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Normal almost contact metric manifolds of dimension three

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Abstract. One of the important classes of almost contact metric (shortly a.c.m.) manifolds is the class consisting of those which are normal. However, the curvature nature of such manifolds is not known in general, except for Sasakian or cosymplectic manifolds. Recently the author derived fundamental curvature identities for quasi-Sasakian manifolds (cf. [6]).

The present paper is devoted to normal a.c.m. structures on a 3-dimensional manifold (denote such a manifold by M). In Section 2 we derive certain necessary and sufficient conditions for an a.c.m. structure on M to be normal and we point out some of their consequences. In Section 3 we completely characterize the local nature of normal a.c.m. structures on M by giving suitable examples. Section 4 concerns the curvature properties of such structures; we prove that they are η-Einstein and in the remaining part of that section we study normal a.c.m. structures on a manifold of constant curvature. We give certain new examples of such structures of rank 1. We also prove that if M is compact, then all such structures are quasi-Sasakian.

1. Preliminaries. Let M be an almost contact manifold and (φ, ξ, η) its almost contact structure. This means, M is an odd-dimensional differentiable manifold and φ, ξ, η are tensor fields on M of types (1, 1), (1, 0), (0, 1), respectively, such that

$$\varphi^2 = -I + \xi\otimes\eta, \quad \eta(\xi) = 1.$$

Then also $\varphi\xi = 0$, $\eta\circ\varphi = 0$. Let $\boldsymbol{R}$ be the real line and t a coordinate on $\boldsymbol{R}$. Define an almost complex structure J on $M\times\boldsymbol{R}$ by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta(X)\frac{d}{dt}\right),$$

where the pair $(X, \lambda d/dt)$ denotes a tangent vector to $M\times\boldsymbol{R}$, X and $\lambda d/dt$ being tangent to M and $\boldsymbol{R}$, respectively.

M and (φ, ξ, η) are said to be *normal* if the structure J is integrable (cf. e.g. [1]). The necessary and sufficient condition for (φ, ξ, η) to be normal is

$$[\varphi, \varphi] + 2\xi\otimes d\eta = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2 [X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

for any $X, Y \in \mathscr{X}(M)$, $\mathscr{X}(M)$ being the Lie algebra of vector fields on M.

We say that the form η has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say that r is the rank of the structure (φ, ξ, η).

A Riemannian metric g on M satisfying the condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \mathscr{X}(M)$, is said to be *compatible* with the structure (φ, ξ, η). If g is such a metric, then the quadruple (φ, ξ, η, g) is called an *almost contact metric* (shortly a.c.m.) *structure* on M and M an *almost contact metric* (shortly a.c.m.) *manifold*. On such a manifold we also have $\eta(X) = g(X, \xi)$ for any $X \in \mathscr{X}(M)$ and we can always define the 2-form Φ by $\Phi(X, Y) = g(X, \varphi Y)$, where $X, Y \in \mathscr{X}(M)$.

It is not hard to see that if $\dim M = 3$, then two Riemannian metrics g and g' are compatible with the same almost contact structure (φ, ξ, η) on M if and only if $g' = \sigma g + (1-\sigma)\eta \otimes \eta$, for a certain positive function σ on M.

AGREEMENT. Through the rest of this paper M always denotes a 3-dimensional differentiable manifold.

2. Preliminary results.

PROPOSITION 1. *For an a.c.m. structure (φ, ξ, η, g) on M we have*

$$(1) \qquad (\nabla_X \varphi) Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi,$$

$$(2) \qquad d\Phi = (\operatorname{div} \xi)\eta \wedge \Phi,$$

$$(3) \qquad d\eta = \eta \wedge (\nabla_\xi \eta) + \tfrac{1}{2}(\operatorname{tr}(\varphi \nabla \xi))\Phi,$$

where ∇ is the Levi-Civita connection on M, $\operatorname{div} \xi$ is the divergence of ξ defined by $\operatorname{div} \xi = \operatorname{trace}\{X \to \nabla_X \xi\}$ and $\operatorname{tr}(\varphi \nabla \xi) = \operatorname{trace}\{X \to \varphi \nabla_X \xi\}$.

