NORM PRINCIPLE FOR REDUCTIVE ALGEBRAIC GROUPS

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1. Introduction

We begin by recalling the norm principles of Knebusch and Scharlau from the algebraic theory of quadratic forms. Let (V, q) be a non-degenerate quadratic form over a field F and let L/F be a finite field extension. Let $D(q)$ denote the subgroup in $F^\times$ generated by the non-zero values of q. Knebusch norm principle asserts that $N_{L/F}\big(D(q \otimes_F L)\big) \subset D(q)$, where $N_{L/F} : L^\times \to F^\times$ is the norm map. Let $G(q)$ be the group of multipliers of the quadratic form q, i.e. the group of all $x \in F^\times$ such that $(V, xq) \simeq (V, q)$. Scharlau norm principle states that $N_{L/F}\big(G(q \otimes_F L)\big) \subset G(q)$.

More generally, let $\varphi : G \to T$ be an algebraic group homomorphism defined over a field F. Assume that the group T is commutative, so that the norm homomorphism $N_{L/F} : T(L) \to T(F)$ is defined for any finite separable field extension L/F. The norm principle for φ and L/F claims that the norm homomorphism $N_{L/F}$ maps the image of the induced homomorphism $\varphi_L : G(L) \to T(L)$ to the image of $\varphi_F : G(F) \to T(F)$. Knebusch and Scharlau norm principles are the special cases of the norm principle for certain group homomorphisms (cf. Examples (3.2) and (3.3)).

In the general setting the validity of the norm principle is an open problem. The main result of the paper is the following

Theorem 1.1. *Let G be a reductive group over a field F. Assume that the Dynkin diagram of G does not contain connected components D_n, $n \geq 4$, E_6 or E_7. Then the norm principle holds for any group homomorphism $G \to T$ to a commutative group T and any finite separable field extension L/F.*

2. Definition and properties of the norm principle

Let T be a commutative algebraic group over a field F and let L/F be a finite separable filed extension. The *norm homomorphism*

$$N_{L/F} : T(L) \to T(F)$$

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can be defined as follows. Assuming $L \subset F_{\mathrm{sep}}$, the separable closure of F, we set for arbitrary $t \in T(L)$:

$$N_{L/F}(t) = \prod \gamma(t),$$

where γ runs over all cosets of $\mathrm{Gal}(F_{\mathrm{sep}}/L)$ in $\mathrm{Gal}(F_{\mathrm{sep}}/F)$. The definition of the norm homomorphism easily extends to the case of an arbitrary étale F-algebra L.

For example, if $T = \mathbb{G}_m$, we get the standard norm homomorphism $N_{L/F} : L^\times \to F^\times$.

Let $\varphi : G \to T$ be an algebraic group homomorphism defined over F. Assume that the group T is commutative. For an étale F-algebra L we have the following diagram:

$$\begin{array}{ccc} G(L) & \xrightarrow{\varphi_L} & T(L) \\ & & \downarrow{\scriptstyle N_{L/F}} \\ G(F) & \xrightarrow{\varphi_F} & T(F). \end{array}$$

We say that *the norm principle holds for φ and L/F* if

$$\mathrm{Im}\big(N_{L/F} \circ \varphi_L\big) \subset \mathrm{Im}\, \varphi_F.$$

If we write L as a product of finite separable field extensions $L_1 \times L_2 \times \cdots \times L_k$, then the norm principle for φ and L/F is clearly equivalent to the norm principles for φ and all the field extensions L_i/F.

We say that *the norm principle holds for φ* if the norm principle holds for φ and any finite separable field extension L/F (or, equivalently, for any étale F-algebra L).

Note that if G is also commutative, the norm principle clearly holds: the norm homomorphism for G makes the diagram (**??**) commutative.

In this section we prove simple properties of the norm principle. The following Lemma can be proven by a straightforward diagram chase.

