Normal Elliptic Scrollar Varieties.

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Summary. – *In his classical memoir on the projective classification of elliptic ruled surfaces Corrado Segre described in particular two most general normal types, of even and odd order respectively, of which the former has precisely two minimum directrix curves, while the latter has an elliptic pencil of such curves. The present paper extends this work to normal elliptic scrollar varieties of dimension k, defining and describing k most general types of such varieties. Particular attention is paid to one of these types, which we call the simploid, in which the points of the variety correspond to the unordered sets of k values of an elliptic parameter (modulo its periods). The paper concludes with the identification of a series of self-dual « linked pairs » of such scrollar varieties, of which the simplest example is that of the elliptic quintic ruled surface and the elliptic quintic scrollar threefold in four-dimensional space.*

1. – In two of his classical memoirs, CORRADO SEGRE [1, 2] discussed the projective classification and properties of elliptic curves and elliptic scrolls (ruled surfaces) respectively. Not much appears to have been done, however, about the projective classification of elliptic scrollar varieties of higher dimension—loci R_k of elliptic systems of spaces of dimension $k-1$, where $k \geqslant 3$; and it is the object of the present paper to contribute to this topic.

For elliptic scrolls R_2^n of order n, Segre showed that such a scroll, if it is not a cone, is normal in $[n-1]$; whereas an elliptic ruled surface R_2^n that is normal in $[n]$ is necessarily a cone. In classifying normal R_2^n, excluding cones, by their directrix curves of minimum order, he identified in particular two general types of special significance, namely the type, for n even, which possesses precisely two minimum directrix curves γ_1, γ_2, each of order $\frac{1}{2}n$, and the type, for n odd, which possesses an elliptic pencil (γ), of index two, of directrix curves of minimum order $\frac{1}{2}(n+1)$. A scroll of either of these types projects from a general point of itself into a scroll of the other type; and it is not difficult to deduce from Segre's work that every other type of normal scroll R_2^n, including every type with a minimum directrix curve of order less than $\frac{1}{2}n$, can be obtained as a projection of a scroll of one or other of the two types in question from a suitably chosen set of points lying on it.

As regards elliptic scrollar varieties of higher dimension, SEGRE [3] later recorded an observation to the following effect:

An elliptic scrollar variety of order n and dimension k, if it is not a cone, is normal in $[n-1]$; *and if it is a cone with a space* $[s]$ *as vertex, then it is normal in* $[n+s]$.

(*) Entrato in Redazione il 18 Maggio 1973.

In this paper, we take it as our limited objective to identify and describe k general types of normal elliptic scrollar k-folds analogous to the two general types of normal elliptic scrolls to which we have already referred. Among the k types so arising we shall be specially interested in one of them—which we call the « simploidal » type (*)—this being a type of elliptic scrollar k-fold whose points are in unexceptional (1,1) correspondence with the *unordered* sets $(u_1, \ldots, u_k)$ (mod $2\omega_1, 2\omega_2$) of k values of an elliptic parameter u with periods $2\omega_1$ and $2\omega_2$.

2. – An elliptic scrollar variety may be thought of primarily as a one-dimensional elliptic system $\{\Pi_{k-1}\}$ of subspaces $[k-1]$ of a space S_N, where k may have any one of the values $1, 2, \ldots, N$. Its order n is the number of its spaces Π_{k-1} that meet a generic $[N-k]$ of S_N or, equivalently, the number of its Π_{k-1} that are co-prime with a generic $[N-k]$. The points of all the Π_{k-1} constitute a k-dimensional algebraic variety of order n — the whole space S_N counted n times if $k=N$; and the primes through the Π_{k-1} constitute an $(N-k+1)$-dimensional algebraic envelope of class n — the whole dual space of S_N counted n times if $k=1$.

In view of Segre's observation quoted in § 1, and because we shall be concerned only with normal elliptic scrollar varieties that are not cones, it will be convenient to adopt the following notation throughout the paper.

(i) A symbol of the form E_k^n will always denote the locus, in its ambient space S_{n-1}, of generators $[k-1]$ of a normal elliptic scrollar variety (not a cone) of order n; and

(ii) a symbol of the form $\mathcal{E}_{n-k}^n$ will always denote the envelope of primes through the $[k-1]$'s of an E_k^n.

A point which lies on s generators of E_k^n will be s-fold on E_k^n; and a prime which contains s generators will be an s-fold prime of the associated $\mathcal{E}_{n-k}^n$. A locus E_{n-k}^n is the dual of an envelope $\mathcal{E}_{n-k}^n$.

If we fix k and consider varieties E_k^n for increasing n, then the exceptional case with which we start is that of an E_k^{k+1}, this being a space S_k counted $k+1$ times, its generating $[k-1]$'s being the primes of an elliptic envelope of class $k+1$ — dual of an elliptic curve ${}^1C^{k+1}$ of S_k. Thereafter, for $n=k+2$, we have a scrollar primal E_k^{k+2} of S_{k+1}, of which the simplest examples, for $k=2$ and $k=3$, are the elliptic quartic scroll E_2^4 of S_3 with two double lines, and the well-known quintic E_3^5 of S_4 with an E_2^5 as its double surface. For $n \leqslant 2k-1$, two generators Π_{k-1} will meet in a space $[2k-n-1]$, and such spaces will generate the double locus V_{2k-n+1} of E_k^n; for $n \leqslant \frac{1}{2}(3k-1)$, E_k^n will possess similarly a triple locus $V_{3k-2n+2}$, locus of the ∞^3 spaces $[3k-2n-1]$ of intersection of triads of generators; and so on. For $n > 2k-1$, the generators Π_{k-1} of E_k^n will in general be skew to one another (though some pairs may meet); and this is the case that we shall have principally in mind.

(*) Properties of elliptic ruled surfaces of this type (the case $k=2$) have been discussed briefly in a paper by DU VAL and SEMPLE [4].

For the envelopes $\mathcal{E}^n_{n-k}$, for fixed k as n increases from the value $k+1$, the aggregates of double primes, triple primes, etc., will come in as n increases, first with primes containing pairs of generators, then with primes containing triads of generators, and so on.

3. – Sections and projections of varieties E^n.

Complete linear sections of an E^n_k, being subnormal if they are irreducible, cannot be varieties $E^n_{k-\alpha}$ $(\alpha > 0)$; but, as will shortly appear, we do get partial linear sections, residual to one or more generators, which are normal elliptic scrollar varieties.

As regards projections of an E^n_k from a space S_β, we are only concerned with those which are birational; and this excludes in particular any projection from a space S_β which contains a directrix curve of E^n_k (i.e. a curve meeting each generator in one point). The special case in which E^n_k is projected into a space S_k counted $k+1$ times can be interpreted appropriately. When projections of varieties E^n_k for fixed k and variable n are considered, it is an advantage to take n large enough (with respect to k) so that complications arising from mutual intersections of generators are avoided.