Proof. Since $\eta \wedge \Phi$ is up to a constant factor the volume element on M, it is parallel with respect to ∇, i.e., $\nabla_X(\eta \wedge \Phi) = 0$. Hence

$$(\nabla_X \eta)(Y)\Phi(Z, W) + \eta(Y)(\nabla_X \Phi)(Z, W) + (\nabla_X \eta)(Z)\Phi(W, Y) +$$
$$+ \eta(Z)(\nabla_X \Phi)(W, Y) + (\nabla_X \eta)(W)\Phi(Y, Z) + \eta(W)(\nabla_X \Phi)(Y, Z) = 0.$$

The last relation for $W = \xi$ gives

$$\begin{aligned}(\nabla_X \Phi)(Z, Y) &= -\eta(Z)(\nabla_X \Phi)(Y, \xi) + \eta(Y)(\nabla_X \Phi)(Z, \xi)\\ &= \eta(Z) g(\varphi \nabla_X \xi, Y) - \eta(Y) g(\varphi \nabla_X \xi, Z),\end{aligned}$$

which leads to (1). To prove (2), we note that $d\Phi = \sigma\eta \wedge \Phi$ for a certain function σ on M. Taking a local φ-basis $\{E_0, E_1, E_2\}$ (i.e., an orthonormal frame such that $E_0 = \xi$ and $E_2 = \varphi E_1$) we find

$$3(\eta \wedge \Phi)(E_0, E_1, E_2) = -1,$$

$$3(d\Phi)(E_0, E_1, E_2) = (\nabla_{E_0}\Phi)(E_1, E_2)+(\nabla_{E_1}\Phi)(E_2, E_0)+(\nabla_{E_2}\Phi)(E_0, E_1)$$
$$= -g(\nabla_{E_1}E_0, E_1)-g(\nabla_{E_2}E_0, E_2) = -\operatorname{div}\xi,$$

which implies $\sigma = \operatorname{div}\xi$. Finally, by elementary argumentations we get $d\eta = \eta \wedge \omega+\varrho\Phi$, where ω is a 1-form orthogonal to η, i.e., $\omega(\xi) = 0$, and ϱ is a scalar function. Consequently

$$\omega(X) = 2(\eta \wedge \omega+\varrho\Phi)(\xi, X) = 2d\eta(\xi, X) = (\nabla_\xi \eta)(X),$$

$$2\varrho = 2(\eta \wedge \omega+\varrho\Phi)(E_2, E_1) = 2d\eta(E_2, E_1) = (\nabla_{E_2}\eta)(E_1)-(\nabla_{E_1}\eta)(E_2)$$
$$= g(\varphi\nabla_{E_1}\xi, E_1)+g(\varphi\nabla_{E_2}\xi, E_2) = \operatorname{trace}\{X \to \varphi\nabla_X\xi\},$$

completing the proof of (3).

PROPOSITION 2. *For an a.c.m. structure* (φ, ξ, η, g) *on* M *the following conditions are mutually equivalent:*

(a) *the structure* (φ, ξ, η) *is normal,*

(b) $\nabla_{\varphi X}\xi = \varphi\nabla_X\xi$,

(c) $\nabla_X\xi = \alpha\{X-\eta(X)\xi\}-\beta\varphi X$,

where $2\alpha = \operatorname{div}\xi$ *and* $2\beta = \operatorname{tr}(\varphi\nabla\xi)$.

Proof. As it is known (cf. [7], p. 171), an a.c.m. structure is normal if and only if

$$\varphi(\nabla_X\varphi)Y-(\nabla_{\varphi X}\varphi)Y-(\nabla_X\eta)(Y)\xi = 0.$$

With the use of (1) we can easily see that in dimension 3 the above condition is equivalent to (b). It is also clear that (c)$\Rightarrow$(b). To prove (b)$\Rightarrow$(c), take a φ-basis $\{E_0, E_1, E_2\}$. Then by (b) we obtain $\nabla_{E_0}\xi = 0$, $\nabla_{E_1}\xi = \alpha E_1-\beta E_2$ and $\nabla_{E_2}\xi = \beta E_1+\alpha E_2$ for certain α, β. This gives our assertion.

COROLLARY 1. *For a normal a.c.m. structure* (φ, ξ, η, g) *on* M *we have* $\nabla_\xi\xi = 0$ *and* $d\eta = \beta\Phi$, *where* $2\beta = \operatorname{tr}(\varphi\nabla\xi)$. *In particular, the rank of such a structure cannot be* 2.