Lemma 2.1. *Let $\varphi : G \to T$ and $\psi : T \to T'$ be two algebraic group homomorphisms over F. Assume that the groups T and T' are commutative. Then the norm principle for φ implies the norm principle for the composition $\psi \circ \varphi$.*

Let $\varphi : G \to T$ be an algebraic group homomorphism over F. Assume that the commutator subgroup G' in G is defined over F. Then φ factors through the natural homomorphism $\varphi' : G \to G/G'$ and, by Lemma 2.1, the norm principle for φ' implies the norm principle for φ, i.e. the norm principle for φ' is the universal one. We call it *the norm principle for the group G*. Thus, the norm principle for G implies the norm principle for any homomorphism $\varphi : G \to T$ to any commutative group T. In the sequel we will consider only algebraic groups with the commutator subgroup defined over F. For example, reductive groups satisfy this property. In this case the commutator subgroup G' is the *semisimple part of G*.

Note that the norm principle for G is equivalent to the norm principle for any homomorphism $\varphi : G \to T$ (not necessarily surjective) to a commutative group T with the kernel G'. If $G' = G$ (for example, if G is semisimple), the norm principle for G trivially holds.

Example 2.2. Assume that the field F is finite and the commutator subgroup G' of G is connected (for example, if G is reductive). Then $H^1(F, G') = 1$ by [11, Ch III, Th.1] and hence the homomorphism $G(F) \to (G/G')(F)$ is surjective. In particular, the norm principle for G trivially holds.

Let $\varphi : G \to T$ and $\psi : T' \to T$ be two algebraic group homomorphisms over F with T and T' commutative. Consider the fiber square

$$\begin{array}{ccc} G' & \xrightarrow{\varphi'} & T' \\ \downarrow & & \downarrow \psi \\ G & \xrightarrow{\varphi} & T, \end{array}$$

so that $G' = G \times_T T'$. A simple diagram chase gives a proof of the following

Lemma 2.3. *The norm principle for φ implies the norm principle for φ'.*

Corollary 2.4. *Let H be a subgroup in G. Assume that the commutator subgroup G' of G coincides with the commutator subgroup H' of H. Then the norm principle for G implies the norm principle for H.*

Proof. Under assumptions, we have a fiber square

$$\begin{array}{ccc} H & \longrightarrow & H/H' \\ \downarrow & & \downarrow \\ G & \longrightarrow & G/G'. \end{array}$$

Hence, the statement follows from Lemma 2.3. □

□

Lemma 2.5. *Let $G \to \widetilde{G}$ be a surjective algebraic group homomorphism with the kernel S. Assume that $H^1(L, S) = 1$ for any field extension L/F (for example, if S is a quasi-trivial torus). Then the norm principle for G implies the norm principle for $\widetilde{G}$.*

Proof. A simple diagram chase using surjectivity of $G(L) \to \widetilde{G}(L)$. □ □

Let E be an étale F-algebra and G an algebraic group over E. Denote by $R_{E/F}(G)$ the corestriction of scalars [6, Ch. 6].

Lemma 2.6. *The norm principle for G implies the norm principle for $R_{E/F}(G)$.*

3. Examples

Example 3.1. Let A be a central simple algebra over F. Denote by $\mathbf{GL}_1(A)$ the algebraic group of invertible elements in A. The reduced norm homomorphism for A [4, §22], [6, §1] defines an algebraic group homomorphism

$$\mathrm{Nrd} : \mathbf{GL}_1(A) \to \mathbb{G}_m.$$

The norm principle for Nrd amounts the inclusion

$$N_{L/F}\big(\mathrm{Nrd}(A \otimes_F L)^\times\big) \subset \mathrm{Nrd}(A^\times)$$

for any finite separable field extension L/F. This statement is an easy implication of the following description of the reduced norms: The group $\mathrm{Nrd}(A^\times)$ is a product of the norm groups $N_{E/F}(E^\times)$ for all finite field extensions E/F such that the E-algebra $A \otimes_F E$ splits.

Example 3.2. (Knebusch norm principle) Let (V, q) be a non-degenerate quadratic form over F, $\mathbf{\Gamma}(V, q)$ the Clifford group and

$$\mathrm{Sn} : \mathbf{\Gamma}(V, q) \to \mathbb{G}_m$$

the spinor norm homomorphism [6, Ch. 6]. The image of Sn_F in $F^\times$ consists of the products of non-zero values of the quadratic form q. The norm principle for Sn is known as *Knebusch norm principle* [7, Ch. 7].

