Normal Bundle to Subvarieties in Quadratics II (*).

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Summary. – *In this paper we prove that the restriction of the tangent bundle of a nonsingular quadrix Q to a subvariety X is ample if and only if X does not contain a straight line. This implies that the normal bundle of a locally complete intersection, reduced and irreducible curve C is ample if and only if C is not a straight line. The result gives information also for higher dimensional subvarieties of Q.*

Introduction.

This paper is a continuation and a generalization of our previous paper [1]. We work always over an algebrically closed field k with $\mathrm{ch}(k) = 0$.

In the first paragraph we consider a nonsingular quadric Q with tangent bundle TQ and a subvariety X of Q. We prove that the restriction $TQ_{|X}$ of TQ to X is an ample vector bundle if and only if X does not contain a straight line. This result implies in particular [1], theorem 1, but applies also to singular curves and to a large class of subvarieties of Q.

It implies that a generically reduced, locally complete intersection curve C in a nonsingular quadric Q has an ample normal bundle $N_{C/Q}$ if and only if it is not a straight line.

Furthermore the normal bundle $N_{X/Q}$ of a nonsingular subvariety X in a nonsingular quadric Q is ample if X does not contain a straight line. If X is contained in a linear space contained in Q, then this condition is also necessary. In fact it is essential for the higher dimensional case to considerer all the singular curve C.

The method used are similar to those of the previous paper [1]; by reduction to the case of a curve we apply the normalization and try to repeat the proofs of [1]. Therefore some proofs are omitted. Similar results were proved by A. PAPANTONOPOULOU for grassmannians [9]. In particular the main result is proved by A. PAPANTONOPOULOU for the case of $G(1, 3)$, the grassmannian of lines in $\boldsymbol{P}_3$, which is a nonsingular quadric of dimension four. His proof is different from our's, but his paper inspired us, showing the possible applications.

The ampleness of the normal bundle has many interesting applications for a complete variety. In particular it applies to formal meromorphic functions [5],

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corollary 6.8, and therefore to the rigidity problem. Therefore it seems worthwhile to study directly the «exceptional case» of a straight line L in a nonsingular quadric Q. This is done in the second paragraph. We prove that the ring of formal regular functions on the completion of Q in L is isomorphic to k if Q has dimension at least three. This is the expected result if Q has dimension at least three. In fact, over the field of complex numbers, if $\dim Q \geqslant 3$, L is a generating subspace of Q in the sense of CHOW [2] and therefore every holomorphic function on a complex trascendental neighborhood of L in Q is constant for [2], theorem 2.

In the third paragraph we study the ampleness of the normal bundle of a curve in a product of quadrics and projective spaces.

1. – Let k be an algebraically closed field with $\mathrm{ch}(k) = 0$. Every variety is defined over k.

We want to prove the following

THEOREM 1. – *Let Q be a nonsingular quadric and X a subvariety of Q. Then the restriction $TQ_{|X}$ of TQ to X is an ample vector bundle if X does not contain a straight line.*

PROOF. – The scheme of the proof is similar to the scheme of the proof of [1], theorem 1.

Let L be a straight line contained in Q. Since $TQ_{|L} = \mathcal{O}_L \oplus \mathcal{O}_L(1)^{\oplus(n-1)}$ if Q has dimension n, the condition is obviously necessary.

Since TQ is generated by global sections, for a theorem of Gieseker [7], prog. 2.1, $TQ_{|X}$ is ample if and only if $TQ_{|C}$ is ample for each curve C contained in X. Therefore it is sufficient to treat the case $X = C$ a curve. We may suppose C reduced because a vector bundle E on C is ample if and only if $E_{|C_{\mathrm{red}}}$ is ample on C_{red} [6], prop. 1.4, pag. 84. Furthermore we may suppose C irreducible for the easy result [6], propr. 1.5, pag. 84.

We distinguish two cases:

a) C is contained in a linear space contained in Q;

b) C is not contained in a linear space contained in Q.

a) Let C be a reduced and irreducible curve contained in a maximal linear space $\boldsymbol{P}_s$ contained in Q. In [1], prop. 2, we proved that the normal bundle of $\boldsymbol{P}_s$ in Q, $N_{\boldsymbol{P}_s/Q}$, is isomorphic to $\Omega^1_{\boldsymbol{P}_s}(2)$ if dim $Q = 2s$ is even and that $N_{\boldsymbol{P}_s/Q}$ is isomorphic to $\Omega^1_{\boldsymbol{P}_s}(2) \oplus \mathcal{O}_{\boldsymbol{P}_s}(1)$ if $\dim Q = 2s+1$ is odd.