Consider first, then, a birational projection of E^n_k from a simple point P of itself into a prime S_{n-2} of its ambient space S_{n-1}. Plainly this projection, not being a cone, will be an E^{n-1}_k normal in S_{n-2}. If Π is the generator of E^n_k through P, then, as is well-known for scrollar varieties in general, the generators of E^n_k other than Π project into all but one of the generators of E^{n-1}_k; the tangent $[k]$ to E^n_k at P projects into the remaining generator of E^{n-1}_k, say Π^*, and Π^* is indistinguishable from the other generators of E^{n-1}_k; and Π itself projects into a $[k-2]$ of Π^*, likewise indistinguishable from the other $[k-2]$'s of Π^* in relation to E^{n-1}_k.

Now consider the projection of E^n_k from a line containing just two simple points P, Q of E^n_k, again supposing that the projection is birational. It follows from the above observations that the projection is an E^{n-2}_k; and this is true whether (*a*) P and Q lie in distinct generators of E^n_k, or (*b*) Q is consecutive to P, so that PQ is a tangent line to E^n_k at P; the same remains true, in fact, even if (*c*) P and Q lie in the same generator, so that their join lies on E^n_k but do s not pass through any multiple point of this variety.

By repeated application of the above observations we obtain in particular the following result:

Proposition 3.1. – *Provided that the projections concerned are birational, an E^n_k projects from one of its generators into an E^{n-k}_k; and it projects from an $(sk-1)$-dimensional space $B^{(s)}$ spanning s of its generators (mutually skew to one another) into an E^{n-sk}_k.*

From this there follows

Proposition 3.2. – *Subject to the restrictions of Prop. 3.1, an $[n-k-1]$ through a generator of E^n_k meets this variety residually in $n-k$ points; and, more generally, an $[n-k-1]$ through s generators meets E^n_k residually in $n-sk$ points.*

If Π is a generator of E_k^n, then any general prime through Π meets E_k^n residually in an E_{k-1}^{n-1} of which one generator lies in Π. For plainly this residual intersection cannot be a cone; and if it belonged to a space of dimension $n-3$, then primes through this space would have to meet E_k^n residually in its system of generators, which is impossible. Similarly, if $t<k$, a general space $[n-1-t]$ through Π will meet E_k^n residually in an E_{k-t}^{n-t} of which one generator $[k-t-1]$ lies in Π; while, if $t=k$, then the residual section is a set of $n-k$ points spanning an $[n-k-1]$.

By a straightforward extension of the above, we have

PROPOSITION 3.3. – *If $B^{(s)}$ is a space, of dimension $sk-1\leqslant n-2$, which spans s (mutually skew) generators $\Pi^{(1)}, \ldots, \Pi^{(s)}$ of an E_k^n and contains no directrix curve of the E_k^n, then any general prime through $B^{(s)}$ meets E_k^n residually in an E_{k-1}^{n-s} of which one generator lies in each $\Pi^{(i)}$. Further, if $t<k$, any general $[n-1-t]$ through $B^{(s)}$ meets E_k^n residually in an E_{k-t}^{n-ts} of which one generator lies in each $\Pi^{(i)}$; while, if $t=k$, the residual intersection is a linearly independent set of $n-sk$ points.*

All the $E_{k-\beta}^{n-\alpha}$ which occur in the above propositions are directrix varieties of E_k^n according to the following

DEFINITION. – A *directrix d-fold of an E_k^n* is any d-dimensional elliptic scrollar subvariety of E_k^n which meets every generator of this variety in a $[d-1]$.

4. – Generation of E_k^n by related directrix curves.

Let $C_0, \ldots, C_{k-1}$ be a set of k equimodular normal elliptic curves, of orders $n_0, \ldots, n_{k-1}$, whose ambient spaces $[n_i-1]$ span a space S_{n-1}, where $n=n_0+\ldots+n_{k-1}$; and let points $P_0, \ldots, P_{k-1}$ describe birationally related ranges on the curves. Each set $(P_0, \ldots, P_{k-1})$ must span a $[k-1]$, say Π_{k-1}, since otherwise the spaces $[n_i-1]$ would not span S_{n-1}. We find easily, then, that as the set $(P_0, \ldots, P_{k-1})$ varies along the curves, its ambient space Π_{k-1} generates an E_k^n. *A priori* some of the curves $C_0, \ldots, C_{k-1}$ may be isolated on E_k^n, while others may belong to algebraic systems of positive dimension.

In accord with our objective, as stated in § 1, we now propose to define, for given k, a set of k general types of E_k^n; and we begin by describing the general type for the case when n is an integral multiple mk of k $(m\geqslant 2)$.

5. – The key variety E_k^{mk} of S_{mk-1}.

From the preceding section it follows that if $n_0+\ldots+n_{k-1}=mk$, then there exist varieties E_k^{mk} generated by birationally related ranges on k equimodular curves ${}^1C^{n_0}, \ldots, {}^1C^{n_{k-1}}$ whose ambient spaces span a space $[mk-1]$; and, in particular, taking $n_i = m$ $(i = 0, ..., k-1)$, there exist E_k^{mk} generated by birationally related ranges on k equimodular normal curves ${}^1C^m$.

DEFINITION. – An E_k^{mk} will be called a *key elliptic scrollar variety* for $k \geqslant 2$ and $m \geqslant 2$ if it possesses precisely k minimal directrix curves ${}^1C^m$, being then generated by joins of corresponding points of birationally related ranges on the curves.

The above definition, as should be noted, excludes the case in which the correspondence between any two of the curves is projective; for in such case the ruled surface generated by the projective correspondence between the two curves would be of the projectively generated type of E_2^{2m}, and this contains a rational pencil of directrix curves ${}^1C^m$ which would also be directrix curves of E_k^{mk}.

We propose now to regard the key E_k^{mk}, as above defined, as the most general type of normal elliptic scrollar k-fold whose order is an integral multiple of its dimension. Some justification of the importance we thus attach to it, apart from its being an obvious generalization of one of the two basic types of elliptic scrollar surface (§ 1), is provided by the following result.

THEOREM 5.1. – *If an E_k^{mk} $(k \geqslant 2, m \geqslant 3)$ possesses no directrix curve of order less than m, then it possesses in general k directrix curves of order m.*

PROOF. – Let $B^{(m-1)}$ denote the space, of dimension $(m-1)k - 1 = (mk-1) - k$, spanned by a set of $m-1$ generators of E_k^{mk}. Plainly, since E_k^{mk} contains no directrix curve of order less than m, we may suppose that $B^{(m-1)}$ contains no directrix curve of the variety; whence, by Prop. 3.3, it meets E_k^{mk}, residually to the $m-1$ generators, in a set of k points spanning a $[k-1]$. Let P be any one of these points. Then through P there passes a unique space $[m-2]$ which meets each of the $m-1$ generators in a point; and we denote this space by π. Since π meets E_k^{mk} in m points, the projection of E_k^{mk} from π will be a scrollar variety R_k^{mk-m} whose ambient space is of dimension $mk - 1 - (m-2) - 1 = mk - m$; whence, by the observation quoted in § 1, R_k^{mk-m} must be a point-cone.