Example 3.3. (Scharlau norm principle) Let $\mathbf{GO}(V, q)$ be the group of similitudes and

$$\mu : \mathbf{GO}(V, q) \to \mathbb{G}_m$$

the multiplier homomorphism [6, Ch. 3]. The image of μ_F consists of all $x \in F^\times$ such that the forms q and xq are isomorphic. The norm principle for μ is known as *Scharlau norm principle* [7, Ch. 7].

4. The envelope of a semisimple algebraic group

Let G be a semisimple algebraic group over F. The center C of G is a finite algebraic group of multiplicative type. Consider an embedding $\rho : C \hookrightarrow S$ over F of C into a quasi-split torus S. We define *the envelope of* G *with respect to* ρ, denoted $e(G, \rho)$, to be the cofiber product of G and S over C:

$$\begin{array}{ccc} C & \longrightarrow & G \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ S & \longrightarrow & e(G, \rho), \end{array}$$

i.e. $e(G, \rho) = (G \times S)/C$. Thus, $e(G, \rho)$ is a reductive group with the semisimple part isomorphic to G and center isomorphic to S. An *envelope of* G is the group $e(G, \rho)$ for some ρ.

Let $\widehat{G}$ be a reductive group with the semisimple part G. Assume that the center S of $\widehat{G}$ is a quasi-split torus. The center C of G is the intersection of S

and G. Denote by ρ the embedding of C into S. Then clearly $\widehat{G}$ is isomorphic to the envelope $e(G,\rho)$.

Let E be an étale F-algebra and G a semisimple algebraic group over E with center C and embedding $\rho : C \hookrightarrow S$ for a quasi-trivial torus S over E. Then $R_{E/F}(C)$ is the center of the semisimple group $R_{E/F}(G)$, $R_{E/F}(\rho) : R_{E/F}(C) \to R_{E/F}(S)$ is an embedding into a quasi-trivial torus $R_{E/F}(S)$ and

$$e\big(R_{E/F}(G), R_{E/F}(\rho)\big) = R_{E/F}\big(e(G,\rho)\big).$$

In the examples below we consider "small" envelopes of some absolutely simple simply connected algebraic groups.

Example 4.1. (Type A_{n-1}) Let G be an absolutely simple simply connected algebraic groups of type A_{n-1}. Then G is the special unitary group $\mathbf{SU}(B,\tau)$, where B is a central simple algebra of degree n over a quadratic extension K/F with a unitary involution τ [6, Ch. 6]. Consider the group $\widehat{G} = \mathbf{GU}(B,\tau)$ of unitary similitudes [6, §23]. It is a reductive group with the semisimple part $G = \mathbf{SU}(B,\tau)$. The center of $\widehat{G}$ is the 2-dimensional quasi-split torus $R_{K/F}(\mathbb{G}_m)$. Hence, $\widehat{G}$ is an envelope of G.

Example 4.2. (Type B_n) Let G be an absolutely simple simply connected algebraic groups of type B_n. Then G is the spinor group $\mathbf{Spin}(V,q)$ of a non-degenerate quadratic form (V,q) of dimension $2n+1$ [6, Ch. 6]. Consider the even Clifford group $\widehat{G} = \mathbf{\Gamma}^+(V,q)$. It is a reductive group with the semisimple part $G = \mathbf{Spin}(V,q)$. The center of $\widehat{G}$ is the split torus $\mathbb{G}_m$. Hence, $\widehat{G}$ is an envelope of G.

Example 4.3. (Type C_n) Let G be an absolutely simple simply connected algebraic groups of type C_n. Then G is the symplectic group $\mathbf{Sp}(A,\sigma)$ of a central simple F-algebra A of degree $2n$ with a symplectic involution σ [6, Ch. 6]. Consider the group of symplectic similitudes $\widehat{G} = \mathbf{GSp}(A,\sigma)$ [6, §23]. It is a reductive group with the semisimple part $G = \mathbf{Sp}(A,\sigma)$. The center of $\widehat{G}$ is the split torus $\mathbb{G}_m$. Hence, $\widehat{G}$ is an envelope of G.