From the exact sequence

$$0 \to T\boldsymbol{P}_s \to TQ_{|\boldsymbol{P}_s} \to N_{\boldsymbol{P}_s/Q} \to 0$$

and the ampleness of $T\boldsymbol{P}_s$ we obtain that $TQ_{|C}$ is simple if and only if $\Omega^1_{\boldsymbol{P}_s}(2)_{|C}$ is ample. Therefore it is sufficient for the proof of case *a*) to prove the following

PROPOSITION 1. – *Let C be a reduced and irreducible curve contained in a projective space $\boldsymbol{P}_s$. Then $\Omega^1_{\boldsymbol{P}_s}(2)_{|C}$ is ample if and only if C is not a straight line.*

PROOF. – We generalize the proof of [1], prop. 3, to the case of a singular curve C.

We use induction on s. For $s = 1$ the result is empty.

Suppose $s \geqslant 1$ and that $\Omega^1_{\boldsymbol{P}_s}(2)_{|C}$ is not ample. Let $p: C' \to C$ be the normalization. A vector bundle E on C is ample if and only if p^*E is ample on C' (see [5], prop. 1.6, pag. 84). As $\Omega^1_{\boldsymbol{P}_k}(2)$ is generated by global sections, $p^*(\Omega^1_{\boldsymbol{P}_s}(2)_{|C})$ is generated by global sections. For a criterion of ampleness of GIESEKER-HARTSHORNE [7], prop. 2.1, $p^*(\Omega^1_{\boldsymbol{P}_s}(2)_{|C})$ has $\mathcal{O}_{C'}$ as a trivial quotient bundle.

We put $\mathcal{O}_{C'}(t) := p^*(\mathcal{O}_C(t))$ and $d := \deg C := \deg \mathcal{O}_C(1) := \deg \mathcal{O}_{C'}(1)$. Dualizing the give surjection from $p^*(\Omega^1_{\boldsymbol{P}^s}(2)_{|C})$ to $\mathcal{O}_{C'}$ we obtain an exact sequence on C':

$$0 \to \mathcal{O}_{C'}(2) \to p^*(T\boldsymbol{P}_{s|C}) \to E \to 0$$

in which E is a rank $(s-1)$-bundle on C' with $\deg E = (s-1)\,d$. If we prove that C is contained in a linear space $\boldsymbol{P}_{s-1} \subset \boldsymbol{P}_s$, the thesis will follow from the exact sequence

$$0 \to \mathcal{O}_{\boldsymbol{P}_{s-1}}(1) \to \Omega^1_{\boldsymbol{P}_s}(2)_{\boldsymbol{P}_{s-1}} \to \Omega^1_{\boldsymbol{P}_{s-1}}(2) \to 0$$

and the inductive hypotheses. Therefore the thesis follows from the following useful

LEMMA 1. – *Let C be a reduced and irreducible curve of degree d contained in a projective space $\boldsymbol{P}_s$ and $p: C' \to C$ the normalization. If E is a vector bundle of rank r on C' which is a quotient of $p^*(T\boldsymbol{P}_{s|C})$, then $h := \deg E \geqslant rd$ and if $h = rd$, then C is contained in a hyperplane of $\boldsymbol{P}_s$.*

PROOF. – If C is nonsingular, then this is lemma 2 of [1]. The general case is similar and we sketch some details. We choose homogeneous coordinates $z_0, \ldots, z_s$ on $\boldsymbol{P}_s$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{C'} \xrightarrow{z} \mathcal{O}_{C'}(1)^{\oplus(s+1)} \longrightarrow p^*(T\boldsymbol{P}_{s|C}) \longrightarrow 0$$

where $z(f) := (fz_0, \ldots, fz_s)$, we obtain a surjective map from $\mathcal{O}_{C'}(1)^{\oplus(s+1)}$ to E. This proves $h \geqslant rd$. We suppose $h = rd$. Any section of $E(-1)$ gives a subline bundle L of $E(-1)$ with $\deg L \geqslant 0$. Therefore after $r-1$ steps we arrive at a surjective map $f: \mathcal{O}_C(1)^{\oplus(s-1)} \to \mathcal{O}_C(1)$ where $\mathcal{O}_{C'}(1)$ is a quotient of E. The map f gives $s+1$ costants $a_0, \ldots, a_s$ such that the section $a_0 p^*(z_0) + \ldots + a_s p^*(z_s)$ of $\mathcal{O}_{C'}(1)$ is the zero-section. Therefore C is contained in the hyperplane

$$H := \{(z_0; \ldots; z_s) \in \boldsymbol{P}_s \colon a_0 z_0 + \ldots + a_s z_s = 0\}. \qquad \text{Q.E.D.}$$

COROLLARY. – *Let $\boldsymbol{P}_s$ be a maximal linear space contained in a nonsingular quadric Q and X be a nonsingular subvariety of $\boldsymbol{P}_s$. Then $N_{X/Q}$ is ample if and only if X does not contain a straight line.*