Remark 1. A normal a.c.m. manifold for which $d\Phi = 2\eta \wedge \Phi$ is said to be a *Kenmotsu manifold* (cf. [4], also [5]). Any Kenmotsu manifold fulfils the relation $\nabla_X\xi = X-\eta(X)\xi$. By Propositions 1 and 2 it can be seen that the above relation is a sufficient condition for an a.c.m. manifold of dimension 3 to be Kenmotsu.

PROPOSITION 3. *Let* (φ, ξ, η, g) *be a normal a.c.m. structure on* M. *Suppose additionally*

(4) $$\varphi' = \varphi, \quad \xi' = \varepsilon\xi, \quad \eta' = \varepsilon\eta, \quad g' = \sigma g+(1-\sigma)\eta\otimes\eta,$$

where $\varepsilon = +1$ *or* -1 *and* σ *is a positive function on* M. *Then* $(\varphi', \xi', \eta', g')$ *is a normal a.c.m. structure on* M, $\operatorname{rank}\eta' = \operatorname{rank}\eta$. *And for functions* α, β, α', β' *defined by* $d\Phi = 2\alpha\eta \wedge \Phi$, $d\eta = \beta\Phi$, $d\Phi' = 2\alpha'\eta' \wedge \Phi'$, $d\eta' = \beta'\Phi'$ *we have*

$$(5)\quad \text{(a)}\quad 2\alpha' = \varepsilon(2\alpha + \xi \log\sigma), \qquad \text{(b)}\quad \beta' = \varepsilon\beta/\sigma,$$

where Φ' *is a 2-form given by* $\Phi'(X, Y) = g'(X, \varphi'Y)$.

Proof. The first two parts of our assertion can be easily obtained by straightforward verification. To see (5), note that $\Phi' = \sigma\Phi$. Consequently, $\beta'\Phi' = d\eta' = \varepsilon d\eta = \varepsilon\beta\Phi = \varepsilon\frac{\beta}{\sigma}\Phi'$, which gives (5) (b). Now take a φ-basis $\{E_0, E_1, E_2\}$ for the structure (φ, ξ, η, g), and its dual basis of 1-forms $\{E_0^* = \eta, E_1^*, E_2^*\}$. Then $\Phi = 2E_2^* \wedge E_1^*$ and $d\sigma = (\xi\sigma)\eta + (E_1\sigma)E_1^* + (E_2\sigma)E_2^*$. Thus, necessarily $d\sigma \wedge \Phi = (\xi\sigma)\eta \wedge \Phi$ and consequently

$$2\alpha'\eta' \wedge \Phi' = d\Phi' = d\sigma \wedge \Phi + \sigma d\Phi = (\xi\sigma + 2\alpha\sigma)\eta \wedge \Phi = \varepsilon(\xi\sigma/\sigma + 2\alpha)\eta' \wedge \Phi',$$

which implies (5) (a).

3. A normal a.c.m. structure (φ, ξ, η, g) satisfying additionally the condition $d\eta = \Phi$ is called *Sasakian*. Of course, any such structure on M has rank 3.

THEOREM 1. *Let* (φ, ξ, η, g) *be a Sasakian structure on* M. *Then an a.c.m. structure* $(\varphi', \xi', \eta', g')$ *given by* (4) *is normal and is of rank* 3. *Conversely, if* $(\varphi', \xi', \eta', g')$ *is a normal a.c.m. structure of rank* 3 *on* M, *then there is a Sasakian structure* (φ, ξ, η, g) *and a function* σ *on* M *such that the two structures are related by* (4).

Proof. We must only prove the second statement. Let $(\varphi', \xi', \eta', g')$ be a normal a.c.m. structure of rank 3 on M. Then $d\eta' = \beta\Phi' \neq 0$ at every point of M with $2\beta = \operatorname{tr}(\varphi' \nabla' \xi')$. Suppose $\varepsilon = \operatorname{sign}\beta$ and

$$\varphi = \varphi', \quad \xi = \varepsilon\xi', \quad \eta = \varepsilon\eta', \quad g = \frac{1}{\varepsilon\beta}g' + \left(1 - \frac{1}{\varepsilon\beta}\right)\eta' \otimes \eta'.$$

In view of Proposition 3 we have $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$. Thus the structure (φ, ξ, η, g) is Sasakian. Finally, we can easily verify that the two structures are related by (4) with $\varepsilon = \operatorname{sign}\beta$ and $\sigma = \varepsilon\beta$.

