NORMAL-CONVEX EMBEDDINGS OF INVERSE SEMIGROUPS

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Normal-convex embeddings are introduced for inverse semigroups, generalizing the group-theoretic concept, due to Papakyriakopoulos [**4**]. It is shown that every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group, a stronger version of a result by O'Carroll [**3**]. A general embedding result for inverse semigroups is also obtained.

1. Preliminaries. The general terminology and notation are those of Petrich [**5**].

Let S be an inverse semigroup and let $R \subseteq S \times S$ be a relation on S. We denote by $R^{\#}$ the congruence on S generated by R, that is, the transitive closure of $\{(aub, avb) : a, b \in S^1 \text{ and } (u,v) \in R \cup R^{-1}\}$. The natural projection $S \to S/R^{\#}$ is denoted by $(R^{\#})^{\natural}$.

Let $\varphi : S \to T$ be a homomorphism of inverse semigroups and let R be a relation on S. The relation

$$R\varphi = \{(u\varphi, v\varphi) : (u,v) \in R\}$$

is said to be the relation on T induced by R and φ. It follows easily that

$$R^{\#}\varphi \subseteq (R\varphi)^{\#}. \tag{1.1}$$

If φ is injective, we say that φ is an embedding of inverse semigroups.

Now let $\varphi : S \to T$ be an embedding of inverse semigroups. We say that φ is *normal-convex* if and only if, for every relation R on S,

$$(R\varphi)^{\#} \cap (S \times S)\varphi \subseteq R^{\#}\varphi.$$

Note that, by (1.1), the inclusion $R^{\#}\varphi \subseteq (R\varphi)^{\#} \cap (S \times S)\varphi$ is always true. Also by (1.1), we know that φ induces a unique homomorphism $\varphi_R : S/R^{\#} \to T/(R\varphi)^{\#}$ such that the canonical diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ {\scriptstyle (R^{\#})^{\natural}}\downarrow & & \downarrow{\scriptstyle [(R\varphi)^{\#}]^{\natural}} \\ S/R^{\#} & \xrightarrow[\varphi_R]{} & T/(R\varphi)^{\#} \end{array} \tag{1.2}$$

commutes. Now we have

LEMMA 1.1. *Let $\varphi : S \to T$ be an embedding of inverse semigroups. Then φ is normal-convex if and only if φ_R is injective for every relation R on S.*

Proof. Suppose that φ is normal-convex and let R be a relation on S. Let $a, b \in S$ be such that $(aR^{\#})_{\varphi_R} = (bR^{\#})_{\varphi_R}$. Since (1.2) commutes, we have $(a\varphi)(R\varphi)^{\#} = (b\varphi)(R\varphi)^{\#}$. Hence $(a\varphi, b\varphi) \in (R\varphi)^{\#} \cap (S \times S)\varphi$. Since φ is normal-convex, this yields $(a\varphi, b\varphi) \in R^{\#}\varphi$. Thus $aR^{\#} = bR^{\#}$ and so φ_R is injective.

Conversely, suppose that φ_R is injective for every relation R on S. Suppose that $(a\varphi, b\varphi) \in (R\varphi)^{\#}$ for some $a, b \in S$. Since (1.2) commutes, we have $(aR^{\#})_{\varphi_R} = (bR^{\#})_{\varphi_R}$,

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and since φ_R is injective, $aR^\# = bR^\#$. Therefore $(a\varphi, b\varphi) \in R^\#\varphi$ and so φ is normal-convex.

The following result shows that the class of normal-convex embeddings is closed under composition.

LEMMA 1.2. *Let $\varphi: S \to T$ and $\psi: T \to U$ be normal-convex embeddings of inverse semigroups. Then $\varphi\psi$ is a normal-convex embedding.*

Proof. It is trivial that $\varphi\psi$ is an embedding. Now let R be a relation on S. Since $(\varphi\psi)_R$ is uniquely defined, we certainly have $(\varphi\psi)_R = \varphi_R\psi_{R\varphi}$ and so $(\varphi\psi)_R$ is injective. Thus, by Lemma 1.1, $\varphi\psi$ is normal-convex.