PROOF. – From the exact sequence

$$0 \to N_{X/\boldsymbol{P}_s} \to N_{X/Q} \to N_{\boldsymbol{P}_s/Q|X} \to 0$$

[E.G.A. IV 16.2.7 and IV 16.9.13], the ampleness of $N_{X/\boldsymbol{P}_s}$ and the explicit form of $N_{\boldsymbol{P}_s/Q}$, it follows that $N_{X/Q}$ is ample if and only if $\Omega^1_{\boldsymbol{P}_s}(2)_{|X}$ is ample. As $\Omega^1_{\boldsymbol{P}_s}(2)$ is generated by global sections, for the cited criterion of Gieseker-Hartshorne [7], prop. 2.1, $\Omega^1_{\boldsymbol{P}_s}(2)$ is ample if and only if $\Omega^1_{\boldsymbol{P}_s}(2)_{|C}$ is ample for any reduced and irreducible curve C contained in X i.e. for proposition 1 if and only if X does not contain a straight line. Q.E.D.

b) For any variety X and any coherent sheaf $\mathcal{F}$ on X, we put $t(\mathcal{F}) :=$ torsion part of $\mathcal{F}$. Let C be a reduced and irreducible curve and $p\colon C' \to C$ the normalization map. We give an ad hoc definition. We say that a coherent algebraic sheaf $\mathcal{F}$ on C is ample if the vector bundle on C' $\mathcal{F}' := p^*(\mathcal{F})/t(p^*(\mathcal{F}))$ is ample; this definition is equivalent to the usual definitions if $\mathcal{F}$ is a locally free sheaf (see [6], prop. 1.6, pag. 84).

Now let Q_{s-1} be an irreducible quadric contained in $\boldsymbol{P}_s$; let C be a reduced and irreducible curve contained in Q_{s-1} but not contained in the singular locus S of Q_{s-1}. The tangent sheaf TX to a variety X is by definition the dual of Ω^1_X. We have an exact sequence of sheaves on Q_{s-1} [E.G.A. IV 16.4.21]:

$$0 \to \mathcal{O}_{Q_{s-1}}(-2) \to \Omega^1_{\boldsymbol{P}_s/Q_{s-1}} \to \Omega^1_{Q_{s-1}} .$$

We dualize the exact sequence above, obtaining another exact sequence

$$(1) \qquad 0 \longrightarrow TQ_{s-1} \longrightarrow T\boldsymbol{P}_{s|Q_{s-1}} \xrightarrow{u'} \mathcal{O}_{Q_{s-1}}(2) .$$

In fact $\operatorname{Hom}(\Omega^1_{\boldsymbol{P}_s|Q_{s-1}}, \mathcal{O}_{\boldsymbol{P}_{s-1}})$ is isomorphic to $(\operatorname{Hom}(\Omega^1_{\boldsymbol{P}_s}, \mathcal{O}_{\boldsymbol{P}_s}))_{|Q_{s-1}}$ because $\Omega^1_{\boldsymbol{P}_s}$ is locally free.

Restricting to C, pulling back to C' and killing torsion we obtain a complex of vector bundles on C':

$$(2) \qquad 0 \to (TC_{|C})' \xrightarrow{i} (T\boldsymbol{P}_{s|C})' \xrightarrow{u} p^*(\mathcal{O}_C(2)) \to 0$$

which is exact except for a finite number of points.

The first map i is necessarly injective because $\ker(i)$, as a torsion-free sheaf with finite support, must be zero. We put $M_{s-1} := \operatorname{Im} u$. M_{s-1} is an invertible sheaf on C' because it is torsion-free and C' is nonsingular. We put $d := \deg C$.

From proposition 1 it follows that M_{s-1} has degree $\geqslant d$ and that if we have $\deg \boldsymbol{M}_{s-1} = d$, then C is contained in a hyperplane.

The sequence of vector bundles on C'

$$0 \to (TQ_{|C})' \to p^*(T\boldsymbol{P}_{s|C}) \to M_{s-1} \to 0 \tag{3}$$

is exact i.e. we have $\mathcal{K}\mathrm{er}\,(u) = \mathrm{Im}\,(i)$. In fact we have an exact sequence

$$p^*(TQ_{|C}) \to p^*(T\boldsymbol{P}_{s|C}) \to p^*(\mathrm{R}) \to 0$$

with $R := \mathrm{Im}\,(u')$. We have $\mathrm{Im}\Big(t\big(p^*(TQ_{|C})\big)\Big) = 0$ in $p^*(T\boldsymbol{P}_{s|C})$ because $p^*(T\boldsymbol{P}_{s|C})$ is locally free.