Remark 2. A normal a.c.m. structure satisfying additionally the condition $d\Phi = 0$ is said to be *quasi-Sasakian* (cf. [2], [3]). In virtue of Propositions 1 and 2 we can state the following: an a.c.m. structure (φ, ξ, η, g) on M is quasi-Sasakian if and only if it fulfils $\nabla_X \xi = -\beta\varphi X$ for some function β on M. Clearly, a quasi-Sasakian structure of rank 1 is cosymplectic. By Theorem 1 and Proposition 3 we have the following statement: any quasi-Sasakian structure of rank 3 on M can be obtained from a Sasakian structure by some deformation of the form (4), and conversely, a deformation of a Sasakian structure given by (4) with $\xi\sigma = 0$ leads to a quasi-Sasakian structure of rank 3.

Before we describe the structure of normal a.c.m. manifolds of rank 1 and dimension 3, we examine the following example.

EXAMPLE 1. Let $N = L \times V$, where L is the circle S^1 or an open interval (a, b), $-\infty \leqslant a < b \leqslant +\infty$, and V is a Kählerian manifold of dimension 2 with (J, G) as its Kählerian structure. Let E be a nowhere vanishing vector field on L, E^* its dual field of 1-forms and σ a positive function on N. Denote by (X_0, X_1) a tangent vector to N, where X_0, X_1 are tangent to L, V respectively. Assume

$$\begin{aligned} &\varphi(X_0, X_1) = (0, JX_1), \qquad \xi = (E, 0), \qquad \eta = (E^*, 0), \\ &g\big((X_0, X_1), (Y_0, Y_1)\big) = E^*(X_0)E^*(Y_0) + \sigma G(X_1, Y_1). \end{aligned} \tag{6}$$

Then it is a matter of an easy straightforward verification that (φ, ξ, η, g) is a normal a.c.m. structure of rank 1 on N.

THEOREM 2. *Let (φ, ξ, η, g) be a normal a.c.m. structure of rank 1 on M. Then for any point $p \in M$, some neighbourhood U of p is identified with the product $(a, b) \times V$, where (a, b) is an open interval and V is a Kählerian manifold of dimension 2, on which the structure (φ, ξ, η, g) is given as in Example 1 with $L = (a, b)$.*

Proof. By our assumption and Proposition 1 we have $d\eta = 0$ on M and $d\Phi = 2\alpha\eta \wedge \Phi$, where $2\alpha = \operatorname{div}\xi$. Fix a point $p \in M$. There is a neighbourhood U' of p on which $\alpha = \xi \log\sqrt{\sigma}$, for a certain positive function σ. Assume $g' = \dfrac{1}{\sigma}g + \left(1 - \dfrac{1}{\sigma}\right)\eta \otimes \eta$. By Proposition 3, (φ, ξ, η, g') is a normal a.c.m. structure of rank 1 on U' satisfying $d\eta = 0$, $d\Phi' = 0$. Thus (φ, ξ, η, g') is a cosymplectic structure on U'. Therefore the point p has a neighbourhood $U = (a, b) \times V \subset U'$. The structure (φ, ξ, η, g') is given on U by

$$\begin{aligned} &\varphi(X_0, X_1) = (0, JX_1), \qquad \xi = (E, 0), \qquad \eta = (E^*, 0), \\ &g'\big((X_0, X_1), (Y_0, Y_1)\big) = E^*(X_0)E^*(Y_0) + G(X_1, Y_1), \end{aligned}$$

where (J, G) is a Kählerian structure on V, E is a nowhere vanishing vector field on (a, b) and E^* its dual. Finally, since $g = \sigma g' + (1 - \sigma)\eta \otimes \eta$, we see that the structure (φ, ξ, η, g) is of the form (6) on U.

It seems to be of interest to know under what conditions we have a global decomposition of the form as in Example 1. A partial answer to this question is given by the following theorem.