If O is the vertex of this cone, then all the generators of E_k^{mk} must meet the $[m-1]$ which joins π to O; and they will meet it in the points of a directrix curve of E_k^{mk}, plainly a ${}^1C^m$ $[m-1]$. It follows then, since one such curve arises from each of the k points such as P (in general distinct), that E_k^{mk} possesses in general k directrix curves of order m, as was to be proved.

We now proceed to investigate the properties of a key E_k^{mk} and also those of the general types of $E_k^{mk-\alpha}$ that can be obtained from it by projection.

6. – Directrix varieties of the key E_k^{mk}.

In what follows we shall be largely concerned with systems of directrix $(k-1)$-folds on an E_k^n residual, within the complete system $|H|$ of prime sections of E_k^n, to a set $G^{(s)}$

of s generators of this variety. In this connection we introduce the

DEFINITION. – An *s-residual variety* $R^{(s)}$ on an E_k^n is the residual intersection of E_k^n with a prime (of its ambient space) which contains s of its generators.

If, as will in general be the case, $R^{(s)}$ is neither reducible nor a cone, then, by Prop. 3.3, it is a directrix variety E_{k-1}^{n-s} of E_k^n.

Any set $G^{(s)}$ of s generators of E_k^n belongs to a linear series $G_{s-1}^s = |G^{(s)}|$ of sets of s generators; primes through $G^{(s)}$, as we may suppose, meet E_k^n residually in a complete linear system of s-residuals $|R^{(s)}| = |E_{k-1}^{n-s}|$; the ∞^{s-1} primes through the ambient $[n-s-1]$ of any one of these E_{k-1}^{n-s} meet E_k^n residually in the sets of the abovementioned G_{s-1}^s; in brief, $|R^{(s)}|$ and G_{s-1}^s are residual within the complete prime section system $|H|$.

Now consider the key variety E_k^{mk}. A set $G^{(m)}$ of m of its generators is not in general contained in any prime; but there exist, nevertheless, certain sets $G^{(m)}$ which are contained in primes, as follows from the following observation:

The key variety E_k^{mk} possesses, along with its k minimal directrix curves ${}^1C^m$, a set of k minimal directrix $(k-1)$-folds. These are, namely, the k key varieties $E_{k-1}^{m(k-1)}$ each of which has $k-1$ of the ${}^1C^m$ as directrix curves and is generated by the original correspondences between these curves.

Residual to each of these $E_{k-1}^{m(k-1)}$ there is a linear series G_{m-1}^m of sets of m generators; and the sets of the k series G_{m-1}^m so arising are the only sets $G^{(m)}$ of E_k^{mk} that lie in primes.

We pass on now to the $(m-1)$-residuals $R^{(m-1)}$ on E_k^{mk}, these being in general varieties E_{k-1}^{mk-m+1} residual to generator sets $G^{(m-1)}$. They form, in aggregate, a nonlinear system which we shall denote by $\{R^{(m-1)}\}$; and this is composed of ∞^1 linear systems $|R^{(m-1)}|$, each residual, within the complete linear system $|H|$ of prime sections of E_k^{mk}, to one of the linear series G_{m-2}^{m-1} of generators.

If $B^{(m-1)}$ is the space $[mk-1-k]$ spanning a generator set $G^{(m-1)}$, then the ∞^{k-1} primes through $B^{(m-1)}$ cut a complete system $|R^{(m-1)}|$ on E_k^{mk}; whence the dimensions of $|R^{(m-1)}|$ and $\{R^{(m-1)}\}$ are $k-1$ and k respectively. As previously noted in § 5, $B^{(m-1)}$ meets E_k^{mk}—residually to $G^{(m-1)}$—in k points $P_1, \dots, P_k$; and these are therefore base points of the associated system $|R^{(m-1)}|$. The points P_i, moreover, lie one on each of the k directrix curves ${}^1C^m$; for primes through $B^{(m-1)}$ already meet each ${}^1C^m$ in $m-1$ known points, and they therefore meet it in one further fixed point, the whole group of m points on this ${}^1C^m$ having the property that the m generators through them are co-prime. This gives

PROPOSITION 6.1. – *The $(m-1)$-residual varieties $R^{(m-1)}$ on E_k^{mk} form an algebraic ∞^k-system $\{R^{(m-1)}\}$ composed of an elliptic ∞^1-system of linear ∞^{k-1}-systems $|R^{(m-1)}|$ residual to the ∞^1 linear series G_{m-2}^{m-1} of generators. Each $|R^{(m-1)}|$ has k base points, one on each of the k directrix curves ${}^1C^m$ of E_k^{mk}.*

Our next result—a generalization of one of Segre's for $k=2$—is as follows:

PROPOSITION 6.2. – *The ∞^k-system $\{R^{(m-1)}\}$, as above defined, is k-atic, in the sense that the number of members of it that pass through k general points of E_k^{mk} is k.*

PROOF. – Let $G^{(m-2)}$ be a fixed set of $m-2$ generators of E_k^{mk}. We observe, then, that every member of $\{R^{(m-1)}\}$ is residual to a unique set of the form $G^{(m-2)}+G^{(1)}$. Let $A_1, \ldots, A_k$ be k points of general position on E_k^{mk}; and let Σ be the $[mk-1-k]$ that contains $G^{(m-2)}$ and $A_1, \ldots, A_k$. By Prop. 3.3, Σ has k further intersections $P_1, \ldots, P_k$ with E_k^{mk}. Through each P_i $(i=1, \ldots, k)$ there passes a generator $G_i^{(1)}$ which, since it meets Σ in a point, lies with Σ in a prime; and this prime meets E_k^{mk}, residually to $G^{(m-2)}+G_i^{(1)}$, in a variety of $\{R^{(m-1)}\}$ which passes through $A_1, \ldots, A_k$. Since one such variety arises from each of $P_1, \ldots, P_k$, and since any member of $\{R^{(m-1)}\}$ that passes through $A_1, \ldots, A_k$ determines one of the P_i, the Proposition is established.

COROLLARY 6.2.1. – *If $\{R_s^{(m-1)}\}$ is the ∞^s-subsystem of $\{R^{(m-1)}\}$ with $k-s$ assigned base points $(1 \leqslant s \leqslant k-1)$, then $\{R_s^{(m-1)}\}$ is also k-atic in the sense that k of its members pass through any general set of s points of E_k^{mk}.*

To obtain further information about directrix varieties on E_k^{mk}, we now make two observations of which the second will be particularly useful:

(*a*) E_k^{mk} $(m \geqslant 3)$ projects from a generator of itself into a key variety $E_k^{(m-1)k}$, the k minimal directrix curves ${}^1C^m$ of the former projecting into the k minimal directrix curves ${}^1C^{m-1}$ of the latter. By this projection the k-atic system $\{R^{(m-1)}\}$ of the former projects into the k-atic system $\{R^{(m-2)}\}$ of the latter.