If Q_s is contained in $\boldsymbol{P}_{s+1}$ and $Q_{s-1} = Q_s \cap \boldsymbol{P}_s$, we obtain a commutative diagram of vector bundles on C' with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (TQ_{s-1|C})' \arrow[r] \arrow[d] & (T\boldsymbol{P}_{s\,C})' \arrow[r] \arrow[d] & M_{s-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (TQ_{s|C})' \arrow[r] \arrow[d] & (T\boldsymbol{P}_{s+1|C})' \arrow[r] \arrow[d] & M_s \arrow[r] & 0 \\
0 \arrow[r] & H_s \arrow[r] \arrow[d] & p^*\big(\mathcal{O}_C(1)\big) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{4}$$

Here H_s is a line bundle on C' because, in the same way as when we defined M_{s-1}, it is contained in $p^*(\mathcal{O}_C(1))$ and any torsion-free algebraic coherent sheaf on a nonsingular curve is locally free. The maps from H_s to $p^*(\mathcal{O}_C(1))$ and from M_{s-1} to M_s are injective because they are injective except for a finite number of points. The exactness of the first column in $(TQ_{s|C})'$ follows, with an easy diagram chasing, from the commutativity of the diagram and the exactness of the rows and of the other columns. From lemma 1 and the definitions we have $d \leqslant \deg M_{s-1} \leqslant \deg M_s \leqslant 2d$. Thus we obtain $\deg H_s = \deg M_{s-1} - \deg M_s + d$. Therefore $\deg H_s \geqslant 0$ and $\deg H_s = 0$, i.e. H_s is not ample, if and only if $\deg M_s = 2d$ and $\deg M_{s-1} = d$. From $\deg M_{s-1} = d$ and lemma 1 it follows that C is contained in a hyperplane of $\boldsymbol{P}_s$. Part *b*) of theorem 1 is a particular case of the following

PROPOSITION 2. – *Let C be an irreducible, reduced curve contained in the irreducible quadric Q but not contained in a linear space contained in Q. Then the vector bundle $(TQ_{|C})'$ is ample i.e. the sheaf $TQ_{|C}$ is ample.*

The proof of the proposition above is a straightforward generalization of the proof of proposition 1 and lemma 4 of [1] and is therefore omitted.

Now the proof of the theorem is finished. Q.E.D.

COROLLARY 1. – *Let X be a nonsingular subvariety of a nonsingular quadric Q. If X does not contain a straight line, then $N_{X/Q}$ is ample.*

PROOF. – Since X is nonsingular, the natural map from $TQ_{|X}$ to $N_{X/Q}$ is surjective. Therefore the thesis follows from the theorem. Q.E.D.

COROLLARY 2. – *Let C be a curve which is a locally complete intersection and generically reduced in a nonsingular quadric Q. If C_{red} does not contain a straight line as irreducible component, then $N_{C/Q}$ is ample.*

PROOF. – The natural map from $TQ_{|C}$ to $N_{C/Q}$ is surjective except for a finite number of points, the singular points of C. Therefore it is sufficient to prove the following useful

LEMMA 2. – *Let C be a complete curve and E, F vector bundles on C with a homomorphism $u\colon E\to F$ which is surjective except for a finite number of points. If E is ample, then F is ample.*

PROOF. – First we may suppose C reduced and then, passing to the normalization, we way suppose C non singular. Let $\mathcal{F}$ be a coherent algebraic sheaf on C and $t_0\in \boldsymbol{N}$ such that for $t\geqslant t_0$ we have $H^1(C, S^t(E)\otimes\mathcal{F})=0$ by [6], theorem 1.1, pag. 83.

We have two exact sequences, with v_t induced by u, which define $\mathcal{G}_t$ and R_t

$$S^t(E)\otimes\mathcal{F}\xrightarrow{v_t}\mathcal{G}_t\to 0\,,$$

$$0\to\mathcal{G}_t\to S^t(F)\otimes\mathcal{F}\to R_t\to 0\,.$$

Here R_t has finite support and thus we have $H^1(C,R_t)=0$. Furthermore we have $H^2(C,\mathcal{L})=0$ for any coherent algebraic sheaf on C because C is a curve. Therefore from the two exact sequences we obtain $H^1(C, S^t(F)\otimes\mathcal{F})=H^1(C,\mathcal{G}_t)=0$ for $t\geqslant t_0$. This proves the ampleness of F by the fundamental criterion of ampleness [6], theorem 1.1, pag. 83. Q.E.D.

Lemma 2 applies also to a curve in a projective space.

COROLLARY 3. – *Let V be a nonsingular subvariety of the nonsingular quadric $Q\subset\boldsymbol{P}_n$. Suppose that for any straight line L contained in V there exists a straight line $L'\subset V$, intersecting L and such that the plane H containing L and L' is not contained in Q. Then $N_{V/Q}$ is an ample vector bundle on V.*

PROOF. – Since $N_{V/Q}$ is generated by global sections, the cited criterion of ampleness of Gieseker-Hartshorne [7], prop. 2.1, shows that it is sufficient to prove that $N_{V/Q|C}$ is ample for every curve C contained in V. From theorem 1 and the surjectivity of the natural map from $TQ_{|V}$ to $N_{V/Q}$, it follows that it is sufficient to prove that $N_{V/Q|L}$ is an ample vector bundle on L for every straight line Γ contained in V. Let L' be a straight line contained in V, intersecting L and such that the plane H generated by L and L' is not contained in V. Let X be the subvariety of Q union of L and L' with the reduced structure, that is to say $X := H \cap Q = H \cap V$.