Example 4.4. (Type D_n) Let G be an absolutely simple simply connected algebraic groups of type D_n, $n \neq 4$. Then G is the spinor group $\mathbf{Spin}(A,\sigma,f)$ for a central simple F-algebra A of degree $2n$ with a quadratic pair (σ,f) [6, Ch. 6]. Consider the extended Clifford group $\widehat{G} = \mathbf{Omega}(A,\sigma,f)$ [6, §23]. It is a reductive group with the semisimple part $G = \mathbf{Spin}(A,\sigma,f)$. The center of $\widehat{G}$ is the 2-dimensional quasi-split torus $R_{Z/F}(\mathbb{G}_m)$ where Z/F is the discriminant quadratic extension. Hence, $\widehat{G}$ is an envelope of G.

Example 4.5. (Types G_2, F_4, E_8) Let G be an absolutely simple simply connected algebraic groups of type G_2, F_4 or E_8. The center of G is trivial, hence the group G can be chosen as an envelope of itself.

5. Reduction

In this section we reduce the statement of Theorem 1.1 to the case of an envelope of an absolutely simple simply connected group.

Proposition 5.1. *The norm principle for a reductive group G follows from the norm principle for any envelope of the semisimple part G' of G.*

Proof. Let $\widehat{G} = e(G', \rho)$ be an envelope of G', the semisimple part of G, where $\rho : C \to S$ is an embedding of the center C of G' into a quasi-trivial torus S. Denote by $\gamma : S \to \widehat{G}$ the natural embedding. Let T be the center of G (T is a group of multiplicative type, neither reduced, nor connected in general), and let $\widetilde{T}$ be the cofiber product of T and S over C:

$$\begin{array}{ccc} C & \longrightarrow & T \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\alpha} \\ S & \xrightarrow{\nu} & \widetilde{T}, \end{array}$$

i.e. $\widetilde{T} = (S \times T)/C$.

Consider the algebraic group $\widetilde{G}$ defined by the cofiber square

$$\begin{array}{ccc} G' \times T & \xrightarrow{\mu} & G \\ {\scriptstyle id\times\alpha}\downarrow & & \downarrow \\ G' \times \widetilde{T} & \xrightarrow{\varepsilon} & \widetilde{G}, \end{array}$$

where μ is the multiplication homomorphism. The cokernel of the embedding β is isomorphic to $\operatorname{Coker}(\alpha) \simeq \operatorname{Coker}(\rho)$ and hence is a torus. Therefore, $\widetilde{G}$ is a reductive group with the semisimple part isomorphic to G'.

Consider the following commutative diagram with exact rows and cofiber left square:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \xrightarrow{(\delta,\nu\rho)} & G' \times \widetilde{T} & \xrightarrow{\varepsilon} & \widetilde{G} & \longrightarrow & 1 \\ & & {\scriptstyle\rho}\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & S & \xrightarrow{(\gamma,\nu)} & \widehat{G} \times \widetilde{T} & \longrightarrow & \widetilde{G} & \longrightarrow & 1, \end{array}$$

where $\delta : C \to G'$ and $\eta : G' \to \widehat{G}$ are the canonical embeddings.

The norm principle for $\widehat{G}$ implies the norm principle for $\widetilde{G}$ by Lemma 2.5 applied to the quotient of $\widehat{G} \times \widetilde{T}$ by the quasi-trivial torus S. Finally, the norm principle for G follows from the norm principle for $\widetilde{G}$ by Corollary 2.4 applied to the embedding $\beta : G \hookrightarrow \widetilde{G}$. □

□

Proposition 5.2. *Let G be a reductive group over F. Assume that for any connected component Φ of the Dynkin diagram of G the norm principle for some envelope of any simply connected semisimple group (defined over a field* *extension of F) with the Dynkin diagram Φ holds. Then the norm principle holds for G.*

Proof. By [10, Lemma 7.6], there is an exact sequence

$$1 \to P \to K \to G \to 1,$$

where P is a quasi-trivial torus and K is a reductive group with simply connected semisimple part. By Lemma 2.5, the norm principle for G follows from the norm principle for K. Thus, we may assume (replacing G by K) that the semisimple part G' of G is simply connected. By [6, Th. 26.8], there is an étale F-algebra E and an absolutely simple simply connected group H' over E such that

$$G' = R_{E/F}(H').$$

Let H be an envelope of H'. By assumption, the norm principle holds for H. By a property of the envelopes, the group $G_1 = R_{E/F}(H)$ is an envelope of G'. The norm principle for G_1 holds by Lemma 2.6. Finally, Proposition 5.1 implies the norm principle for G. □ □