The next result shows an application of the concept of normal-convex embedding.

Given a semigroup S and a relation R on S, the *word problem* for R consists in finding an algorithm which determines, for every $a, b \in S$, whether or not $(a, b) \in R^\#$.

THEOREM 1.3. *Let $\varphi: S \to T$ be a normal-convex embedding of inverse semigroups and let R be a relation on S. Then the word problem for R is solvable if the word problem for $R\varphi$ is solvable.*

Proof. Suppose that the word problem for $R\varphi$ is solvable. Let $a, b \in S$. By Lemma 1.1, φ_R is injective and so $aR^\# = bR^\# \Leftrightarrow (aR^\#)\varphi_R = (bR^\#)\varphi_R$. Since (1.2) commutes, we have $(aR^\#)_{\varphi_R} = (bR^\#)_{\varphi_R} \Leftrightarrow (a\varphi)(R\varphi)^\# = (b\varphi)(R\varphi)^\#$. Since the word problem for $R\varphi$ is solvable, we can determine whether or not this latter equality holds, hence the word problem for R is solvable and the theorem is proved.

Now let S be an inverse semigroup with semilattice of idempotents $E(S)$. The *least group congruence* on S is defined by

$$(a, b) \in \sigma \Leftrightarrow \exists e \in E(S): ae = be.$$

We say that S is *E-unitary* if

$$\forall a \in S, \qquad a\sigma = 1 \Rightarrow a \in E(S).$$

Let M denote an inverse monoid with least group congruence σ. Then M is said to be *F-inverse* if every σ-class of M has a maximal element under the natural partial order. It is well-known that every F-inverse monoid is E-unitary [**5**, §VII.5].

Let G be a group and let K be a semilattice. An *action* of G on K by left automorphisms is a map $G \times K \to K: (g, A) \mapsto gA$ such that, for every $g, h \in G$ and $A, B \in K$,

$$g(hA) = (gh)A,$$
$$g(AB) = (gA)(gB),$$
$$1A = A.$$

It follows easily that, for every $g \in G$ and $A, B \in K$, we have

$$A \leqslant B \Rightarrow gA \leqslant gB.$$

The *semidirect product* of K by G induced by this action is the inverse semigroup $K \times G$ with the operation given by $(A, g)(B, h) = (A(gB), gh)$. When no ambiguity arises about the action, we shall denote this semigroup by $K \overline{\times} G$.

Now suppose that L is an ideal of K such that $GL = K$. Then we say that (G, K, L) is a *strong McAlister triple* and

$$P(G, K, L) = \{(A, g) \in L \times G : g^{-1}A \in L\}$$

is an inverse subsemigroup of $K \mathbin{\overline{\times}} G$ [**1**].

LEMMA 1.4 [**1**]. *Let M be an inverse monoid. Then M is F-inverse if and only if $M \simeq P(G, K, L)$ for some strong McAlister triple (G, K, L) such that L has a unity.*

Let S be an inverse semigroup and let τ be a congruence on S. We say that τ is *idempotent-pure* if, for every $(a, b) \in \tau$,

$$a \in E(S) \Rightarrow b \in E(S).$$

We say that τ is *idempotent-separating* if, for every $(a, b) \in \tau$,

$$a \in E(S) \Rightarrow b \notin E(S).$$

Finally, an inverse semigroup S is said to be *quasi-free* if $T \simeq F/\tau$ for some free inverse semigroup F and some idempotent-pure congruence τ on F.

LEMMA 1.5 [**2**]. *Let S be a quasi-free inverse semigroup. Then $S \simeq P(G, K, L)$ for some strong McAlister triple (G, K, L) with G free.*

2. Strong McAlister triples. In this section we show that, for every strong McAlister triple (G, K, L), there exists a canonical embedding of $P(G, K, L)$ into a semidirect product of a semilattice by a group.

