. Sullivan, Contractive projections with contractive complement in L_p space, *J. Multivariate Anal.* **2** (1972), 1-13.

37. C. L. Byrne, H. B. Cohen and F. E. Sullivan, A note on contractive projections on L^p spaces, *Rev. Roumaine Math. Pures Appl.* **17** (1972), 1569-1574.

38. B. Calvert, Convergence sets in reflexive Banach spaces, *Proc. Amer. Math. Soc.* **47** (1975), 423-428.

39. B. Calvert, The range of a contractive projection, *Math. Chronicle* **6** (1977), 68-71.

40. B. Calvert, When T-accretive implies accretive *Houston J. Math.* **14** (1988), 147-155.

41. B. Calvert and S. Fitzpatrick, Characterizing ℓ_p and c_0 by projections onto hyperplanes, *Boll. Un. Mat. Ital.* (VI) **5** (1986), 405-410.

42. B. Calvert and S. Fitzpatrick, A Banach lattice characterization of c_0 and l_p, *Math. Chronicle* **16** (1987), 25-40.

43. B. Calvert and S. Fitzpatrick, Consequences of the duality map taking planes to planes, in: *Nonlinear Analysis and Applications*, *Arlington*, *Tex.*, *1986*, Dekker, New York, 1987, pp. 105-114.

44. B. Calvert and S. Fitzpatrick, Nonexpansive projections onto two–dimensional subspaces of Banach spaces, *Bull. Austral. Math. Soc.* **37** (1988), 149-160.

45. B. Calvert and S. Fitzpatrick, Characterising nonsmooth norms by nonexpansive projections, *Math. Chronicle*, **18** (1989), 15-30.

46. S. Campbell, G. Faulkner and R. Sine, Isometries, projections and Wold decompositions, in: *Operator Theory and Functional Analysis, Papers, Summer Meeting, Amer. Math. Soc.*, *Providence, R.I.*, *1978*, Pitman, Boston, Mass., 1979, pp. 85-114.

47. P. G. Casazza, Approximation properties, in: *Handbook on the Geometry of Banach Spaces*, W. B. Johnson and J. Lindenstrauss, eds., North Holland, to appear.

48. P. G. Casazza and N. J. Kalton, Notes on approximation properties in separable Banach spaces, in: *Geometry of Banach Spaces*, Strobl, 1989, Cambridge Univ. Press, Cambridge, 1990, pp. 49-63.

49. S. Chen, Geometry of Orlicz spaces, *Dissertationes Math.* (Rozprawy Mat.) **356** (1996), 1-204.

50. E. W. Cheney and K. H. Price Minimal projections, in: *Approximation Theory*, Proc. Sympos., Lancaster, 1969, Academic Press, London, 1970, pp. 261-289.

51. H. B. Cohen and F. E. Sullivan, Projecting onto cycles in smooth, reflexive Banach spaces, *Pacific J. Math.* **34** (1970), 355-364.

52. F. Deutsch, Linear selections for the metric projection, *J. Funct. Anal.* **49** (1982), 269-292.

53. F. Deutsch, A survey of metric selections, in: *Fixed Points and Nonexpansive Mappings*, Cincinnati, Ohio, 1982, Amer. Math. Soc., Providence, R. I., 1983, pages 49-71.

54. N. Dinculeanu, Conditional expectations for general measure spaces, *J. Multivariate Anal.* **1** (1971), 347-364.

55. N. Dinculeanu and M. M. Rao, Contractive projections and conditional expectations, *J. Multivariate Anal.* **2** (1972), 362-381.

56. S. Z. Ditor, Averaging operators in $C(S)$ and lower semicontinuous sections of continuous maps, *Trans. Amer. Math. Soc.* **175** (1973), 195-208.

57. P. G. Dodds, C. B. Huijsmans and B. de Pagter, Characterizations of conditional expectation-type operators, *Pacific J. Math.* **141** (1990), 55-77.

58. J. C. Doob, *Stochastic Processes*, John Wiley & Sons, Inc., 1953.

59. L. E. Dorembus [L. E. Dor], Transfinite bases of subspaces in Hausdorff linear topological spaces, *Math. Ann.* **192** (1971), 71-82.

60. L. E. Dor, A note on monotone bases, *Israel J. Math.* **14** (1973), 285-286. 1973.

61. L. E. Dor and E. Odell, Monotone bases in L_p. *Pacific J. Math.* **60** (1975), 51-61.

62. R. G. Douglas, Contractive projections on an L_1 space, *Pacific J. Math.* **15** (1965), 443-462.

63. I. Doust, Contractive projections on Banach spaces, in: *Workshop/Miniconference on Functional Analysis and Optimization*, Canberra, 1988, Austral. Nat. Univ., Canberra, 1988, pp. 50-58.