In order to proof Theorem 1.1, in view of Proposition 5.2, it is sufficient to prove the norm principle for some envelopes of all absolutely simple simply connected groups of types A_n, B_n, C_n, G_2, F_4 and E_8. As noticed in Example 4.5, for the exceptional types the envelopes can be chosen semisimple, hence the norm principle trivially holds. In the remaining sections we consider the classical types A_n, B_n and C_n.

6. Case A_{n-1}

By (4.1), an envelope $\widehat{G}$ of an absolutely simple simply connected algebraic group of type A_{n-1} can be chosen as the group of unitary similitudes $\mathbf{GU}(B,\tau)$, where B is a central simple algebra of degree n over a quadratic extension K/F with a unitary involution τ. Denote by T the torus $\mathbb{G}_m \times R_{K/F}(\mathbb{G}_m)$ and consider the homomorphism

$$\varphi : \mathbf{GU}(B,\tau) \to T$$

taking a similitude b to the pair $\big(\tau(b)b, \mathrm{Nrd}_B(b)\big)$. The kernel of φ is the special unitary group $\mathbf{SU}(B,\tau)$, the semisimple part of $\mathbf{GU}(B,\tau)$. Thus, it is sufficient to prove the norm principle for the homomorphism φ.

By Wedderburn theorem, $B = \mathrm{End}_D(V)$ for a central division algebra D over K and a vector space V over D. By [6, Th. 4.2], D has a unitary involution, say ρ, over K/F, and the involution τ is adjoint with respect to a Hermitian form h on V over (D,ρ). Thus, the group $\widehat{G}$ is the group of similitudes $\mathbf{GU}(V,h)$ of the Hermitian form (V,h).

Let L be an étale F-algebra, $b \in B_L = B \otimes_F L$ a similitude of $(V_L, h_L) = (V \otimes_F L, h \otimes_F L)$ with the multiplier $x = \tau(b)b \in L^\times$, i.e. b is an isometry between Hermitian forms (V_L, h_L) and (V_L, xh_L). The element $\varphi_L(b) \in T(L)$ is the pair

$$\big(x, \mathrm{Nrd}_{B_L}(b)\big) \in L^\times \times (K \otimes_F L)^\times.$$

We need to find a similitude b' of the Hermitian form (V,h) with the multiplier $N_{L/F}(x)$, i.e. an isometry b' between (V,h) and $(V, N_{L/F}(x)h)$, such that

$$\mathrm{Nrd}_B(b') = N_{K\otimes L/K}\big(\mathrm{Nrd}_{B_L}(b)\big).$$

Any D_L-endomorphism of V_L can be considered as just an D-endomorphism, i.e. we can consider B_L as a subalgebra in the central simple K-algebra $\widetilde{B} = \mathrm{End}_D(V_L)$. Clearly, B_L is the centralizer of L in $\widetilde{B}$. By the reduced tower formula [4, p. 150],

$$N_{K\otimes L/K}\big(\mathrm{Nrd}_{B_L}(b)\big) = \mathrm{Nrd}_{\widetilde{B}}(b). \tag{6.1}$$

Assume first that the degree $m = [L:F]$ is odd, $m = 2k+1$, and x generates L as F-algebra. Thus, the elements $1, x, x^2, \dots, x^{m-1}$ form a basis of L over F. Consider the F-linear map $s : L \to F$ given by $s(1) = 1$ and $s(x^i) = 0$ for $i = 1, 2, \dots, m-1$. We also denote by s the extended D-linear map $D_L \to D$. Composing h with s we get a Hermitian form h' on V_L over D. Clearly, h' is the tensor product of (V,h) and (L,p), where p is the bilinear form on L given by $p(y,z) = s(yz)$. The orthogonal complement of $(F, \langle 1\rangle)$ in (L,p) is the subspace with the basis $x, x^2, \dots, x^{m-1}$. The restriction of p on this space is metabolic since it contains a totally isotropic subspace generated by $x, x^2, \dots, x^k$. Thus, there is an isometry