THEOREM 2.1. *Let (G, K, L) be a strong McAlister triple. Then the inclusion map $\varphi : P(G, K, L) \to K \mathbin{\overline{\times}} G$ is normal-convex.*

Proof. Let $S = P(G, K, L)$ and let $T = K \mathbin{\overline{\times}} G$. Let R be a relation on S, say $R = \{((A_i, g_i), (B_i, h_i)) : i \in I\}$. Without loss of generality, we can assume that R is symmetric. Let $(U, u), (V, v) \in S$ be such that $(U, u)(R\varphi)^{\#} = (V, v)(R\varphi)^{\#}$. We want to prove that $(U, u)R^{\#} = (V, v)R^{\#}$. Since R is symmetric, we know that there exist $(W_0, w_0), \ldots, (W_n, w_n) \in T$ such that

$$(W_0, w_0) = (U, u)$$
$$(W_n, w_n) = (V, v)$$
$$\forall j \in \{1, \ldots, n\}\ \exists (P_j, p_j), (Q_j, q_j) \in T\ \exists i_j \in I:$$
$$(W_{j-1}, w_{j-1}) = (P_j, p_j)(A_{i_j}, g_{i_j})(Q_j, q_j)$$

and

$$(W_j, w_j) = (P_j, p_j)(B_{i_j}, h_{i_j})(Q_j, q_j).$$

Now we show that, for every $m \in \{0, \ldots, n\}$,

$$\begin{aligned} &\exists P'_m, Q'_m, W'_m \in L: \\ &\quad (W'_m, w_m) \in S, \\ &\quad (W'_m, w_m)R^{\#} = (U, u)R^{\#}, \\ &\quad (W'_m, w_m) = (P'_m, 1)(W_m, w_m)(Q'_m, 1). \end{aligned} \tag{2.1}$$

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We use induction on m. Defining $P'_0 = U$, $Q'_0 = u^{-1}U$ and $W'_0 = U$, we see that (2.1) holds for $m = 0$.

Now suppose that (2.1) holds for $m = j - 1$, with $j \in \{1, \dots, n\}$. Then

$$\begin{aligned}(W'_{j-1}, w_{j-1}) &= (W'_{j-1}, 1)(W'_{j-1}, w_{j-1})(w_{j-1}^{-1}W'_{j-1}, 1)\\ &= (W'_{j-1}, 1)(P'_{j-1}, 1)(W_{j-1}, w_{j-1})(Q'_{j-1}, 1)(w_{j-1}^{-1}W'_{j-1}, 1)\\ &= (W'_{j-1}, 1)(P'_{j-1}, 1)(P_j, p_j)(A_{i_j}, g_{i_j})(Q_j, q_j)(Q'_{j-1}, 1)(w_{j-1}^{-1}W'_{j-1}, 1).\end{aligned}$$

It is clear that

$$W'_{j-1} \leqslant P'_{j-1}P_j \tag{2.2}$$

and so

$$(W'_{j-1}, 1)(P'_{j-1}, 1)(P_j, p_j) = (W'_{j-1}, p_j).$$

Similarly,

$$W'_{j-1} \leqslant (p_j g_{i_j} Q_j)(p_j g_{i_j} q_j Q'_{j-1})$$

and so

$$g_{i_j}^{-1} p_j^{-1} W'_{j-1} \leqslant Q_j(q_j Q'_{j-1}). \tag{2.3}$$

Hence

$$(Q_j, q_j)(Q'_{j-1}, 1)(w_{j-1}^{-1}W'_{j-1}, 1) = (g_{i_j}^{-1} p_j^{-1} W'_{j-1}, q_j).$$

Thus

$$(W'_{j-1}, w_{j-1}) = (W'_{j-1}, p_j)(A_{i_j}, g_{i_j})(g_{i_j}^{-1} p_j^{-1} W'_{j-1}, q_j).$$