THEOREM 3. *Let (φ, ξ, η, g) be a normal a.c.m. structure of rank 1 on M. Suppose additionally that $d(\operatorname{div}\xi)$ is linearly dependent of η (or equivalently, $(\operatorname{div}\xi)\eta$ is a closed 1-form) and M is simply connected and complete. Then*

$M = \mathbf{R} \times V$, where $\mathbf{R}$ is the real line, V is a Kählerian manifold of dimension 2 *and the structure (φ, ξ, η, g) is given on M as in Example* 1 *with $L = \mathbf{R}$.*

Proof. Under our assumptions we have $(\operatorname{div} \xi)\eta = d \log \sigma$ for a certain positive function σ on M. Let $g' = \frac{1}{\sigma} g + \left(1 - \frac{1}{\sigma}\right) \eta \otimes \eta$. By Proposition 3, (φ, ξ, η, g') is a normal a.c.m. structure of rank 1 on M. Moreover, by Proposition 1 and $\sigma \Phi' = \Phi$, we have

$$(\operatorname{div} \xi)\eta \wedge \Phi = d\Phi = d\sigma \wedge \Phi' + \sigma d\Phi' = \sigma (\operatorname{div} \xi)\eta \wedge \Phi' + \sigma d\Phi'$$
$$= (\operatorname{div} \xi)\eta \wedge \Phi + \sigma d\Phi',$$

whence $d\Phi' = 0$. So (φ, ξ, η, g') is a cosymplectic structure on M. Taking into account the topological assumptions we obtain $M = \mathbf{R} \times V$, where V is a 2-dimensional Kählerian manifold. Argument concerning the structure (φ, ξ, η, g) is the same as in the proof of Theorem 2.

In the above considerations only normal a.c.m. manifolds of pure rank 1 or 3 have appeared. However, there are such manifolds having rank 3 only on subsets, as it is illustrated, by the following example.

Example 2. Let N, L, V, J, G, E, E^* be as in Example 1. Let σ be a positive function on N and ω a 1-form on V. Then $d\omega = \bar{\beta}\Omega$, where $\bar{\beta}$ is a function on V and Ω the fundamental 2-form of the Kählerian structure defined by $\Omega(X_1, Y_1) = G(X_1, JY_1)$. Define

$$\varphi(X_0, X_1) = (-\omega(JX_1)E, JX_1), \qquad \xi = (E, 0), \qquad \eta = (E^*, \omega),$$
$$g((X_0, X_1), (Y_0, Y_1)) = \{E^*(X_0) + \omega(X_1)\}\{E^*(Y_0) + \omega(Y_1)\} + \sigma G(X_1, Y_1). \tag{7}$$

By straightforward verification we can see that (φ, ξ, η, g) is an a.c.m. structure on N. To prove the normality of (φ, ξ, η), we take the metric $g' = \frac{1}{\sigma} g + \left(1 - \frac{1}{\sigma}\right) \eta \otimes \eta$ as a new metric compatible with this structure. We shall show that $\nabla'_X \xi = -\beta\varphi X$, where ∇' is the Levi-Civita connection of g' and β is the function on N defined by $\beta(t, q) = \bar{\beta}(q)$ for any $(t, q) \in L \times V = N$. On account of Proposition 2 this fact suffices for the normality of (φ, ξ, η). So, using the explicit formula for ∇' we get

$$2g'(\nabla'_{(\lambda E, X_1)} \xi, (\mu E, Y_1)) = X_1 \omega(Y_1) - Y_1 \omega(X_1) - \omega([X_1, Y_1])$$
$$= 2d\omega(X_1, Y_1) = 2\beta\Omega(X_1, Y_1) = -2\beta G(JX_1, Y_1)$$
$$= -2\beta g'(\varphi(\lambda E, X_1), (\mu E, Y_1)),$$

where X_1, Y_1 are vector fields on V and λ, μ arbitrary constants. This implies $\nabla'_X \xi = -\beta\varphi X$. Now, in virtue of Propositions 1–3, for the structure (φ, ξ, η, g) we obtain $d\eta = \frac{\beta}{\sigma} \Phi$ and $d\Phi = (\xi \log \sigma)\eta \wedge \Phi$. Therefore rank η

$= 3$ on the set $U = \{q \in N |\ \beta(q) \neq 0\}$ and $\operatorname{rank} \eta = 1$ on $N \setminus U$. Note also that the structure (φ, ξ, η, g) is quasi-Sasakian if $\xi\sigma = 0$.

Now we prove that the converse holds locally.