$$\alpha : (L,p) \xrightarrow{\sim} (F, \langle 1\rangle) \perp (F^{2k}, p'),$$

where p' is a metabolic bilinear form. We have

$$\det(\alpha)^2 = \frac{\det p}{\det p'}. \tag{6.2}$$

Composing xh with s we get a Hermitian form h'' on V_L over D. Clearly, h'' is the tensor product of (V,h) and (L,q) where q is the bilinear form on L given by $q(y,z) = s(xyz)$. The subspace generated by $1, x, \dots, x^{k-1}$ is totally isotropic for q. Hence, (L,q) contains a metabolic subspace of dimension $2k$. As computed in [7, p. 196], the orthogonal complement of it is the 1-dimensional space $(F, \langle N_{L/F}(x)\rangle)$. Thus, there is an isometry

$$\beta : (L,q) \xrightarrow{\sim} (F, \langle N_{L/F}(x)\rangle) \perp (F^{2k}, q'),$$

where q' is a metabolic bilinear form. We have

$$\det(\beta)^2 = \frac{\det q}{N_{L/F}(x)\det q'}. \tag{6.3}$$

An easy computation shows that

$$\det(q) = N_{L/F}(x)\cdot \det(p).$$

Hence, (6.2) and (6.3) imply

$$\left(\frac{\det\beta}{\det\alpha}\right)^2 = \frac{\det p'}{\det q'}. \tag{6.4}$$

Note that if $\mathrm{char}(F) \neq 2$, the forms p' and q' are hyperbolic and hence are isomorphic. If $\mathrm{char}(F) = 2$, the forms p' and q' may not be isomorphic.

Nevertheless, after tensoring with h these forms become isomorphic. We will need a more precise statement.

Lemma 6.5. *There is an isometry*

$$c : (V^{2k}, h \otimes p') \to^{\sim} (V^{2k}, h \otimes q')$$

such that $\mathrm{Nrd}(c) = f^n$ *and* $\frac{\det p'}{\det q'} = f^2$ *for some* $f \in F^{\times}$.

Proof. Let $\{e_i, f_i\}$, $i = 1, 2, \ldots, k$ be a metabolic basis of (F^{2k}, p'), i.e. $p'(e_i, e_j) = 0$, $p'(e_i, f_j) = \delta_{ij}$ for all i, j, and $p'(f_i, f_j) = 0$ for $i \neq j$. Choose also a metabolic basis $\{e_i', f_i'\}$ of (F^{2k}, q'). If a is the linear automorphism of F^{2k} taking e_i to f_i and e_i' to f_i', then $\det(a)^2 = \frac{\det p'}{\det q'}$.

By [6, Prop. 2.17], any ρ-symmetric element of D is of the form $\rho(d) + d$ for some $d \in D$. Choose elements $d_i \in D$ such that

$$p'(f_i, f_i) = q'(f_i', f_i') + \rho(d_i) + d_i$$

for any i. Let v_j, $j = 1, 2, \ldots, l$, be a D-basis of V. Define an automorphism c of $V^{2k} = V \otimes_F F^{2k}$ over D by

$$c(v_j \otimes e_i) = v_j \otimes e_i',$$
$$c(v_j \otimes f_i) = v_j \otimes f_i' + d_i v_j \otimes e_i'.$$

Clearly, c is an isometry between $(V^{2k}, h \otimes p')$ and $(V^{2k}, h \otimes q')$. Finally, $\mathrm{Nrd}(c) = \det(a)^n$ and we can set $f = \det(a)$. □ □