Since $W'_{j-1} \leqslant p_j A_{i_j}$, we have $p_j^{-1}W'_{j-1} \leqslant A_{i_j} \in L$. But L is an ideal of K and so $p_j^{-1}W'_{j-1} \in L$. Since $W'_{j-1} \in L$, we obtain $(W'_{j-1}, p_j) \in S$. Similarly, we have $g_{i_j}^{-1}p_j^{-1}W'_{j-1} \leqslant g_{i_j}^{-1}p_j^{-1}(p_j A_{i_j}) = g_{i_j}^{-1}A_{i_j} \in L$, and $q_j^{-1}g_{i_j}^{-1}p_j^{-1}W'_{j-1} = w_{j-1}^{-1}W'_{j-1} \in L$. Hence

$$(g_{i_j}^{-1}p_j^{-1}W'_{j-1}, q_j) \in S.$$

Let $P'_j = W'_{j-1}$, $Q'_j = w_{j-1}^{-1}W'_{j-1}$ and $W'_j = W'_{j-1}(p_j B_{i_j})(w_j w_{j-1}^{-1} W'_{j-1})$. Obviously, $P'_j, Q'_j \in L$ and since L is an ideal of K, we have $W'_j \in L$ as well. We have $(W'_j, w_j) = (W'_{j-1}, p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1}, q_j)$, that is, (W'_j, w_j) is a product of elements of S. Therefore $(W'_j, w_j) \in S$. Moreover,

$$\begin{aligned}(W'_j, w_j)R^{\#} &= [(W'_{j-1}, p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1}, q_j)]R^{\#}\\ &= [(W'_{j-1}, p_j)(A_{i_j}, g_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1}, q_j)]R^{\#} = (W'_{j-1}, w_{j-1})R^{\#} = (U, u)R^{\#}.\end{aligned}$$

It follows from (2.2) that $(W'_{j-1}, p_j) = (W'_{j-1}, 1)(P_j, p_j)$. Similarly, (2.3) yields $(g_{i_j}^{-1}p_j^{-1}W'_{j-1}, q_j) = (Q_j, q_j)(w_{j-1}^{-1}W'_{j-1}, 1)$. Hence

$$\begin{aligned}(W'_j, w_j) &= (W'_{j-1}, p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1}, q_j)\\ &= (W'_{j-1}, 1)(P_j, p_j)(B_{i_j}, h_{i_j})(Q_j, q_j)(w_{j-1}^{-1}W'_{j-1}, 1) = (P'_j, 1)(W_j, w_j)(Q'_j, 1)\end{aligned}$$

and so (2.1) holds for $m = j$.

Thus (2.1) holds for every $m \in \{0, \dots, n\}$. In particular, we have $(W'_n, v)R^{\#} = (W'_n, w_n)R^{\#} = (U, u)R^{\#}$ and $(W'_n, v) = (P'_n, 1)(W_n, w_n)(Q'_n, 1) = (P'_n, 1)(V, v)(Q'_n, 1)$. Therefore $W'_n \leqslant V$ and so $(W'_n, v) = (W'_n, 1)(V, v)$. It follows that $(U, u)R^{\#} = (W'_n, 1)R^{\#}(V, v)R^{\#}$ and so $(U, u)R^{\#} \leqslant (V, v)R^{\#}$. Similarly, we obtain $(V, v)R^{\#} \leqslant (U, u)R^{\#}$ and so $(U, u)R^{\#} = (V, v)R^{\#}$. Thus φ is normal-convex.

Now, Lemma 1.5 and Theorem 2.1 immediately yield

COROLLARY 2.2. *Every quasi-free inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a free group.*

Since every free inverse semigroup is quasi-free, we also obtain

COROLLARY 2.3. *Every free inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a free group.*

3. E-unitary inverse semigroups. In this section we prove that every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group.

Let S be an E-unitary inverse semigroup. Let $M(S)=\{\emptyset\notin A\subseteq S:E(S).A\subseteq A\subseteq a\sigma$ for some $a\in S\}$ with the operation described by $AB=\{ab:a\in A$ and $b\in B\}$. The following result is due to O'Carroll.