(*b*) E_k^{mk} $(m \geqslant 3)$ also projects into a key $E_k^{(m-1)k}$ from any general set $A_1, \ldots, A_k$ of points of itself, the k members of $\{R^{(m-1)}\}$ that pass through $A_1, \ldots, A_k$ projecting into the k minimal directrix varieties $E_{k-1}^{(m-1)(k-1)}$ of $E_k^{(m-1)k}$. In this projection, by which the orders of directrix varieties of E_k^{mk} that pass (simply) through $A_1, \ldots, A_k$ are reduced by k, the minimal directrix curves ${}^1C^{m-1}$ of $E_k^{(m-1)k}$ are projections of the curves of intersection of sets of $k-1$ of the k members of $\{R^{(m-1)}\}$ that pass through $A_1, \ldots, A_k$.

From (*b*) in particular, we deduce at once the following comprehensive result:

PROPOSITION 6.3. – *The $(m-1)$-residual system $\{R^{(m-1)}\}$ of E_k^{mk} has the following properties:*

(i) *k members of $\{R^{(m-1)}\}$ meet, in general, in precisely k points;*

(ii) *$k-1$ members of $\{R^{(m-1)}\}$ meet, in general, in a curve ${}^1C^{m+k-1}$; and, of the $\infty^{k(k-1)}$ curves so arising, precisely k pass through k general points of E_k^{mk};*

(iii) *more generally, s members of $\{R^{(m-1)}\}$ $(1 \leqslant s \leqslant k-1)$ meet in general in a directrix variety $E_{k-s}^{(m-1)(k-s)+k} \equiv E_{k-s}^{m(k-s)+s}$.*

As regards (ii), however, we should note that ∞^k of the $\infty^{k(k-1)}$ intersection curves ${}^1C^{m+k-1}$ are exceptional, namely those curves each of which is an intersection of $k-1$ members of a linear system $|R^{(m-1)}|$ and is therefore the base curve of a linear ∞^{k-2}-subsystem of the $|R^{(m-1)}|$.

If we put $s = k-2$ in (iii), it appears that the members of $\{R^{(m-1)}\}$ meet by sets of $k-2$ in $\infty^{k(k-2)}$ ruled surfaces E_2^{2m+k-2} of general type; and each of these contains, if k is even, two directrix curves of order $\frac{1}{2}(2m+k-2)$, or, if k is odd, an elliptic pencil of directrix curves of order $\frac{1}{2}(2m+k-1)$.

We now ask, generally, what directrix curves ${}^1C^{m+\alpha}$ $(1 \leqslant \alpha \leqslant k-2)$ exist on E_k^{mk}, of orders intermediate between that of the minimal directrix curves ${}^1C^m$ and that of the complete intersection curves ${}^1C^{m+k-1}$. Any such curve ${}^1C^{m+\alpha}$ will be a residual intersection of $k-1$ members of $\{R^{(m-1)}\}$ that contain $k-1-\alpha$ lines lying each in a generator of E_k^{mk}. A straightforward calculation, on this basis, of the dimension of the system of ${}^1C^{m+\alpha}$ on E_k^{mk} leads to the result:

PROPOSITION 6.4. – *For $0 \leqslant \alpha \leqslant k-1$, the directrix curves ${}^1C^{m+\alpha}$ on E_k^{mk} form an $\infty^{k\alpha}$ system, there being ∞^α of them that pass through α general points of E_k^{mk}.*

The case $k=3$. For elliptic scrollar threefolds, the first two key varieties that arise are $E_3^6\,[5]$ and $E_3^9\,[8]$, given respectively by $m=2$ and $m=3$.

The variety $E_3^6\,[5]$ has three (equimodular) elliptic double lines as its minimal directrix curves. This means, if these are carried by lines p, q, r (spanning S_5), that E_3^6 is generated by the planes that join corresponding points in certain (2, 2) correspondences between the lines, say between p and q and between p and r. E_3^6 has three minimal directrix scrolls E_2^4, each having two of p, q, r as double lines; and it possesses an ∞^3-system $\{R^{(1)}\}$ of scrolls E_2^5 which is composed of ∞^1 linear nets $|R^{(1)}|$, each residual to a unique generating plane of E_3^6. The directrix ${}^1C^3$ on E_3^6 form an ∞^3-system, while the surfaces of $\{R^{(1)}\}$ meet by pairs in ∞^6 directrix curves ${}^1C^4$.

The variety $E_3^9\,[8]$ has three minimal directrix cubics ${}^1C^3$ and three minimal directrix scrolls E_2^6. Sections residual to pairs of generating planes form an ∞^3-system $\{R^{(2)}\}$ of septimic scrolls E_2^7; and this, being again 3-atic, is composed of ∞^1 linear nets $|R^{(2)}|$, each residual to a G_1^2 of generators. The ${}^1C^4$ on E_3^9 form an ∞^3-system; and the surfaces of $\{R^{(2)}\}$ meet by pairs in ∞^6 directrix curves ${}^1C^5$. E_3^9 projects from any generating plane of itself into an E_3^6.

7. – The general types $E_k^{mk-s}(1 \leqslant s \leqslant k-1)$.

As previously remarked, a general projection of a key variety E_k^{mk} $(m \geqslant 3)$ from k of its points (or from a generator) is another key variety $E_k^{(m-1)k}$. We now propose to fill the gap between these two varieties as follows:

DEFINITION. – For $m \geqslant 2$, we shall say that an E_k^{mk-s} $(1 \leqslant s \leqslant k-1)$ is of *general type* (or briefly *general*) it it is a projection of a key E_k^{mk} from s points of general position on this variety.

We now summarize briefly the principal properties of such E_k^{mk-s}.

When a key E_k^{mk} $(m \geqslant 2)$ is projected into an E_k^{m-ks} $(1 \leqslant s \leqslant k-1)$ from a set of s points $P_1, \ldots, P_s$ of general position on itself, the ∞^{k-s} members of the k-atic system $\{R^{(m-1)}\}$ that pass through $P_1, \ldots, P_s$ project into an ∞^{k-s}-system $\{R_{(s)}^{(m-1)}\}$ of directrix varieties $E_{k-1}^{m(k-1)+1-s}$ on E_s^{mk-s}, this being plainly the system of $(m-1)$-residual varieties (residual to sets of $m-1$ generators) on E_k^{mk-s}. Further, this system $\{R_{(s)}^{(m-1)}\}$ is k-atic in the sense that k of its members pass through $k-s$ general points of E_k^{mk-s}; also $k-1$ of its members meet in general in a directrix curve ${}^1C^{m+k-1-s}$ of E_k^{mk-s}. More generally, from Prop. 6.4, we derive

PROPOSITION 7.1. – *For $0 \leqslant \alpha \leqslant k-1$, the directrix curves ${}^1C^{m+\alpha}$ on the general type of variety E_k^{m-ks} $(1 \leqslant s \leqslant k-1)$ form an $\infty^{k\alpha+s}$-system, there being $\infty^{\alpha+s}$ of them that pass through α general points of E_k^{mk-s}.*

In particular, the intersection curves ${}^1C^{m+k-s-1}$ of sets of $k-1$ members of $\{R_{(s)}^{(m-1)}\}$ form an $\infty^{(k-s)(k-1)}$-system on E_k^{mk-s}; and the dimension of the system of minimal directrix curves ${}^1C^m$ on E_k^{mk-s} increases steadily from 1 to $k-1$ as s increases from 1 to $k-1$. A further projection (putting $s=k$) sees the first appearance of directrix curves ${}^1C^{m-1}$ on a key variety $E_k^{(m-1)k}$.