64. I. Doust, Well-bounded and scalar-type spectral operators on L^p-spaces, *J. London Math. Soc.* (*2*), **39** (1989), 525-534.

65. I. Doust, Contractive projections on Lebesgue-Bochner spaces, in: *Function Spaces* Edwardsville, IL, 1994, Dekker, New York, 1995, pp. 101-109.

66. N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Interscience Publishers, Inc., New York, 1958.

67. C. V. Duplissey, *Contractive Projections in Abstract Banach Function Spaces*, Ph.D. Dissertation, University of Texas at Austin, 1971.

68. J. Dye, M. A. Khamsi and S. Reich, Random products of contractions in Banach spaces, *Trans. Amer. Math. Soc.* **325** (1991), 87-99.

69. J. Dye and S. Reich, Random products of nonexpansive mappings, in: *Optimization and Nonlinear Analysis*, Haifa, 1990, Longman Sci. Tech., Harlow, 1992, pp. 106-118.

70. G. D. Faulkner and J. E. Huneycutt, Jr., Orthogonal decomposition of isometries in a Banach space, *Proc. Amer. Math. Soc.* **69** (1978), 125-128.

71. D. G. de Figueiredo and L. A. Karlovitz, On the extention of contractions of normed spaces, in: *Nonlinear Functional Analysis*, Proceedings of Symposia in Pure Mathematics, Volume 18, Part 1, Amer. Math. Soc. Providence, 1970, pp. 95-104.

72. S. Fitzpatrick and B. Calvert, Convex bodies in $\mathbb{R}^3$ invariant under projections, *Math. Chronicle* **20** (1991), 89-108.

73. S. Fitzpatrick and B. Calvert, Sets invariant under projections onto one-dimensional subspaces, *Comment. Math. Univ. Carolin.* **32** (1991), 227-232.

74. S. Fitzpatrick and B. Calvert, Sets invariant under projections onto two-dimensional subspaces, *Comment. Math. Univ. Carolin.* **32** (1991), 233-239.

75. P. Flinn, On a theorem of N. J. Kalton and G. V. Wood concerning 1-complemented subspaces of spaces having an orthonormal basis, in: *Texas Functional Analysis Seminar 1983-1984*, Austin, Tex., Univ. Texas Press, Austin, TX, 1984, pp. 135-144.

76. C. Franchetti and E. M. Semënov, A Hilbert space characterization among function spaces, *Anal. Math.* **21** (1995), 85-93.

77. G. Godefroy and N. J. Kalton, Approximating sequences and bidual projections, *Quart. J. Math. Oxford Ser.* (*2*), **48** (1997), 179-202.

78. G. Godini, On minimal points, *Comment. Math. Univ. Carolin.* **21** (1980), 407-419.

79. W. T. Gowers and B. Maurey, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6** (1993), 851-874.

80. A. Grothendieck, Une characterisation vectorielle metrique des espaces L^1, *Canad. J. Math.* **7** (1955), 552-561.

81. I. Halperin, The product of projection operators, *Acta Sci. Math.* (*Szeged*) **23** (1962), 96-99.

82. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin-Heidelberg-New York, 1993.

83. N. Herrndorf, Counterexamples to results of M. M. Rao, *Z. Wahrsch. Verw. Gebiete* **53** (1980), 283-290.

84. R. C. James, Orthogonality in normed linear spaces, *Duke Math. J.* **12** (1945), 291-302.

85. R. C. James, Orthogonality and linear functionals in normed linear spaces, *Trans. Amer. Math. Soc.* **61** (1947), 265-292.

86. R. C. James, Inner products in normed linear spaces, *Bull. Amer. Math. Soc.* **53** (1947), 559-566.

87. W. B. Johnson, H. P. Rosenthal and M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, *Israel J. Math.* **9** (1971), 488-506.

88. S. Kakutani, Some characterizations of Euclidean space, *Japan J. Math.* **16** (1939), 93-97.

89. N. J. Kalton and B. Randrianantoanina, Surjective isometries of rearrangement-invariant spaces, *Quart. J. Math. Oxford Ser.* (*2*) **45** (1994), 301-327.

90. N. J. Kalton and G. V. Wood, Orthonormal systems in Banach spaces and their applications, *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 493-510.