LEMMA 3.1 [3]. *Let S be an E-unitary inverse semigroup. Then $M(S)$ is an F-inverse monoid and the map $\varphi:S\to M(S):s\mapsto\{t\in S:t\leq s\}$ is an embedding. Moreover, if σ_S and $\sigma_{M(S)}$ denote respectively the least group congruences of S and $M(S)$, then $\sigma_{M(S)}\cap(S\times S)\varphi=\sigma_S\varphi$.*

We prove that this embedding is in fact normal-convex.

LEMMA 3.2. *Let S be an E-unitary inverse semigroup. Then the embedding $\varphi:S\to M(S):s\mapsto\{t\in S:t\leq s\}$ is normal-convex.*

Proof. Let R be a relation on S. Without loss of generality, we can assume that R is symmetric. Let $a,b\in S$ be such that $(a\varphi,b\varphi)\in(R\varphi)^{\#}$. We want to prove that $(a,b)\in R^{\#}$.

Since $(a\varphi,b\varphi)\in(R\varphi)^{\#}$, there exist $W_0,\ldots,W_n\in M(S)$ such that

$$W_0=a\varphi;$$
$$W_n=b\varphi;$$
$$\forall i\in\{1,\ldots,n\}\ \exists P_i,Q_i\in M(S)\ \exists(u_i,v_i)\in R:$$
$$W_{i-1}=P_i(u_i\varphi)Q_i\quad\text{and}\quad W_i=P_i(v_i\varphi)Q_i.$$

We prove the following result. Let $z\in S$ and $C,D\in M(S)$ be such that $C(z\varphi)D\in S\varphi$. Then

$$\exists c,d\in S:c\varphi\subseteq C,\qquad d\varphi\subseteq D\quad\text{and}\quad(czd)\varphi=C(z\varphi)D. \tag{3.1}$$

Since $C(z\varphi)D\in S\varphi$, there exists some $w\in S$ such that $C(z\varphi)D=w\varphi$. Since $w\in w\varphi$, there exist $c\in C$, $z'\in z\varphi$ and $d\in D$ such that $cz'd=w$. Since $c\varphi\subseteq C$, $z'\varphi\subseteq z\varphi$ and $d\varphi\subseteq D$, we obtain $w\varphi=(cz'd)\varphi=(c\varphi)(z'\varphi)(d\varphi)\subseteq(c\varphi)(z\varphi)(d\varphi)\subseteq C(z\varphi)D=w\varphi$. Therefore $(czd)\varphi=C(z\varphi)D$ and (3.1) holds.

Since S is E-unitary, it is clear that

$$\forall A\in M(S),\qquad AA^{-1}\subseteq 1\sigma\subseteq E(S). \tag{3.2}$$

Now we show that, for every $j\in\{0,\ldots,n\}$

$$\exists w_j\in S:w_j\varphi\subseteq W_j\quad\text{and}\quad(a,w_j)\in R^{\#}. \tag{3.3}$$

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Let $w_0 = a$. It follows that (3.3) holds for $j = 0$.

Now suppose that (3.3) holds for $j = i - 1$, with $i > 0$. Then $w_{i-1}\varphi \subseteq W_{i-1}$ and so, since S is inverse, $w_{i-1}\varphi \subseteq W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi)$. By (3.2), we also have $W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi) \subseteq w_{i-1}\varphi$. Hence $w_{i-1}\varphi = W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi) = P_i(u_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$. Now we can apply (3.1) with $z = u_i$, $C = P_i$ and $D = Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$. Hence there exist $p_i, q_i \in S$ such that $p_i\varphi \subseteq P_i$, $q_i\varphi \subseteq Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$ and $(p_iu_iq_i)\varphi = P_i(u_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi) = w_{i-1}\varphi$. We define $w_i = p_iv_iq_i$. Now $w_i\varphi = (p_i\varphi)(v_i\varphi)(q_i\varphi) \subseteq P_i(v_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi) = W_iW_{i-1}^{-1}(w_{i-1}\varphi) \subseteq W_iW_{i-1}^{-1}W_{i-1}$ and so, by (3.2), we have $w_i\varphi \subseteq W_i\,.\,E(S)$. For every $s \in S$ and $e \in E(S)$, we have $ae = aea^{-1}a$, and hence $W_i\,.\,E(S) \subseteq E(S)\,.\,W_i$. Therefore $w_i\varphi \subseteq W_i\,.\,E(S) \subseteq E(S)\,.\,W_i \subseteq W_i$. Moreover, $w_iR^{\#} = (p_iv_iq_i)R^{\#} = (p_iu_iq_i)R^{\#} = w_{i-1}R^{\#} = aR^{\#}$ and so (3.3) holds for $j = i$. Thus (3.3) holds for every $j \in \{0, \ldots, n\}$.