We have the following exact sequence [E.G.A. IV 16.2.7]:

$$0 \to N_{X/V} \to N_{X/Q} \to N_{V/Q|X} .$$

Furthermore $N_{X/Q}$ is ample and in fact we have $N_{X/Q} \cong \mathcal{O}_X(1)^{\oplus(n-2)}$. From this exact sequence and lemma 2, we obtain the ampleness of $N_{V/Q|X}$ and $N_{V/Q|L}$, as restriction of an ample vector bundle to a closed subset, is ample, too. Q.E.D.

2. – The ampleness of the normal bundle of a subvariety of a variety has many well-known applications, for example to formal rational functions (see [5], [1]) and therefore to the rigidity problem.

Therefore it is natural to study directly the case of a straight line contained in a nonsingular quadric from the point of view of formal rational functions.

Here we recall some definitions. Let Z be a noetherian scheme, X a closed subscheme of Z defined by the ideal sheaf. The formal completion of Z along X is the ringed space $(X, \mathcal{O}_{Z/X})$ where we have $\mathcal{O}_{Z/X} := \operatorname{inv\,lim} \mathcal{O}_Z/\mathcal{J}^n_{|Z}$. We say as in [2] that X is G-1 in Z if the natural map from $H^0(Z, \mathcal{O}_Z)$ to $H_0(X, \mathcal{O}_{Z/X})$ is an isomormorphism.

Let $\mathcal{K}_{Z/X}$ be the total quotient sheaf of $\mathcal{O}_{Z/X}$ and put $K(Z/X) := H^0(X, \mathcal{K}_{Z/X})$. We say as in [2] that X is G-3 in Z if the canonical map $K(Z) \to K(Z/X)$ is an isomorphism. We say that X is G-2 in Z if for the canonical map $K(Z) \to K(Z/X)$, $K(Z/X)$ is a finite module over $K(Z)$.

From the results of the first paragraph, a theorem by Hartshorne and an explicit calculation for the formal completion of a straight line in a nonsingular quadric, we obtain the following

THEOREM 2. – *Let X be a reduced and irreducible locally complete intersection in a nonsingular quadric Q of dimension at least three. Then X is G-1 in Q. Furthermore if X is not a straight line, then X is G-3 in Q.*

PROOF. – It is well-known, see [8], Remark 2.10, that G-3 implies G-2 and G-2 implies G-1 because the field k is algebraically closed and we have $H^0(Q, \mathcal{O}_Q) = k$. Furthermore if X contains C and C is G-3 in Q, then X is G-3 in Q, too. If $\dim X = 1$ and X is not a straight line, then its normal bundles is ample for Corollary 2 to Theorem 1.

From a theorem of Hartshorne [5], corollary 6.8, it follows that X is G-2 in Q; as X is also a generating subspace of Q, then X is G 3 in Q (see [1], theorem 2).

If $\dim X \geqslant 2$, let H a generic linear space on the projective space which contains Q and such that $X \cap H$ is an irreducible and reduced curve (Bertini's theorem). Then $X \cap H$ is a locally complete intersection on Q. If $X \cap H$ is not a straight line, we have proved the theorem in this case. If $X \cap H$ is a straight line, then X has degree 1 i.e. it is a linear space and contains a lot of nonsingular curves which are not straight lines. Therefore X is G-3 in Q. Now we have to prove that a straight line L in Q is G-1 in Q.

The proof of the theorem is therefore reduced to the proof of the following

PROPOSITION 3. – *Let L be a straight line contained in a nonsingular quadric Q with* $\dim Q \geqslant 3$. *Then L is G-1 in Q.*

PROOF (OF THE PROPOSITION). – Let Q be a quadric hypersurface in $\boldsymbol{P}_n$ with $n \geqslant 4$. We choose a system of homogeneous coordinate $(x_0; \ldots; x_n)$ on $\boldsymbol{P}_n$ such that L has equations $x_2 = \ldots = x_n = 0$ and Q has equation

$$x_0 x_2 + x_1 x_3 + \sum_{s=4}^{n} x_s^2 = 0 \,.$$

This can be done because the orthogonal group $O(n+1)$ acts transitevely on the straight lines contained into Q. In fact it is easy to prove first that $O(n+1)$ acts transitevely on the set of maximal linear subspace of Q. Let H be a maximal linear space in Q containing L. Then it is easy to prove that the elements of $O(n+1)$ which leave H fixed act transitevely on the set of lines contained into H.