91. S. Kinnunen, On projections and Birkhoff-James orthogonality in Banach spaces, *Nieuw Arch. Wisk.* (*4*), **2** (1984), 235-255.

92. A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitrechnung*, Vol. 2, No. 3, Ergebuisse der Mathematik, 1933. (English translation: Foundations of the Theory of Probability, Chelsea Publ. Co., New York, 1950).

93. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, P. Noordhoff LTD., Groningen, The Netherlands, 1961.

94. H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer, Berlin-Heidelberg-New York, 1974.

95. H. E. Lacey and S. J. Bernau, Characterizations and classifications of some classical Banach spaces, *Adv. Math.* **12** (1974), 367-401.

96. D. R. Lewis, Ellipsoids defined by Banach ideal norms, *Mathematika* **26** (1979), 18-29.

97. P.-K. Lin, Remarks on linear selections for the metric projection, *J. Approx. Theory* **43** (1985), 64-74.

98. K. Lindberg, On subspaces of Orlicz sequence spaces, *Studia Math.* **45** (1973), 119-146.

99. J. Lindenstrauss, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964).

100. J. Lindenstrauss, On nonlinear projections in Banach spaces, *Michigan Math. J.* **11** (1964), 263-287.

101. J. Lindenstrauss, On projections with norm 1 – an example, *Proc. Amer. Math. Soc.* **15** (1964) 403-406.

102. J. Lindenstrauss, A remark on symmetric bases, *Israel J. Math.* **13** (1972), 317-320.

103. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, *Vol. 1*, *Sequence spaces*. Springer–Verlag, Berlin–Heidelberg–New York, 1978.

104. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, *Vol. 2*, *Function Spaces*, Springer–Verlag, Berlin–Heidelberg–New York, 1979.

105. E. R. Lorch, On a calculus of operators in reflexive vector spaces, *Trans. Amer. Math. Soc.* **45** (1939), 217-234.

106. C. G. Lorentz, Some new functional spaces, *Ann. of Math.* **51** (1950), 37-55.

107. W. A. J. Luxemburg, Review of the paper by N. Herrndorf "Counterexamples to results of M. M. Rao", [Z. Wahrsch. Verw. Gebiete **53** (1980), 283-290], *Math. Reviews*, MR82f:46030.

108. S.T.C. Moy, Characterizations of conditional expectations as a transformation on function spaces, *Pacific J. Math.* **4** (1954), 47-64.

109. F. J. Murray, On complementary manifolds and projections in spaces L_p and l_p, *Trans. Amer. Math. Soc.* **41** (1937), 138-152.

110. V. P. Odinec, The uniqueness of the projection with norm equal to 1 in a Banach space, *Izv. Vyssh. Uchebn. Zaved. Mat.* **140** (1974), 82-89.

111. V. P. Odinec, Smoothness subspaces, normal bases, and the uniqueness of the projection with norm 1, *Rev. Roumaine Math. Pures Appl.* **20** (1975), 429-437.

112. V. P. Odinec, Conditions for uniqueness of a projector with norm one, *Mat. Zametki* **22** (1977), 45-49.

113. W. Odyniec [V. P. Odinec] and G. Lewicki, *Minimal Projections in Banach Spaces*, *Problems of Existence and Uniqueness and Their Application*, Springer-Verlag, Berlin, 1990.

114. W. Orlicz, Über eine gewisse Klasse von Räumen von Typus B, *Bull. Intern. Acad.*, *Sér. A*, *Cracovie*, **8** (1932), 207-220.

115. W. Orlicz, Über Raume (L^M). *Bull. Intern. Acad. Pol.*, *Sér. A*, *Cracovie*, 1936.

116. P. L. Papini, Minimal and closest points, nonexpansive and quasinonexpansive retractions in real Banach spaces, in: *Convexity and its applications*, Birkhäuser, Basel, 1983, pp. 248-263.

117. P. L. Papini, Some questions related to the concept of orthogonality in Banach spaces. Orthogonal projections, *Bol. Un. Mat. Ital.* (4) **9** (1974), 386-401.

118. A. Pełczyński, Projections in certain Banach spaces, *Studia Math.* **19** (1960), 209-228.

119. A. Pełczyński, Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis, *Studia Math.* **40** (1971), 239-243.

120. A. Pełczyński (in collaboration with Cz. Bessaga), Some aspects of the present theory of Banach spaces, in: *S. Banach, Oeuvres*, Vol. II, *Travaux sur l'Analyse Fonctionelle*, PWN – Éditions Scientiques de Pologne, Warsaw, 1979, pp. 223-302.