The ∞^s minimal directrix curves ${}^1C^m$ on E_k^{mk-s} $(1 \leqslant s \leqslant k-1)$ generate a variety N_{s+1} which we may call the *nuclear variety* of E_k^{mk-s} — a surface if $s=1$ and the variety $E_k^{mk-(k-1)}$ itself if $s=k-1$.

For $k=3$, the general type E_3^5 (projection of E_3^6 from a point) is the well-known scrollar quintic primal of S_4 whose double surface is an E_2^5 (its dual); its ∞^1 elliptic double lines (minimal directrix curves) lie along the generators of E_2^5, and its generating planes are those which contain the ∞^1 directrix cubic curves of E_2^5.

The most interesting member of the sequence E_k^{mk-s} $(1 \leqslant s \leqslant k-1)$ is undoubtedly the last, given by $s=k-1$; and we now consider this in more detail.

8. – The simploidal variety $E_k^{mk-(k-1)} \equiv E_k^{(m-1)k+1}$.

The general type that we now discuss is such that $m-1$ of its generators always span a prime of its ambient space $S_{(m-1)k}$. Its $(m-1)$-residual varieties form therefore an elliptic ∞^1-system $\{R_{(k-1)}^{(m-1)}\}$ which we shall denote briefly by $\{\Phi\}$; and this system has the following properties:

(i) each Φ is a directrix variety $E_{k-1}^{(m-1)(k-1)-1}$ of $E_k^{(m-1)k+1}$, and it is residual to a linear generator series G_{m-2}^{m-1};

(ii) $\{\Phi\}$, being of dimension 1, is such that k of its members meet in one point, and k of its members (distinct or otherwise) pass through any point of $E_k^{(m-1)k+1}$.

By virtue of (ii) we shall call $\{\Phi\}$ a *simploidal system* on $E_k^{(m-1)k+1}$; and we shall say that $E_k^{(m-1)k+1}$ is a *simploidal variety* with $\{\Phi\}$ as its system of *simploids*.

If u (mod $2\omega_1, 2\omega_2$) is an elliptic parameter for the generators of $E_k^{(m-1)k+1}$, then the linear generator series G_{m-2}^{m-1} also form an elliptic system; and this system has an elliptic parameter w with the same periods $2\omega_1$, $2\omega_2$ as u. Plainly then, by (i) and (ii) above, w is also an elliptic parameter for $\{\Phi\}$, and each point of $E_k^{(m-1)k+1}$ is uniquely associated with the unordered k-ad $(w_1, \ldots, w_k)$ of values of w (mod $2\omega_1, 2\omega_2$) that correspond to the k members of $\{\Phi\}$ that pass through it. Furthermore, the correspondence between the points of $E_k^{(m-1)k+1}$ and the unordered k-ads $(w_1, \ldots, w_k)$ is (1,1) without exception. We have thus proved

PROPOSITION 8.1. – *The general type of elliptic scrollar variety $E_k^{(m-1)k+1}$ is simploidal, possessing an elliptic ∞^1-system $\{\Phi\}$ of directrix $(k-1)$-folds $E_{k-1}^{(k-1)(m-1)+1}$ such that k members of $\{\Phi\}$ meet in one point, while k members of $\{\Phi\}$, distinct or otherwise, pass through any point of $E_k^{(m-1)k+1}$. The points of this variety are in unexceptional* (1,1) *correspondence with the unordered sets of k values of an elliptic parameter w (mod periods $2\omega_1, 2\omega_2$).*

From the above, and with reference to Prop. 7.1 (for $s=k-1$ and $\alpha=0$) we derive

COROLLARY 8.2. – *The variety $E_k^{(m-1)k+1}$ possesses ∞^{k-1} minimal directrix curves ${}^1C^m$, and these are the curves of intersection of sets of $k-1$ members of $\{\Phi\}$—the* simploidal curves *of the variety. Each of them is given by equations of the form $w_i=\alpha_i$ (constant) (mod $2\omega_1, 2\omega_2$) for $i=1, \ldots, k-1$; and k of them, distinct or otherwise, pass through any point of $E_k^{(m-1)k+1}$.*

Finally, in the above connection, we ask what kind of relation between $w_1, \ldots, w_k$ represents a generator Π_{k-1} of $E_k^{(m-1)k+1}$. If we think of w as the parameter for points on a (non-singular) elliptic curve C, then the points of Π_{k-1} correspond to the sets of an algebraic series γ_{k-1}^k on C; and this γ_{k-1}^k is (a) rational, and (b) such that $k-1$ points of C belong to a unique set of the series. This last. in fact, follows from the observation that the ${}^1C^m$ common to $k-1$ members of $\{\Phi\}$ is a directrix curve of $E_k^{(m-1)k+1}$ and therefore meets Π_{k-1} in one point. It follows, then, by the Castelnuovo-Humbert Theorem (cf. ENRIQUES and CHISINI [5], Vol. III, p. 32 and p. 476) that γ_{k-1}^k is a linear series; and it is therefore given by a parametric equation of the form $w_1+\ldots+w_k \equiv \text{const.}$ (mod $2\omega_1, 2\omega_2$). This gives

PROPOSITION 8.3. – *In the representation of the points of a simploidal variety $E_k^{(m-1)k+1}$ by the unordered k-ads $(w_1, \ldots, w_k)$ of values of an elliptic parameter w (mod $2\omega_1, 2\omega_2$), the generators of $E_k^{(m-1)k+1}$ are each given by a parametric equation of the form $w_1+\ldots+$ $+w_k \equiv$ const. (mod $2\omega_1, 2\omega_2$).*

For $k=2$, the simploidal character and parametric representation of the general type of elliptic scroll E_2^{2m-1} have already been noted by DU VAL and SEMPLE [4].

For $k=3$, the simplest example ($m=3$) is that of an E_3^7 in S_6 for which $\{\Phi\}$ is the ∞^1-system of scrolls E_2^5 residual to pairs of generating planes of E_3^7; and this variety possesses ∞^2 minimal directrix curves ${}^1C^3$ which are the intersections of pairs of members of $\{\Phi\}$. The simploids Φ have parametric equations of the form $w_1 \equiv$ const., simploidal curves ${}^1C^3$ have equations of the form $w_1 \equiv$ const., $w_2 \equiv$ $\equiv$ const., and generating planes have equations of the form $w_1+w_2+w_3 \equiv$ const. (*).