L is covered by two affine open sets $B_0 = Q \cap \{x_0 \neq 0\}$ and $B_1 = Q \cap \{x_1 \neq 0\}$. We put $a_s = x_s/x_0$ and $b_s = x_s/x_1$. Then we have $k[B_0] = k[a_1, a_3, a_4, \ldots, a_n]$ and $k[B_1] = k[b_0, b_2, b_4, \ldots, b_n]$. $B_2 := B_0 \cap B_1$ is an affine open subset of Q with $k[B_2] = k[a_1, a_1^{-1}][a_3, a_4, \ldots, a_n]$. We have a natural injective map of restriction $i\colon k[B_0] \to k[B_2]$ induced from the inclusion of B_2 into B_0 with $i(a_s) = a_s$. We have also a natural injective map $j\colon k[B_1] \to k[B_2]$ induced from the inclusion of B_2 into B_1, with $j(b_0) = a_1^{-1}$, $j(b_2) = -a_3 - \sum_{s=4}^{n} a_s^2 a_1^{-1}$ and $j(b_s) = a_s a_1^{-1}$ for $s \geqslant 4$. Let A_s, $s = 0, 1, 2$, the ring of coordinate of the completion of B_s in $L \cap B_s$. We have $A_0 = k[a_1][\![a_3, a_4, \ldots, a_n]\!]$, $A_1 = k[b_0][\![b_2, b_4, \ldots, b_n]\!]$ and $A_2 = k[a_1, a_1^{-1}][\![a_3, a', \ldots, a_n]\!]$. We have inclusions i, j between the A_s's. We have $k[Q/L] = i(A_0) \cap j(A_1)$ for the sheaves' axioms.

$$f = \sum_{cd} P_{cd}(b_0)(b_2)^c (b_4)^{d_4} \ldots (b_n)^{d_n}$$

with $d = (d_4, \ldots, d_n)$ a multiindex and P_{cd}'s polynomials. Therefore we have

$$j(f) = \sum_{cd} P_{cd}(a_1^{-1})\Big(-a_3 - \sum_{s=4}^{n} a_s^2\, a_1^{-1}\Big)^c \cdot (a_4\, a_1^{-1})^{d_4} \ldots (a_4\, a_1^{-1})^{d_n} \,.$$

Suppose $j(f) \in i(A_0)$. First we have that P_{0c} is a constant. From the monomial $P_{c0} a_3^c$ we obtain that P_{c0} is a costant. From the monomial $cP_{c0} a_3^{c-1} a_4^2 a_1^{-1}$ we obtain $P_{c0} = 0$ for $c > 0$ because every other monomial with a_4^2 has at least a_1^{-2} as a factor (recall that k has $\mathrm{ch}(k) = 0$). We prove by induction on $|d| = d_4 + \ldots + d_n$ that every term P_{cd} with $|d| > 0$ vanishes. In fact the monomial $P_{cd} a_3^c a_4^{d_n} \ldots a_n^{d_n} a_2^{-|d|}$ can be deleted only by terms with a P_{uv} as a factor with $u > c$ and therefore $|v| < |d|$. By induction we have $P_{uv} = 0$ and therefore $P_{cd} = 0$. Thus f is costant. Q.E.D.

The proposition above is true also if L is a straight line contained in a three dimensional nonsingular quadric contained in the singular quadric Q, while it is false if the singular locus of Q is bigger. I do not know if a straight lines is G-3 in a nonsingular quadric.

This result is the expected one. Consider the problem over the field of complex numbers. Then it is known that a straight line L, as any subspace, in a nonsingular quadric Q of dimension at least 3 is a generating subspace in the sense of [2]. We recall Chow's definition. Let G be an algebraic group and X be a projective variety which is homogeneous under a regular action of G; let Y be a closed connected subvariety of X; let p be a point of Y. We put $G_{p,Y} = \{g \in G\colon gp \in Y\}$. The subgroup G_Y of G generate by $G_{p,Y}$ does not depend upon the choice of the point p. We say [2] that Y is a generating subspace of X if we have $G_Y = G$. Then by [2], theorem 2, any holomorphic function in a neighborhood of L in Q for the trascendental topology is constant. Therefore only the constants among the formal regular functions on the completion of L in Q can be extended to a complex neighborhood of L as holomorphic functions; therefore theorem 2 means that any formal regular functions on the completion of L in Q converges.

REMARK. – The proof of theorem 2 works for every connected subvariety of Q which contains a locally complete intersection curve which is not a straight line. In particular this occur for Bertini's theorem for any connected subvariety which is non singular in codimension 1, e.g. for a connected normal variety of dimension at least two.

3. – In this paragraph we want to study the ampleness of the normal bundle of a curve in a product of quadrics and projective spaces. Let $V = X_1 \times \ldots \times X_r$ be a product of varieties; we take an index i, $1 \leqslant i \leqslant r$, and a point $x \in X_i$; the slice of V corresponding to the point x is simply $X_1 \times \ldots \times X_{i-1} \times \{x\} \times X_{i+1} \times \ldots \times X_r$ with the induced structure. We recall some notation of the first paragraph. Let C be a reduced curve; we put $p\colon C' \to C$ the normalization map; if E is a sheaf on C, $E' := p^*(E)/t(p^*(E))$ i.e. $p^*(E)$ modulo the torsion part, is a vector bundle on C'; for definition E is an ample sheaf if and only if E' is an ample vector bundle on C'. This definition is given only because it is useful, for example in the proof of theorem 3 below, although probably it is not the good one or an interesting one.