THEOREM 4. *Let (φ, ξ, η, g) be a normal a.c.m. structure on M. Then, for any point $p \in M$, some neighbourhood U of p is identified with the product $(-a, a) \times V$, where $(-a, a)$ is an open interval and V is a Kählerian manifold of dimension 2, on which the structure (φ, ξ, η, g) is given as in Example 2.*

Proof. Fix a point $p \in M$. Let U' be a neighbourhood of p and σ a function on U' such that $\operatorname{div} \xi = \xi \log \sigma$. Assume $g' = \frac{1}{\sigma} g + \left(1 - \frac{1}{\sigma}\right) \eta \otimes \eta$. On account of Propositions 1 and 3, (φ, ξ, η, g) is a normal a.c.m. structure on U' and $d\Phi' = 0$. Hence this structure is quasi-Sasakian and consequently (cf. [2])

$$\mathscr{L}_\xi g' = 0, \qquad \mathscr{L}_\xi \eta = 0, \qquad \mathscr{L}_\xi \varphi = 0. \tag{8}$$

A certain coordinate neighbourhood of p, say U, with coordinates (x^0, x^1, x^2), is the product $(-a, a) \times V$ such that $\xi = \partial/\partial x^0$ on U, x^0 is the coordinate on $(-a, a)$ and (x^1, x^2) are the coordinates on V. Let g'_{ij}, η_i, φ^i_j be the components of g', η, φ in the coordinates (x^0, x^1, x^2). The Latin (resp. Greek) indices take on values 0, 1, 2 (resp. 1, 2). Thus, assuming the summation convention, we have

$$\begin{gathered} \varphi^i_a \varphi^a_j = -\delta^i_j + \delta^i_0 \eta_j, \qquad \eta_0 = 1 = g'_{00}, \qquad \eta_\alpha = g'_{0\alpha}, \\ \varphi^0_\alpha + \eta_1 \varphi^1_\alpha + \eta_2 \varphi^2_\alpha = 0, \qquad \varphi^i_0 = 0, \qquad g'_{ab} \varphi^a_i \varphi^b_j = g'_{ij} - \eta_i \eta_j. \end{gathered} \tag{9}$$

By (8) the components g'_{ij}, η_i, φ^i_j are independent of the coordinate x^0. Therefore they can be used to a description of an almost Hermitian structure on V. For, define

$$J\partial/\partial x^\alpha = \varphi^1_\alpha \partial/\partial x^1 + \varphi^2_\alpha \partial/\partial x^2, \qquad G(\partial/\partial x^\alpha, \partial/\partial x^\beta) = g'_{\alpha\beta} - \eta_\alpha \eta_\beta. \tag{10}$$

Using relations (9) one verifies that (J, G) is such a structure on V. And since $\dim V = 2$, it is Kählerian. Define additionally $\omega(\partial/\partial x^\alpha) = \eta_\alpha$, $E = \partial/\partial x^0$, $E^* = dx^0$. Knowing (9), (10) and $g = \sigma g' + (1-\sigma)\eta \otimes \eta$, we verify that our structure (φ, ξ, η, g) is of the form (7) on U.

4. The aim of this section is to find curvature properties of a normal a.c.m. structure on M. Let (φ, ξ, η, g) be such a structure. By Proposition 2 we have

$$\nabla_Y \xi = \alpha\{Y - \eta(Y)\xi\} - \beta\varphi Y, \tag{11}$$

which applied to (1) gives

$$(\nabla_X \varphi) Y = \{\beta g(\varphi X, \varphi Y) - \alpha g(X, \varphi Y)\} \xi + \eta(Y)\{\beta\varphi^2 X - \alpha\varphi X\}. \tag{12}$$

Differentiating (11) covariantly and using (12) we find

$$\nabla_X \nabla_Y \xi = \alpha\{\nabla_X Y - \eta(\nabla_X Y)\xi\} - \beta\varphi\nabla_X Y - (\alpha^2+\beta^2)g(\varphi X, \varphi Y) - \\ -(X\alpha)\varphi^2 Y - (X\beta)\varphi Y + (\alpha^2-\beta^2)\eta(Y)\varphi^2 X + 2\alpha\beta\eta(Y)\varphi X.$$

Therefore, for the curvature transformation $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ we obtain

$$(13)\quad R_{XY}\xi = \{Y\alpha + (\alpha^2-\beta^2)\eta(Y)\}\varphi^2 X - \{X\alpha + (\alpha^2-\beta^2)\eta(X)\}\varphi^2 Y + \\ + \{Y\beta + 2\alpha\beta\eta(Y)\}\varphi X - \{X\beta + 2\alpha\beta\eta(X)\}\varphi Y,$$

which for the Ricci curvature tensor S leads to

$$(14)\quad S(Y, \xi) = -Y\alpha - (\varphi Y)\beta - \{\xi\alpha + 2(\alpha^2-\beta^2)\}\eta(Y).$$