From the results of this section, it will be noted that any simploidal variety $E_k^{(m-1)k+1}$ ($k \geqslant 2$, $m \geqslant 3$) is an unexceptional model of all the sets of k points of a (non-singular) elliptic curve. We now note further that the simploids Φ on such a variety are themselves in general simploidal varieties $E_{k-1}^{(m-1)(k-1)+1}$; and it will appear later that their ambient spaces $[(m-1)(k-1)]$ generate an elliptic scrollar variety $E_{(m-1)(k-1)+1}^{(m-1)k+1}$ which has $E_k^{(m-1)k+1}$ as its k-ple locus.

9. – The focal curve of an $E_k^{(m-1)k+1}$.

In extension of certain observations of Segre for the case $k=2$, we now remark that a simploidal variety $E_k^{(m-1)k+1}$ has various types of *coincidence loci*—loci of points of the variety for which coincidences of different kinds occur in the sets of k members of $\{\Phi\}$ that pass through them. More especially, $E_k^{(m-1)k+1}$ possesses a *focal curve* f such that the k simploids through any point of f all coincide. The equations of f are

$$w_1 \equiv w_2 \equiv \ldots \equiv w_k \pmod{2\omega_1, 2\omega_2} .$$

As in the case $k=2$, the simploids Φ envelop the curve f, each of them having k-point contact with f at its only common point with this curve; and the points of f are thereby in (1,1) correspondence with the members of $\{\Phi\}$.

If Π_{k-1} is the generator of $E_k^{(m-1)k+1}$ with parametric equation

$$w_1+\ldots+w_k \equiv c \pmod{2\omega_1, 2\omega_2} ,$$

then Π_{k-1} meets f in the k^2 points $(w, \ldots, w)$ for which $kw \equiv c \pmod{2\omega_1, 2\omega_2}$. Further f has k coincident intersections with any simploid Φ, and it has k^2 intersections with each of the $m-1$ generators in which $E_k^{(m-1)k+1}$ is met residually by a prime through Φ; so that the order of f is $(m-1)k^2+k$. This gives

PROPOSITION 9.1. – *A simploidal variety* $E_k^{(m-1)k+1}$ ($k \geqslant 2$, $m \geqslant 3$) *has an elliptic focal curve f, locus of points P for which the k simploids Φ through P all coincide. This curve f*

(*) It may be shown that the surfaces on E_3^7 with equations of the form $w_1+w_2 \equiv$ const constitute an ∞^1-system $\{J\}$ of elliptic sextic scrolls E_2^6, of which only one is residual to each generating plane of E_3^7. Each J, unlike the general E_2^6 residual to a generating plane, is projectively generated, possessing a rational pencil of minimal directrix curves ${}^1C^3$. Through each point of E_3^7 there pass three of the scrolls J; but three such scrolls, on the other hand, meet in general in four points.

is met by each Φ in k consecutive points; it meets each generator of $E_k^{(m-1)k+1}$ in k^2 points; and its order is $(m-1)k^2+k$.

Plainly $E_k^{(m-1)k+1}$ projects from any point of f into a specialization of the key $E_k^{(m-1)k}$ for which the k minimal directrix curves ${}^1C^{m-1}$ all coincide.

10. – Principal curves on the simploidal $E_k^{(m-1)k+1}$.

For the case $k=2$, Segre pointed out the existence on the general (simploidal) type of scroll E_2^{2m-1} of three normal elliptic curves ${}^1C^{2m-1}$, bisecant to the generators, to which he gave the name *principal curves* of E_2^{2m-1}. In the parametric representation of E_2^{2m-1} by unordered parameter pairs (w_1, w_2) (mod $2\omega_1, 2\omega_2$), the three curves in question have parametric equations $w_2 \equiv w_1+\omega_1$, $w_2 \equiv w_1+\omega_2$ and $w_2 \equiv w_1+ +\omega_1+\omega_2$. We now extend this result to $E_k^{(m-1)k+1}$.

Let ν be the number of cyclic elliptic involutions that are properly of order k on an elliptic ∞^1-system with parameter w (mod $2\omega_1, 2\omega_2$). For a discussion of these involutions we refer the reader to ENRIQUES and CHISINI [5], Vol. IV, 91-94. The general set of any one of them can be defined by a parameter set

$$(w, w+\alpha, w+2\alpha, \ldots, w+(k-1)\alpha) \qquad (\mathrm{mod}\, 2\omega_1, 2\omega_2)\,,$$

where α is a k-th part of a period,

$$\alpha=\frac{2m\omega_1+2n\omega_2}{k} \qquad (m,\ n \text{ integers},\ 0\leqslant m,\ n\leqslant k-1)\,,$$

such that the k numbers $0, \alpha, 2\alpha, \ldots, (k-1)\alpha$ are distinct (mod $2\omega_1$, $2\omega_2$). For any one involution the choice of an α which generates it as above is by no means unique, the number of equivalent choices for α being largely dependent on the prime factor decomposition of k. The values of ν for the values 2, 3, 4, 5 of k are 3, 4, 6, 6 respectively.

Now let w, as in the preceding section, be the elliptic parameter for $\{\Phi\}$ on $E_k^{(m-1)k+1}$ and let $\tau(\alpha)$ be the involution—properly of order k—on $\{\Phi\}$ generated as above by a suitably chosen α. As w varies, the point of $E_k^{(m-1)k+1}$ with parameter k-ad $(w, w+\alpha, \ldots, w+(k-1)\alpha)$ describes a curve $C(\alpha)$ which we shall call a *principal curve* on $E_k^{(m-1)k+1}$. The number of such curves is ν.

We observe first, then, that $C(\alpha)$ meets the simploid Φ given by $w_1\equiv\beta$ in the unique point with parameter k-ad $(\beta, \beta+\alpha, \ldots, \beta+(k-1)\alpha)$. Further, if Π_{k-1} is the generator of $E_k^{(m-1)k+1}$ with parametric equation $w_1+\ldots+w_k\equiv c$, then the points of $C(\alpha)$ that lie on Π_{k-1} are those for which

$$kw \equiv c-\tfrac{1}{2}k(k-1)\alpha \qquad (\mathrm{mod}\, 2\omega_1, 2\omega_2)\ ;$$

and as all the values of w that satisfy this congruence relation arrange themselves $(\bmod 2\omega_1, 2\omega_2)$ into k sets of the involution $\tau(\alpha)$, it follows that $C(\alpha)$ meets Π_{k-1} in k points. Finally, by considering the intersections of $C(\alpha)$ with a prime that meets $E_k^{(m-1)k+1}$ in a simploid Φ and $m-1$ generators, we see that $C(\alpha)$ is of order $(m-1)k+1$, equal to the order of $E_k^{(m-1)k+1}$. This gives

PROPOSITION 10.1. – *If v is the number of cyclic elliptic involutions that are properly of order k on an elliptic ∞^1-system, then $E_k^{(m-1)k+1}$ possesses v principal curves, each a normal ${}^1C^{(m-1)k+1}$ and each k-secant to the generators. Thus $E_k^{(m-1)k+1}$ can be envisaged as the locus of $[k-1]$'s spanning the sets of an elliptic involution of order k on any one of the v principal curves.*

This completes our outline of the principal properties of simploidal elliptic scrollar varieties.