121. A. Pełczyński and H. P. Rosenthal, Localization techniques in L^p spaces, *Studia Math.* **52** (1974/75), 263-289.

122. R. S. Phillips, A characterization of Euclidean spaces, *Bull. Amer. Math. Soc.* **46** (1940), 930-933.

123. N. Popa, Complemented sublattices in l_p, $0 < p < 1$, and an extension of Ando's theorem, *Arch. Math.* (*Basel*) **32** (1979), 276-288.

124. N. Popa, On the complemented subspaces of l_p, $0 < p < 1$, *Rev. Roumaine Math. Pures Appl.* **26** (1981), 287-299.

125. M. M. Popov, On norms of projectors in $L_p(\mu)$ with "small" kernels, *Funktsional. Anal. i Prilozhen.* **21** (1987), 84-85.

126. B. Randrianantoanina, Contractive projections in sequence Orlicz spaces, in preparation.

127. B. Randrianantoanina, Contractive projections in nonatomic function spaces, *Proc. Amer. Math. Soc.* **123** (1995), 1747-1750.

128. B. Randrianantoanina, 1-complemented subspaces of spaces with 1-unconditional bases, *Canad. J. Math.* **49** (1997), 1242-1264.

129. B. Randrianantoanina, Contractive projections and isometries in sequence spaces, *Rocky Mountain J. Math.* **28** (1998), 323-340.

130. B. Randrianantoanina, One-complemented subspaces of real sequence spaces, *Results Math.* **33** (1998), 139-154.

131. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.

132. C. J. Read, Different forms of the approximation property, unpublished notes.

133. S. Reich, A limit theorem for projections, *Linear and Multilinear Algebra* **13** (1983), 281-290.

134. P. Renaud, A Korovkin theorem for abstract Lebesgue spaces, *J. Approx. Theory* **102** (2000), 13-20.

135. H. P. Rosenthal, On one-complemented subspaces of complex Banach spaces with one-unconditional basis, according to Kalton and Wood, in: *Israel Seminar on Geometrical Aspects of Functional Analysis*, No. IX, 1983-4.

136. H. P. Rosenthal, Contractively complemented subspaces of Banach spaces with reverse monotone (transfinite) bases, *Longhorn Notes*, *The University of Texas Functional Analysis Seminar*, 1984-5, pp. 1-14.

137. H. P. Rosenthal, Functional Hilbertian sums, *Pacific J. Math.* **124** (1986), 417-467.

138. G.-C. Rota, On the representation of averaging operators, *Rend. Sem. Mat. Univ. Padova* **30** (1960), 52-64.

139. W. Rudin, *Functional Analysis*, 2nd ed., McGraw–Hill, New York, 1991.

140. Z. Šidák, On relations between strict sense and wide sense conditional expectations, *Teoriya Veroyatnost. i Primenen.* **2** (1957), 283-288.

141. R. Sine, Rigidity properties of nonexpansive mappings, *Nonlinear Anal.* **11** (1987), 777-794.

142. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Publ. House Acad. Soc. Rep. Romania, Bucharest, 1967, (In Romanian); English translation: Publ. House Acad. Soc. Rep. Romania, Bucharest and Springer-Verlag, Grundlehren der math. Wiss., Band 171, Berlin-Heidelberg-New York, 1970.

143. I. Singer, *Bases in Banach Spaces. I*, Springer-Verlag, New York, 1970.

144. I. Singer, *The Theory of Best Approximation and Functional Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1974.

145. I. Singer, *Bases in Banach Spaces. II*, Springer-Verlag, Berlin-Heidelberg-New York; Editura Academiei Republicii Socialiste România, Bucharest, 1981.

146. W. H. Song, The linear selections of metric projections in the L_p spaces, *J. Approx. Theory* **66** (1991), 53-57.

147. M. Spivack, Contractive projections on Banach space, *Bull. Austral. Math. Soc.* **34** (1986), 271-274.

148. R. S. Strichartz, L^p contractive projections and the heat semigroup for differential forms, *J. Funct. Anal.* **65** (1986), 348-357.

149. S. J. Szarek, A Banach space without a basis which has the bounded approximation property, *Acta Math.* **159** (1987), 81-98.

150. L. Tzafriri, Remarks on contractive projections in L_p-spaces, *Israel J. Math.* **7** (1969), 9-15.

151. D. E. Wulbert, A note on the characterization of conditional expectation operators, *Pacific J. Math.* **34** (1970), 285-288.

152. D. E. Wulbert, Contractive Korovkin approximations, *J. Funct. Anal.* **19** (1975), 205-215.

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