Denoting the curvature tensor by $R_{XYZW} = g(R_{XY}Z, W)$ we have by (13)

$$R_{\xi YZ\xi} = -(\xi\alpha + \alpha^2 - \beta^2)g(\varphi Y, \varphi Z) - (\xi\beta + 2\alpha\beta)g(Y, \varphi Z),$$

whence it follows that

$$(15)\quad R_{\xi YZ\xi} = -(\xi\alpha + \alpha^2 - \beta^2)g(\varphi Y, \varphi Z),$$

$$(16)\quad \xi\beta + 2\alpha\beta = 0.$$

On the other hand, the curvature tensor in dimension 3 always satisfies

$$(17)\quad R_{XYZW} = g(X, W)S(Y, Z) - g(X, Z)S(Y, W) + g(Y, Z)S(X, W) - \\ -g(Y, W)S(X, Z) - \tfrac{1}{2}r\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\},$$

where r is the scalar curvature. By (14), (15), (17) we can derive

$$(18)\quad S(Y, Z) = (\tfrac{1}{2}r + \xi\alpha + \alpha^2 - \beta^2)g(\varphi Y, \varphi Z) - \eta(Y)\{Z\alpha + (\varphi Z)\beta\} - \\ -\eta(Z)\{Y\alpha + (\varphi Y)\beta\} - 2(\alpha^2-\beta^2)\eta(Y)\eta(Z).$$

Thus we have the following theorem.

THEOREM 5. *The Ricci tensor of a 3-dimensional normal a.c.m. manifold is given by* (18). *Therefore any such manifold is η-Einstein, that is,* $S(Y, Z) = \lambda g(Y, Z)$, *for any vectors Y, Z orthogonal to ξ, where λ is a function.*

In the remaining part of this section we consider 3-dimensional normal a.c.m. manifolds of constant curvature. Concerning examples of such manifolds, recall that $\boldsymbol{R}^3$ admits standard flat cosymplectic structure, the unit sphere $\boldsymbol{S}^3$ admits a Sasakian structure of constant curvature 1, and generally, by Theorem 6.1 of [6], any complete Riemannian manifold (M, g) of constant positive curvature admits a quasi-Sasakian structure (φ, ξ, η, g) (with the same g) of rank 3. Below we give another example of normal a.c.m. structure of constant curvature and of rank 1. However, as we shall prove, any compact normal a.c.m. manifold of constant curvature is quasi-Sasakian.

EXAMPLE 3. In Example 1 we put $L=(a, b)$, $V=S^2$; (J, G) is the standard Kählerian structure of constant curvature 1 on S^2, x^0 is the coordinate on (a, b), $E = c\partial/\partial x^0$, $E^* = \frac{1}{c}dx^0$, $\sigma = \varrho/c^2$, where $c = \text{const} \neq 0$ and ϱ is a positive function of x^0 only. On $N=(a, b)\times S^2$ we have the normal a.c.m. structure (φ, ξ, η, g) defined by (6). Now we specialize the function ϱ so that this structure will be of constant curvature. Let (x^1, x^2) be local coordinates in S^2. Then denoting by $G_{\alpha\beta}$ $(1 \leqslant \alpha, \beta \leqslant 2)$ and g_{ij} $(0 \leqslant i, j \leqslant 2)$ the local components of G and g, respectively, we have $g_{00} = 1/c^2$, $g_{0\alpha} = g_{\alpha 0} = 0$, $g_{\alpha\beta} = \frac{\varrho}{c^2}G_{\alpha\beta}$. Consequently the components of the curvature tensor are of the form

$$R_{1221} = \frac{c^2(4\varrho-\varrho'^2)}{4\varrho^2}(g_{11}g_{22}-g_{12}{}^2),$$

$$R_{\alpha 00\beta} = \frac{c^2(\varrho'^2-2\varrho\varrho'')}{4\varrho^2}(g_{\alpha\beta}g_{00}-g_{\alpha 0}g_{0\beta}),$$

$$R_{\alpha 012} = 0.$$

Hence, if $(a, b) = (0, +\infty)$, $c = 1$ and $\varrho(x^0) = (x^0)^2$, then g is locally flat; if $(a, b) = (0, \pi)$ and $\varrho(x^0) = \sin^2 x^0$, then g is of constant curvature c^2; and if $(a, b) = (0, +\infty)$ and $\varrho(x^0) = \operatorname{sh}^2 x^0$, then g is of constant curvature $(-c^2)$.