11. – The nuclear surface N of a general E_k^{mk-1}.

In § 7 we pointed out that an E_k^{mk-1} of general type—projection of a key E_k^{mk} from a general point of itself—possesses ∞^1 minimal directrix curves ${}^1C^m$, and we defined the nuclear surface of E_k^{mk-1} to be the surface N generated by these ${}^1C^m$. This surface, as will appear, has the remarkable property that it is simply generated not only by the ${}^1C^m$ but also by each of two other elliptic pencils of elliptic curves, each intrinsic to the geometry of E_k^{mk-1}. We proceed now to investigate N; but since the arguments to be used will be adequately illustrated by their application to a typical particular case, it will be sufficient to set them out for the variety E_3^8 (the case $k=m=3$) and then to state the results for general k and m.

Consider then the variety E_3^8 which is the projection of a key E_3^9 from a general point A of the latter (not lying, in particular, on any one of the three directrix cubic curves of the E_3^9). We shall be concerned with three ∞^1-systems of curves on the nuclear surface N of E_3^8, namely (i) the directrix cubics, which we shall call the curves α, (ii) the curves traced by these on generating planes of E_3^8, which we shall call the curves β, and (iii) a third system of special directrix curves, to be called the curves γ, which we now proceed to define.

The 3-atic system $\{R^{(2)}\}$ of E_3^9 is composed of ∞^1 linear systems $|R^{(2)}|$ each of which is residual to every pair of generating planes of a G_1^2 of E_3^9. Among the ∞^6 intersection curves ${}^1C^5$ of pairs of members of $\{R^{(2)}\}$ there are ∞^3 special curves, to be denoted by ${}^1\bar{C}^5$, each of which is the base of a pencil in one of the systems $|R^{(2)}|$; and each such ${}^1\bar{C}^5$ lies in a $[6]$ with each pair of the associated G_1^2, being in fact the residual intersection of this $[6]$—other than the plane pair—with E_3^9. The point A defines a unique pencil in each $|R^{(2)}|$—of those members of $|R^{(2)}|$ that pass through A—and hence A lies on ∞^1 curves ${}^1\bar{C}^5$ (an elliptic system), each associated with one of the systems $|R^{(2)}|$, and each therefore associated with a unique G_1^2 in such a way that it lies in a [6] with every pair of planes of this G_1^2. On projection from A, then, it appears that

E_3^8 contains a unique elliptic pencil of special directrix curves ${}^1\bar{C}^4$—the curves γ that were to be defined—each associated with a unique G_1^2 of generators of E_3^8 in such a way that it lies in the [5] *joining the planes of any pair of this G_1^2.*

Consider then a pair of generating planes π_1, π_2 of E_3^8. Let B be the [5] containing them; let γ be the ${}^1\bar{C}^4$ in which B meets E_3^8 residually to π_1, π_2; and let T_1, T_2 be the points in which γ meets π_1, π_2. Further, let P be a variable point of γ, and let H_1, H_2 be the points in which π_1 and π_2 are met by the unique transversal from P

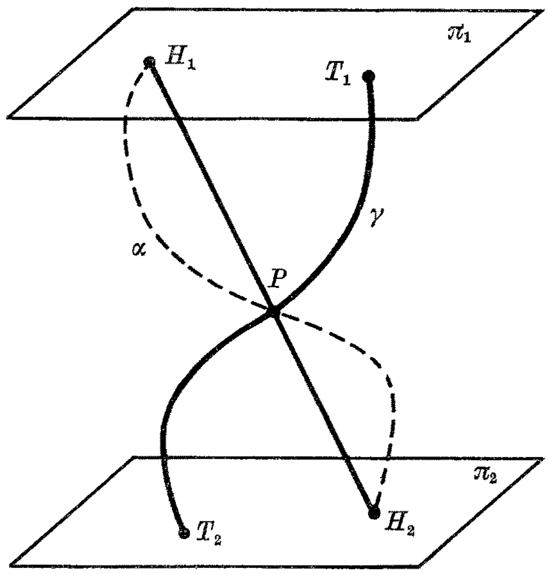

Figura 1.

to these two planes. Since this line PH_1H_2 is a trisecant of E_3^8, it must lie in the plane of one of the directrix cubics α; and as P describes γ, the point H_1, for example, describes a curve β in π_1, locus of the intersections of π_1 with all the curves α. Further, this curve β, being the projection of γ from π_2 into π_1, is a ${}^1C^3$ through T_1. It follows now that the ∞^1 minimal directrix curves α of E_3^8 each meet every γ in one point; and they meet every generating plane of E_3^8 in a plane cubic curve β. Hence:

The surface N on E_3^8 is simply generated by each of the three elliptic pencils (α), (β) *and* (γ).

To find the order of N we consider the section of it by a prime Σ through a pair of generating planes π_1, π_2. Σ meets E_3^8 residually in an elliptic sextic scroll E_2^6—itself a key variety and possessing therefore two directrix cubic curves which are curves α

of E_3^8. Further, as previously shown, Σ contains just one curve γ (lying in the [5] joining π_1 to π_2); and finally it contains the two curves β that lie in π_1 and π_2 respectively. Hence the order of N is $2.3 + 2.3 + 4 = 16$. We find then, by a well-known formula (cf. SEMPLE and ROTH [6], p. 414, (7)) that the section genus of N is 9. Hence:

The nuclear surface of E_3^8 is a surface ${}^9N^{16}$.

When we now follow step-by-step the same procedure as we have used above to investigate the nuclear surface N of an E_k^{mk-1} of general type, we find with little difficulty the following general results:

PROPOSITION 11.1. *The nuclear surface N of an E_k^{mk-1} of general type is simply generated by each of three elliptic pencils of curves, namely*

(i) *the pencil (α) of minimal directrix ${}^1C^m$ of E_k^{mk-1},*

(ii) *the pencil (β) of curves ${}^1C^k$ in which the generators of E_k^{mk-1} are met by the curves α, and*

(iii) *a pencil (γ) of special directrix curves ${}^1C^{m+k-2}$, each of which is the fixed residual intersection of E_k^{mk-1} with any space spanning a set of $m-1$ generators of a G_{m-2}^{m-1} associated with the ${}^1C^{m+k-2}$ in question.*

Further the order of N is $2(mk-1)$ and its section genus is mk.

It may be noted, finally, that the curves α, β and γ on N satisfy the intersection equations

$$\alpha^2 = \beta^2 = \gamma^2 = 0 , \qquad \beta\gamma = \gamma\alpha = \alpha\beta = 1 ;$$

and the order ν of N is expressed by the formula

$$\begin{aligned}\nu &= [(k-1)\alpha + (m-1)\beta + \gamma]^2 \\ &= 2(k-1)(m-1) + 2(m-1) + 2(k-1) = 2(mk-1) .\end{aligned}$$

12. – Review of the general types of E_k^n.