Choose an isometry c and $f \in F^{\times}$ as in Lemma 6.5. It follows from (6.4) that $\det \beta = \pm f \cdot \det \alpha$. Modifying α by an isometry (reflection) of (L, p) we may assume that

$$f = \frac{\det \beta}{\det \alpha}. \tag{6.6}$$

Consider the composition

$$b_1 : (V, h) \perp (V^{2k}, h \otimes p') \to^{\mathrm{id} \otimes \alpha^{-1}} (V_L, h') \to^{b} (V_L, h'') \to^{\mathrm{id} \otimes \beta} (V, N_{L/F}(x)h) \perp (V^{2k}, h \otimes q')$$

as an automorphism of the space V^m over D. Then, by Lemma 6.5, (6.1) and (6.6),

$$\mathrm{Nrd}(b_1) = \mathrm{Nrd}_{\widetilde{B}}(b) \cdot \left(\frac{\det \beta}{\det \alpha}\right)^n = N_{K \otimes L/K}\big(\mathrm{Nrd}_{B_L}(b)\big) \cdot \mathrm{Nrd}(c). \tag{6.7}$$

By [3, Ch IX, §4, Th.1], the isometry c can be extended to an isometry

$$b_2 : (V, h) \perp (V^{2k}, h \otimes p') \to^{\sim} (V, N_{L/F}(x)h) \perp (V^{2k}, h \otimes q')$$

which must be of the form $b_2 = b'' \perp c$ for some isometry

$$b'' : (V, h) \to^{\sim} (V, N_{L/F}(x)h).$$

Thus,

$$\mathrm{Nrd}(b_2) = \mathrm{Nrd}_B(b'') \cdot \mathrm{Nrd}(c).$$

Hence, by (6.7),

$$\mathrm{Nrd}_B(b'') = \mathrm{Nrd}(u) \cdot N_{K\otimes L/K}\big(\mathrm{Nrd}_{B_L}(b)\big) \tag{6.8}$$

for the isometry $u = b_2 b_1^{-1}$ of the space $(V, N_{L/F}(x)h) \perp (V^{2k}, h \otimes q')$.

By [9, Prop. 6.1], the group of reduced norms of isometries of a Hermitian form over D depends only on D and does not depend on the space of the form (a characteristic free proof is given in [1, Th.5.1.3]). Hence, there is an isometry u' of the form (V, h) such that $\mathrm{Nrd}_B(u') = \mathrm{Nrd}(u)$. Finally, we set $b' = b'' \cdot (u')^{-1}$. By (6.8), b' is a desired isometry between (V, h) and $(V, N_{L/F}(x)h)$ such that

$$\mathrm{Nrd}_B(b') = N_{K\otimes L/K}\big(\mathrm{Nrd}_{B_L}(b)\big).$$

We will need the following

Lemma 6.9. *Let L be an étale algebra over an infinite field F, let H be a reductive algebraic group over F and $\psi : H \to R_{L/F}(\mathbb{G}_m)$ a surjective algebraic group homomorphism. Denote by X the set of all $h \in H(F)$ such that the element $x = \psi_F(h) \in L^\times$ generates L as F-algebra. Then X generates the group $H(F)$.*

Proof. There is a non-empty open subset $U \subset R_{L/F}(\mathbb{G}_m)$ defined over F such that an element x belongs to $U(F)$ if and only if x generates L as F-algebra. Denote by U' the open set $\psi^{-1}(U) \subset H$ so that $U'(F) = X$. The set U' is non-empty since g is surjective. Let $h \in H(F)$; set $W = hU' \cap U'$. The open set W is non-empty (H is connected). Since F is infinite, the group $H(F)$ is dense in H [2, Cor. 18.3], and therefore $W(F)$ is non-empty. Choose $w \in W(F)$. Then $w = hu \in U'(F) = X$ for some $u \in U'(F)$. Finally, $h = wu^{-1}$ and $u, w \in X$. □ □

Assume now that the degree $[L : F]$ is still odd but x may not generate L as F-algebra. Consider the corestriction

$$\psi : R_{L/F}(\widehat{G}_L) \to R_{L/F}(\mathbb{G}_m)$$

of the composition of $\varphi_L : \widehat{G}_L \to T_L$ with the first projection $T_L \to \mathbb{G}_m$ and the subset

$$X \subset R_{L/F}(\widehat{G}_L)(F) = \widehat{G}(L) = \mathrm{GU}(B_L, \tau_L)$$

consisting of all b such that the element $x = \psi_F(b) = \tau(b)b$ generates L as F-algebra. In the first part of the proof the norm principle was proven for all elements in X. By Example 2.2, we may assume that the field F is infinite. Lemma 6.9, applied to ψ, shows that X generates the group $\widehat{G}(L)$, whence the norm principle.