THEOREM 6. *Let (φ, ξ, η, g) be a normal a.c.m. structure of constant curvature K on M, M being compact. Then $2\alpha = \operatorname{div}\xi = 0$, i.e. the structure is quasi-Sasakian, and $K \geqslant 0$. Moreover,* $\operatorname{rank}\eta = 1$ *(i.e., the structure is cosymplectic) if $K = 0$, and* $\operatorname{rank}\eta = 3$ *if $K > 0$.*

Proof. At first, by (14) and $S = 2Kg$ we get

$$\xi\alpha + K + \alpha^2 - \beta^2 = 0, \tag{19}$$

$$Y\alpha + (\varphi Y)\beta + (K+\alpha^2-\beta^2)\eta(Y) = 0 \tag{20}$$

Rewriting (20) in the form

$$d\alpha(Y) + g(\operatorname{grad}\beta, \varphi Y) + (K+\alpha^2-\beta^2)\eta(Y) = 0$$

and differentiating covariantly we get

$$(\nabla_X d\alpha)(Y) + g(\nabla_X \operatorname{grad}\beta, \varphi Y) + g(\operatorname{grad}\beta, (\nabla_X\varphi)Y) + \\ + \{X(\alpha^2-\beta^2)\}\eta(Y) + (K+\alpha^2-\beta^2)(\nabla_X\eta)(Y) = 0.$$

Hence, by antisymmetrization with respect to X, Y, we have

$$g(\nabla_X \operatorname{grad}\beta, \varphi Y) - g(\nabla_Y \operatorname{grad}\beta, \varphi X) + \{(\nabla_X\varphi)Y - (\nabla_Y\varphi)X\}\beta + \\ + \{X(\alpha^2-\beta^2)\}\eta(Y) - \{Y(\alpha^2-\beta^2)\}\eta(X) + 2(K+\alpha^2-\beta^2)d\eta(X, Y) = 0. \tag{21}$$

Let $\{E_0, E_1, E_2\}$ be a φ-basis. Taking $X = E_1$, $Y = E_2$ in (21) and considering relation (12) and $d\eta = \beta\Phi$ we find that

(22) $$g(\nabla_{E_1} \operatorname{grad}\beta, E_1) + g(\nabla_{E_2} \operatorname{grad}\beta, E_2) = 2\alpha\xi\beta - 2\beta(K + \alpha^2 - \beta^2).$$

On the other hand, (16) yields $g(\operatorname{grad}\beta, \xi) = -2\alpha\beta$, whence by covariant differentiation we get, on account of (11) and (19),

(23) $$g(\nabla_\xi \operatorname{grad}\beta, \xi) = -2\alpha\xi\beta - 2\beta\xi\alpha = -2\alpha\xi\beta + 2\beta(K + \alpha^2 - \beta^2).$$

Denoting by Δ the Laplacian defined by $\Delta = \operatorname{div}\operatorname{grad}$, in view of (22) and (23) we have $\Delta\beta = 0$. Since M is compact, $\beta = \text{const}$. If $\beta \neq 0$, then by (16) $\alpha = 0$ everywhere on M. So let $\beta = 0$. Relation (20) leads to $Y\alpha + (K + \alpha^2)\eta(Y) = 0$, or $\operatorname{grad}\alpha + (K + \alpha^2)\xi = 0$. Hence it follows that

$$\nabla_X \operatorname{grad}\alpha + (X\alpha^2)\xi + (K + \alpha^2)\alpha\{X - \eta(X)\xi\} = 0,$$

and consequently

$$\Delta\alpha = \sum_{i=0}^{2} g(\nabla_{E_i} \operatorname{grad}\alpha, E_i) = -2\alpha(\xi\alpha + K + \alpha^2),$$

or $\Delta\alpha = 0$ in virtue of (19) and $\beta = 0$. Hence $\alpha = \text{const}$. But $2\alpha = \operatorname{div}\xi = \text{const}$, in view of the compactness of M, implies $\alpha = 0$.

Thus we have $\alpha = 0$ and $\beta = \text{const}$, in any case. From (19) we get $K = \beta^2 \geqslant 0$. This clearly completes the proof.

For quasi-Sasakian structures of constant curvature compare also [6], Theorem 6.2.

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