In particular, $w_n\varphi \subseteq W_n = b\varphi$ and $(a, w_n) \in R^{\#}$. Hence $w_n \leq b$ and $aR^{\#} = w_nR^{\#} \leq bR^{\#}$. Similarly, we prove that $bR^{\#} \leq aR^{\#}$. Thus $(a, b) \in R^{\#}$ and the lemma is proved.

Now we obtain

THEOREM 3.3. *Every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group.*

Proof. Let S be an E-unitary inverse semigroup. By Lemma 3.2, the embedding $\varphi: S \to M(S): s \mapsto \{t \in S : t \leq s\}$ is normal-convex. By Lemma 3.1, $M(S)$ is F-inverse and so, by Lemma 1.4 and Theorem 2.1, there exists a normal-convex embedding $\psi: M(S) \to P$, where P is a semidirect product of a semilattice by a group. By Lemma 1.2, the composition $\varphi\psi: S \to P$ is a normal-convex embedding and the theorem is proved.

4. Inverse semigroups. The results of Section 2 can be used to obtain a general embedding result on inverse semigroups. We shall make use of the following result on quasi-free covers, due to Munn and Reilly.

LEMMA 4.1 [**2**]. *Let S be an inverse semigroup. Then there exists a quasi-free inverse semigroup F and an idempotent-separating congruence τ on F such that $S \simeq F/\tau$.*

Now we have

THEOREM 4.2. *Every inverse semigroup admits a normal-convex embedding into an idempotent-separating homomorphic image of a semidirect product of a semilattice by a free group.*

Proof. Let S be an inverse semigroup. By Lemma 4.1, we can assume that $S = F/\tau$, with F quasi-free and τ idempotent-separating. By Lemma 1.5, we can assume that $F = P(G, K, L)$ for some strong McAlister triple (G, K, L), with G free. By Theorem 2.1, the inclusion $\varphi: F \to K \rtimes G$ is normal-convex. Therefore, by Lemma 1.1, the induced map $\psi: F/\tau \to (K \rtimes G)/(\tau\varphi)^{\#}$ defined by $(a\tau)\psi = a(\tau\varphi)^{\#}$ is injective. We must prove that ψ is normal-convex and $(\tau\varphi)^{\#}$ is idempotent-separating.

First we prove that ψ is normal-convex. Let $T = (K \rtimes G)/(\tau\varphi)^{\#}$. Let R be a relation on S. We want to show that $(R\psi)^{\#} \cap (S \times S)\psi \subseteq R^{\#}\psi$.

Let μ be the congruence on F such that $\mu/\tau = R^{\#}$. It follows that, for every $a, b \in F$, $(a, b) \in \mu$ if and only if $(a\tau, b\tau) \in R^{\#}$. We prove that

$$(R\psi)^{\#} \subseteq (\mu\varphi)^{\#}/(\tau\varphi)^{\#}. \tag{4.1}$$

Since $\tau \subseteq \mu$, we have $\tau\varphi \subseteq \mu\varphi$ and so $(\tau\varphi)^{\#} \subseteq (\mu\varphi)^{\#}$. Hence $(\mu\varphi)^{\#}/(\tau\varphi)^{\#}$ is a congruence on T and we only need to show that $R\psi \subseteq (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$. Let $a, b \in F$ be such

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that $(a\tau, b\tau) \in R$. Then $(a\tau, b\tau) \in R^{\#}$ and so, by definition of μ, we have $(a, b) \in \mu$. Hence $(a\varphi, b\varphi) \in \mu\varphi \subseteq (\mu\varphi)^{\#}$. Therefore $(a\varphi(\tau\varphi)^{\#}, b\varphi(\tau\varphi)^{\#}) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$, that is, $((a\tau)\psi, (b\tau)\psi) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$. Hence (4.1) holds.