PROPOSITION 4. – *Let C be an irreducible and reduced curve contained in the variety $V := \boldsymbol{P}_{n_1} \times \ldots \times \boldsymbol{P}_{n_r}$ with $r \geqslant 1$, $n_i \geqslant 1$ for all index i.*

The following conditions are equivalent:

i) *$TV_{|C}$ is ample vector bundle;*

ii) *$N_{C/V}$ is an ample sheaf;*

iii) *C is not contained in a slice of Q.*

PROOF. – i) $\Rightarrow$ ii). This assertion follows from the natural map $TV_{|C} \to N_{C/V}$ which is surjective except for a finite number of points and from lemma 2.

ii) $\Rightarrow$ iii). If C is contained in a slice, then $TV_{|C}$ and $N_{C/V}$ have a trivial quotient bundle because all factors are positive-dimensional. Therefore they cannot be ample.

iii) $\Rightarrow$ i). Suppose that $TV_{|C}$ is not ample and therefore $p^*(TV_{|C})$ is not ample, too. $TV_{|C}$ is generated by global sections and therefore also $p^*(TV_{|C})$ is generated by global sections. Therefore for the cited criterion of ampleness of GIESEKER-HARTSHORNE [7], prop. 2.1, $(TV_{|C})'$ has $\mathcal{O}_{C'}$ as a quotient line bundle. We put $p_i \colon V \to \boldsymbol{P}_{n_i}$ the projection. TV is a quotient bundle of $\bigoplus_{i=1}^{r} p_i^*(\mathcal{O}_{\boldsymbol{P}_{n_i}}(1))^{n_i+1}$ and therefore we have an index i and a non zero map $p^*\left(p_i^*(\mathcal{O}_{\boldsymbol{P}_{n_i}}(1))_{|C}\right) \to p^*(\mathcal{O}_C)$. This means that the pull-back to C' of the restriction to C of the pull-back to V of any homogeneous form of degree one on $\boldsymbol{P}_{n_i}$ is costant. This means that there exists a point x in $\boldsymbol{P}_{n_i}$ such that C is contained in the slice of V corresponding to the point x. Q.E.D.

PROPOSITION 5. – *Let V be a product of at least two among projective spaces and quadrics (even singular). Let C be an irreducible and reduced curve contained in V but not contained in a slice through a point of a factor $\boldsymbol{P}_{n_i}$ or in the product of a linear space contained in a quadric factor and of the other factors. Then $TV_{|C}$ and $N_{C/V}$ are ample sheaves.*

PROOF. – If $TV_{|C}$ is ample, then it follows from lemma 2 that $N_{C/V}$ is ample, too. Suppose that $TV_{|C}$ is not ample i.e. $(TV_{|C})'$ is not an ample vector bundle on C'. If V has no quadric as a factor, then the thesis is the proposition above. The general case is by induction on the sum t of the dimension of the quadrics which are factors of V.

If $t = 1$, we have only one quadric Q. If Q is nonsingular, then it is isomorphic to a projective line and this case is covered by the proposition above. If Q is singular then it is the union of two lines intersecting in a point. C is contained in the product of one of the two lines for the remaining factors because C is reduced and irreducible. Therefore this case cannot occur.

Now suppose $t > 1$. TV is a quotient of a direct sum of sheaves of the type $p_j^*(TQ_j)$, $p_j^*(T\boldsymbol{P}_{n_j})$ where p_j's are the projections. We use a criterion of ampleness proved by HARTSHORNE [7], theorem 2.4. Let C' be a non-singular, complete curve and F be a vector bundle on C'; then F is ample if and only if for every quotient bundle R of F we have $\deg R > 0$. Suppose that $(TV_{|C})'$ is not ample. Let R be a quotient bundle of $(TV_{|C})'$, with $\deg R \leqslant 0$ and rank $R > 0$. If for a factor $\boldsymbol{P}_{n_j}$ the induced map from $(p_j^*(T\boldsymbol{P}_{n_j})_{|C})'$ to R is not zero, then we obtain that C is contained in a slice corresponding to a point of $\boldsymbol{P}_{n_j}$ in the same way as in the proof of the proposition above. Therefore there exists a quadric Q which is a factor of V and such that the induce map $(p_i^*(TQ)_{|C})' \to R$ is not zero. Here p_i is the projection from V to Q. Suppose that the quadric Q is a hypersurface of $\boldsymbol{P}_s$. On Q we have the exact sequence (1) and, as C is not contained in the singular locus of V, we obtain an exact sequence on C which looks like the sequence (2):