Finally, assume that the degree $[L : F]$ is even. The norm principle in this case follows from the norm principle for the étale algebra $F \times L$ of odd degree over F.

7. Case B_n

By (4.2), an envelope $\widehat{G}$ of an absolutely simple simply connected algebraic group of type B_n can be chosen as the even Clifford group $\widehat{G} = \mathbf{\Gamma}^+(V,q)$ of a non-degenerate quadratic form (V,q) of dimension $2n+1$. The kernel of the spinor norm homomorphism

$$\varphi : \mathbf{\Gamma}^+(V,q) \to \mathbb{G}_m$$

is the semisimple part $\mathbf{Spin}(V,q)$ of $\mathbf{\Gamma}^+(V,q)$. Thus, it is sufficient to prove the norm principle for the universal homomorphism φ. It follows from Knebusch norm principle (Example 3.2). The proof is given in [7, Ch. 7] (it is valid also if $\mathrm{char}(F) = 2$).

8. Case C_n

By (4.3), an envelope $\widehat{G}$ of an absolutely simple simply connected algebraic group of type C_n can be chosen as the group of symplectic similitudes $\mathbf{GSp}(A,\sigma)$ for a central simple F-algebra A of degree $2n$ with a symplectic involution σ. The kernel of the multiplier homomorphism

$$\mu : \mathbf{GSp}(A,\sigma) \to \mathbb{G}_m, \qquad \mu(a) = \sigma(a)a$$

is the semisimple part $\mathbf{Sp}(A,\sigma)$ of $\mathbf{GSp}(A,\sigma)$. Thus, it is sufficient to prove the norm principle for the universal homomorphism μ.

By Wedderburn theorem, $A = \mathrm{End}_D(V)$ for a central division algebra D over F and a vector space V over D. By [6, Th. 4.2], D has an involution of the first kind, say ρ, and the involution σ is adjoint with respect to an ε-Hermitian form h on V over D. Thus, the group $\widehat{G}$ is the group of similitudes $\mathbf{GSp}(V,h)$ of the ε-Hermitian form (V,h).

Let L be an étale F-algebra, $a \in A_L$ a similitude of (V_L,h_L) with the multiplier $x = \sigma(a)a \in L^\times$, i.e. a is an isometry between ε-Hermitian forms (V_L,h_L) and (V_L,xh_L). We need to find a similitude a' of the ε-Hermitian form (V,h) with the multiplier $N_{L/F}(x)$, i.e. an isometry between (V,h) and $(V,N_{L/F}(x)h)$.

Assume first that the degree $m = [L:F]$ is odd, $m = 2k+1$, and x generates L as F-algebra. As in Section 6, we get an isometry

$$(V,h) \perp (V^{2k}, h \otimes p') \to^{\sim} (V, N_{L/F}(x)h) \perp (V^{2k}, h \otimes q')$$

where p' and q' are metabolic bilinear forms over F. Since the involution σ is symplectic, any metabolic ε-Hermitian form is hyperbolic. In particular, the forms $(V^{2k}, h\otimes p')$ and $(V^{2k}, h\otimes q')$ are isomorphic. By the cancelation property [5, Cor. 6.4.2], there is an isometry b' between (V,h) and $(V,N_{L/F}(x)h)$.

Assume now that the degree $[L:F]$ is still odd but x may not generate L as F-algebra. Consider the corestriction

$$\psi : R_{L/F}(\widehat{G}_L) \to R_{L/F}(\mathbb{G}_m)$$

of the homomorphism μ_L and the subset

$$X \subset R_{L/F}(\widehat{G}_L)(F) = \widehat{G}(L) = \mathrm{GSp}(A_L, \sigma_L)$$

consisting of all a such that the element $x = \psi_F(a) = \sigma(a)a$ generates L as F-algebra. By Example 2.2 we may assume that the field F is infinite. Lemma 6.9, applied to ψ, shows that X generates the group $\widehat{G}(L)$, whence the norm principle.

Finally assume that the degree $[L : F]$ is even. The norm principle in this case follows from the norm principle for the étale algebra $F \times L$ of odd degree over F.

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