Now suppose that $a, b \in F$ and $((a\tau)\psi, (b\tau)\psi) \in (R\psi)^{\#}$. Then, by (4.1), we have $((a\tau)\psi, (b\tau)\psi) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$. Hence $(a\varphi(\tau\varphi)^{\#}, b\varphi(\tau\varphi)^{\#}) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$ and so $(a\varphi, b\varphi) \in (\mu\varphi)^{\#}$. Since φ is normal-convex and μ is a congruence on F, we have $(\mu\varphi)^{\#} \cap (F \times F)\varphi \subseteq \mu\varphi$. Hence $(a\varphi, b\varphi) \in \mu\varphi$ and so $(a, b) \in \mu$ and $(a\tau, b\tau) \in R^{\#}$. Therefore $((a\tau)\psi, (b\tau)\psi) \in R^{\#}\psi$ and so ψ is normal-convex.

Now we prove that $(\tau\varphi)^{\#}$ is idempotent-separating. Obviously, $E(K \overline{\times} G) = \{(A, 1) : A \in K\}$. Suppose that $A, B \in K$ are such that $(A, 1)(\tau\varphi)^{\#} = (B, 1)(\tau\varphi)^{\#}$. Since $GL = K$, there exists $g \in G$ and $C \in L$ such that $gC = A$. Hence $g^{-1}A = C \in L$ and we have

$$(g^{-1}A, 1)(\tau\varphi)^{\#} = [(g^{-1}A, g^{-1})(A, 1)(A, g)](\tau\varphi)^{\#} = [(g^{-1}A, g^{-1})(B, 1)(A, g)](\tau\varphi)^{\#}$$
$$= ((g^{-1}A)(g^{-1}B), 1)(\tau\varphi)^{\#}.$$

Since $(g^{-1}A)(g^{-1}B) \leq g^{-1}A \in L$ and L is an ideal of K, we have $(g^{-1}A)(g^{-1}B) \in L$. Hence $(g^{-1}A, 1), ((g^{-1}A)(g^{-1}B), 1) \in F$. But

$$[(g^{-1}A, 1)\tau]\psi = (g^{-1}A, 1)(\tau\varphi)^{\#} = ((g^{-1}A)(g^{-1}B), 1)(\tau\varphi)^{\#} = [((g^{-1}A)(g^{-1}B), 1)\tau]\psi$$

and so, since ψ is injective, $(g^{-1}A, 1)\tau = ((g^{-1}A)(g^{-1}B), 1)\tau$. Since τ is idempotent-separating, we obtain $(g^{-1}A, 1) = ((g^{-1}A)(g^{-1}B), 1)$, that is, $g^{-1}A = (g^{-1}A)(g^{-1}B)$. Hence $A = AB$ and $A \leq B$. Similarly, we obtain $B \leq A$ and so $A = B$. Thus $(A, 1) = (B, 1)$ and $(\tau\varphi)^{\#}$ is idempotent-separating.

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REFERENCES

1. D. B. McAlister, Groups, semilattices and inverse semigroups II, *Trans. Amer. Math. Soc.* **196** (1974), 351–370.

2. W. D. Munn and N. R. Reilly, *E*-unitary congruences on inverse semigroups, *Glasgow Math. J.* **17** (1976), 57–75.

3. L. O'Carroll, Reduced inverse and partially ordered semigroups, *J. London Math. Soc.* (2)**9** (1974), 293–301.

4. C. D. Papakyriakopoulos, A new proof for the invariance of the homology groups of a complex, *Bull. Soc. Math. Grèce* **22** (1943), 1–154.

5. M. Petrich, *Inverse semigroups*, (Wiley, 1984).

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