$$0 \to (p_i^* TQ_{|Q})' \xrightarrow{i} \big((p_i^*(T\boldsymbol{P}_{s|Q}))_{|C}\big) \xrightarrow{u} p^*(\mathcal{O}_C(2)) \to 0$$

which is exact except for a finite number of points. We put $M := \operatorname{Im}(u)$. We obtain the following sequence which looks like the exact sequence (3)

$$0 \to (p_i^* TQ_{|C})' \xrightarrow{i} \big((p_i^*(T\boldsymbol{P}_{s|Q}))_{|C}\big) \xrightarrow{u} M \to 0 \tag{7}$$

which is exact. The proof of the exactness of (7) is the same as the proof of the exactness of (3). Furthermore M, as a torsion-free sheaf on a nonsingular curve C is locally free. Therefore i injects as a map of vector bundles. We put

$$d := \deg \mathcal{O}_C(1) := \deg p^*(p_i^* \mathcal{O}_Q(1)_{|C}) .$$

From the quotient map $(p_i^* TQ_{|C}) \to R$, from the Euler's sequence defining $T\boldsymbol{P}_s$ and from the injectivity of i as a map of vector bundles, we obtain an exact sequence of vector bundles on C

$$p^*(p_i^* \mathcal{O}_Q(1)_{|C})^{i+1} \to E \to 0 \tag{8}$$

with $\operatorname{rank} E = 1 + \operatorname{rank} R \geqslant 2$ and $\deg E \leqslant 2d$. If we prove that C is contained in the product of a hyperplane section of Q for the remaining factors, then we will be gone for the inductive hypotheses and a natural generalization of the diagram (4).

The surjective map h induces a non zero section of $E \otimes p^*(\mathcal{O}_Q(-1)_{|C})$ which gives a commutative diagram with exact rows and columns and which defines a line bundle L of positive degree

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & \mathcal{O}_C & \to & E \otimes p^*(\mathcal{O}_Q(-1)_{|C}) & \to & \big(E \otimes p^*(\mathcal{O}_Q(-1)_{|C})/\mathcal{O}_C\big) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & L & \to & E \otimes p^*(\mathcal{O}_Q(-1)_{|C}) & \to & \big(E \otimes p^*(\mathcal{O}_Q(-1)_{|C})/\mathcal{O}_C\big)' & \to & 0 . \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

In a finite number of steps we reduce to the case in which the vector bundle E is a line bundle E of degree d, necessarly isomorphic to $p^*\big(p_i^*(\mathcal{O}_Q(1))_{|C}\big)$.

Now the thesis follows as in the last part of Lemma 1. Q.E.D.

THEOREM 3. – *Let V be isomorphic to $Q\times \boldsymbol{P}_{n_1}\times\ldots\times\boldsymbol{P}_{n_r}$ with $r\geqslant 1$, $n_i\geqslant 1$. and C be a reduced and irreducible curve contained in V.*

Then the following properties are equivalent:

i) *C is not contained in a slice of V;*

ii) *$TV_{|C}$ is an ample vector bundle;*

iii) *$N_{C/V}$ is an ample sheaf.*

PROOF. – The assertion iii) $\Rightarrow$ i) is obvious.

From Lemma 2 it follows in the usual way that ii) $\Rightarrow$ iii).

We prove the assertion i) $\Rightarrow$ ii). From proposition 5 it follows that it is sufficient to consider the following situation: C is contained in $\boldsymbol{P}_s\times\boldsymbol{P}_{n_1}\times\ldots\times\boldsymbol{P}_{n_r}$ where $\boldsymbol{P}_s$ is a maximal linear space of Q. By [1], prop. 2, we have $N_{\boldsymbol{P}_s/Q}\cong\Omega^1_{\boldsymbol{P}_s}(2)$ if $\dim Q=2s$ is even and we have $N_{\boldsymbol{P}_q/Q}\cong\Omega^1_{\boldsymbol{P}_s}(2)\oplus\mathcal{O}_{\boldsymbol{P}_s}(1)$ if $\dim=2s+1$ is odd. We have the exact sequence

$$0\to p_i^*\,T\boldsymbol{P}_s\oplus p_i^*\,T\boldsymbol{P}_{n_1}\oplus\ldots\otimes p_{n_r}^*\,T\boldsymbol{P}_{n_r}\to TV_{|\boldsymbol{P}_s\times\ldots\times\boldsymbol{P}_{n_r}}\to N_{\boldsymbol{P}_s/Q}\to 0\,.$$

The proof of proposition 4 shows that $\big(p_i^*\mathcal{O}_{\boldsymbol{P}_s}(1)\big)_{|C}$ is ample if C is not contained in a slice. Therefore the thesis in this case is equivalent to prove that $p^*\big(\Omega^1_{\boldsymbol{P}_s}(2)_{|C}\big)$ is ample if C is not contained in a slice.

The proof of this assertion is a straight forward generalization of the proof of proposition 1 in the first paragraph and is therefore omitted. Q.E.D.

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