We have now largely completed our limited objective which was to isolate and study what we take to be the most general types of E_k^n for all possible values of k and n. From our discussion it has emerged that, for any given value of k, the general types of E_k^n can be arranged in sets of k members

$$\mathfrak{S}_k(m): \qquad E_k^{mk-s}(0 \leqslant s \leqslant k-1) ,$$

each set being defined by a value of the integer $m \geqslant 2$. The leading member of a set $\mathfrak{S}_k(m)$, given by $s = 0$, is a key variety with precisely k minimal directrix curves ${}^1C^m$;

while the last member, given by $s = k - 1$, is a simploidal variety with ∞^{k-1} minimal directrix curves ${}^1C^m$.

By way of illustration we now look briefly at the first two sets for $k = 3$.

The set $\mathfrak{S}_3(2)$ consists of the three varieties

$E_3^6[5]$, with three elliptic double lines as minimal directrix curves,

$E_3^5[4]$, which is the well-known quintic planar threefold of S_4, and

$E_3^4[3]$, which is the 4-ple space S_3 defined by an elliptic envelope, dual of a curve ${}^1C^4$ of S_3.

As regards E_3^6, we note that it possesses three minimal directrix scrolls E_2^4, each having two double lines; and its 3-atic system consists of the ∞^3 quintic scrolls E_2^5 residual to generating planes of the variety. As regards E_3^5, we note that this is the dual of its double surface E_2^5; and its nuclear surface is E_2^5 counted doubly, the ∞^1 double directrix lines of E_3^5 lying along the generators of E_2^5.

The set $\mathfrak{S}_3(3)$ consists of the three varieties

$E_3^9[8]$, with three minimal directrix cubic curves $\alpha_1, \alpha_2, \alpha_3$,

$E_3^8[7]$, which possesses an elliptic pencil (α) of directrix curves ${}^1C^3$, and

$E_3^7[6]$, which is simploidal, having ∞^3 scrolls E_2^5 (residual to pairs of generating planes) as its simploidal system.

Details of E_3^9 and E_3^8 have been given in § 11; and here we add only two remarks concerning the three elliptic pencils of curves (α), (β) and (γ) that generate the nuclear surface ${}^9N^{16}$ of E_3^8 (cf. § 11). In the first place the planes of the directrix cubics α of E_3^8 plainly generate a second planar scroll $\bar{E}_3^8$ with the same nuclear surface as E_3^8, the roles of the curves α and β being interchanged for the two varieties. Secondly, when E_3^8 is projected from one of its generating planes (the plane of a curve β) into an E_3^5, its nuclear surface projects doubly into the double surface E_2^5 of E_3^5; the curves α project doubly into the generators of E_2^5; and the pencils of curves β and γ both project into the same set of directrix cubics on E_2^5. As regards E_3^7, besides the details given in § 8, we note that this variety has a focal curve ${}^1C^{21}$ which is enveloped by the the simploids on the variety and is 9-secant to its generating planes; and it possesses four principal curves, each a normal ${}^1C^7$ and each trisecant to the generating planes.

13. – Note on linked simploidal varieties and envelopes.

In this final section, we add a note on simploidal envelopes defined by simploidal elliptic varieties. We whall find it convenient, however, for a reason that will shortly appear, to replace $m - 1$ in our previous notation by μ, so that *the typical simploidal variety is now taken to be an* $E_k^{\mu k+1}$.

We recall then (§ 8) that the ∞^1 simploids on $E_k^{\mu k+1}$ are varieties $E_k^{\mu(k-1)+1}$, and we now consider the aggregate of primes that each contain the ambient space $\Pi_{\mu(k-1)}$ of one of these simploids. These primes, namely, are all those that span sets of μ generators of $E_k^{\mu k+1}$, those of them that pass through a given space $\Pi_{\mu(k-1)}$ being those that span sets of a generator series $G_{\mu-1}^{\mu}$ of $E_k^{\mu k+1}$. Thus, if u (mod $2\omega_1, 2\omega_2$) is an elliptic parameter for the generators, it follows that the primes in question can be represented without exception by the unordered sets $(u_1, \ldots, u_\mu)$ of μ values of u (mod $2\omega_1, 2\omega_2$); and they form accordingly a *simploidal envelope* $\mathcal{E}_\mu^{\mu k+1}$ whose generators (in the dual sense) are the ∞^1 spaces $\Pi_{\mu(k-1)}$, each defined by an equation of the form

$$u_1 + \ldots + u_\mu \equiv \text{const.} \qquad (\text{mod}\, 2\omega_1, 2\omega_2)\,.$$

It follows then (cf. §2(ii)) that the locus of these same spaces $\Pi_{\mu(k-1)}$ is a variety $E_{\mu(k-1)+1}^{\mu k+1}$.

Recalling now that any k simploids of $E_\mu^{\mu k+1}$ meet in a point, it follows that any k generators $\Pi_{\mu(k-1)}$ of $E_{\mu(k-1)+1}^{\mu k+1}$ meet in a point of $E_k^{\mu k+1}$. This gives

PROPOSITION 13.1. – *The ambient spaces $\Pi_{\mu(k-1)}$ of the ∞^1 simploids on a simploidal variety $E_k^{\mu k+1}$ generate an $E_{\mu(k-1)+1}^{\mu k+1}$ with $E_k^{\mu k+1}$ as its k-ple locus. The associated envelope of $E_{\mu(k-1)+1}^{\mu k+1}$ is a simploidal $\mathcal{E}_\mu^{\mu k+1}$ whose primes are all those spanning sets of μ generators of $E_k^{\mu k+1}$.*

A pair of varieties $E_k^{\mu k+1}$ and $E_{\mu(k-1)+1}^{\mu k+1}$ related in the above manner, such that the former is the k-ple locus of the latter, while the latter is the μ-ple envelope (aggregate of μ-ple primes) of the former, will be called a *linked pair.*

Recalling then, generally, that the associated envelope of an E_k^n is an $\mathcal{E}_{n-k}^n$, we may conveniently define the *formal dual* of an E_k^n to be an E_{n-k}^n. In this sense the formal duals of the two varieties described above are an $E_{(\mu-1)k+1}^{\mu k+1}$ and an $E_\mu^{\mu k+1}$ respectively; and these two, in reversed order, will also form a linked pair—the dual of the former. It is notable, then, that the two linked pairs

$$(E_k^{\mu k+1}, E_{\mu(k-1)+1}^{\mu k+1}) \qquad \text{and} \qquad (E_\mu^{\mu k+1}, E_{(\mu-1)k+1}^{\mu k+1})$$

arise from each other by interchange of μ and k.

By taking $\mu = k$ we get a simple sequence

$$(E_k^{k^2+1}, E_{k^2-k+1}^{k^2+1})$$

of *self-dual linked pairs* for $k = 2, 3, \ldots$. For $k = 2$ this gives the well-known dual pair E_2^5 and E_3^5 of S_4; and the next such pair, for $k = 3$, is E_3^{10} and E_7^{10} of S_9.

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