NORMAL FAMILIES OF HOLOMORPHIC MAPPINGS

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Introduction

The main purpose of this paper is to give some applications of the concept of a normal family of holomorphic mappings between complex manifolds. This notion has long proved its importance in the theory of holomorphic functions of one variable, but its study in the general setting of complex manifolds was begun only recently by Grauert and Reckziegel, [10]. The two main problems contemplated here can be briefly described. If $\mathfrak{F}: M \rightarrow N$

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is a family of holomorphic mappings between complex manifolds M and N, the first problem is: if $\mathfrak{F}$ is normal, or compact (in the compact-open topology), or equicontinuous, what can be said about the behavior of the individual members of $\mathfrak{F}$? The second problem, which is important in applications, is to attach suitable invariants to $\mathfrak{F}$ itself (or, if necessary, to N) to insure the normality, or compactness, or equicontinuity of $\mathfrak{F}$.

This paper is divided into three parts. Roughly speaking, the first two parts are devoted to these two problems respectively. The first theorem to be proved in Part I (Theorem A) essentially asserts the truth of Bloch's theorem for a relatively compact family of holomorphic mappings. It will be basic in the proof of the general Bloch theorem as presented in Part III and the Appendix. Theorems C, D and E are respectively the abstract analogues of the theorems of H. Cartan–Caratheodory, H. Cartan and Liouville on bounded domains. (See, for instance, Chu–Kobayashi [9] and Bochner-Martin [6].) The informed reader will readily detect the debt I owe the original theorems. But the proofs of Theorems C, D and E given here may perhaps claim the credit of shedding some light on these classical theorems themselves. The material in Part I is essentially a topic in the pointset topology of function spaces (see Kelley [19], Chapter 7) and is, accordingly, written from this point of view. In Part II, it will be necessary to bring in hermitian differential geometry. The main idea here is to show how the holomorphic curvature of the image hermitian manifold N controls the topology of the space of all holomorphic mappings from M into N. In slightly less precise terms, this was already done by Grauert and Reckziegel in [10]. Their main theorem will be proved here in a generalized form (the basic theorem of § 8). To be sure, there is essentially nothing new to be found in the proof of this more general version, but the latter does yield a result (Corollary 8.3) which is inaccessible in the original formulation. From the basic theorem, it is a routine matter to render more tangible and geometric the general theorems of Part I. Some of the more significant consequences are recorded in § 9.

Part III, § 10, is offered by way of an apology. It will be quite clear from the proofs in Part I that a few of the theorems there are susceptible of considerable generalizations. The reason for not striving for maximum generality in Part I is that these theorems appear most naturally in the setting of holomorphy. Nevertheless, it is inevitable that such generalizations will be pointed out, and so I take the liberty of giving some such indications. One of them, Lemma 10.1, will be seen to be crucial to the proof of the general Bloch theorem, Theorem 11.4. In a special case, this theorem was already proved by Bochner in 1946, [5]. One interesting consequence (Corollary 11.5) of Theorem 11.4 is that the original theorem of Bloch can be extended to include a large class of the pseudo-analytic functions of Lipman Bers, [4]. This fact appears to be new.

The most important and useful special case of the general Bloch theorem is, without doubt, that of quasiconformal holomorphic mappings in several complex variables. This special case was already covered by the Bochner result. However, the proof of Theorem 11.4 as presented here, when restricted to this special case, becomes exceedingly simple, and is dependent solely on the well-known theorem of Montel that a uniformly bounded family of holomorphic functions is relatively compact. But the machinery needed for the degree of generality of Theorem 11.4 somewhat obscures this simplicity. I therefore feel justified in producing a geodesic to get at this special case of Theorem 11.4, one that operates entirely within the category of holomorphic functions and which, even when restricted to the classical case of one complex variable, seems to make the original Bloch theorem appear less mysterious. This is done in the Appendix.

In closing, I would like to raise a few basic questions suggested by the results of this paper:

(1) Is a complete simply connected Kähler manifold with nonpositive riemannian curvature and with holomorphic curvature negative and bounded away from zero biholomorphic to a bounded domain of $\mathbf{C}^n$?

(2) Which domains in $\mathbf{C}^n$ can be equipped with a complete hermitian metric whose holomorphic curvature is negative and bounded away from zero?

(3) If a hermitian manifold is compact and has negative holomorphic curvature, is the set of all holomorphic mappings into itself with nonsingular differential at one point finite?

It is obvious how to make these questions more inclusive by a trivial variation of the terms. The reason for stating them in such restrictive forms is that they are more immediately accessible this way. This is so for (1) because a complete simply connected Kähler manifold with nonpositive riemannian curvature is necessarily Stein, [28]. For (2), we know by Theorem ζ of § 9 that such a domain must be a domain of holomorphy, and for (3), the theorem of Peters–Kobayashi ([27], Satz 3) states that the answer is yes for compact complex manifolds with negative definite first Chern class. It should be pointed out that our present knowledge of the control over the topology of (even) Kähler manifolds by the holomorphic curvature is very meagre. In particular, let me pose a very simple-minded problem. If a complete Kähler manifold has positive holomorphic curvature bounded below from zero, then it is simply connected and compact. However:

(4) If a Kähler manifold is complete and simply connected and has holomorphic curvature negative and bounded above from zero, is it noncompact?

This paper is based on an earlier manuscript with the longer title, "Normal families of holomorphic mappings and the theorem of Bloch in several complex variables". In between these two manuscripts, there has appeared a paper by Kobayashi, [22], which presents (among other things) an entirely different approach to Theorems δ, ε and ζ of § 9. He pointed out to me that the "Kähler" condition assumed throughout the earlier manuscript could be replaced by "hermitian", that the completeness requirement of the metric in the original version of Theorem δ and the assumption of C^2 boundary in the original version of Theorem ζ could be removed. I am very indebted to him for these as well as many informative discussions. I have also been the beneficiary of much valuable advice from Professors Chern and Griffiths. It was mainly through conversations with Phil Griffiths that the dreadful task of revision was transformed into something better than pure drudgery. To all of them, I tender my most sincere thanks.

PART I

1. Basic definitions

The convention in force throughout this paper is that all complex manifolds are *connected* and *second countable* and all objects defined on them (differential forms, hermitian metrics, etc.) are C^∞ unless stated to the contrary. Let M, N be complex manifolds. Then by definition:

(i) $\mathcal{C}(M, N)$ is the set of all *continuous* maps from M to N.

(ii) $\mathcal{A}(M, N)$ is the set of all *holomorphic* maps from M to N.

For convenience, $\mathcal{C}(M) \equiv \mathcal{C}(M, M)$, and $\mathcal{A}(M) \equiv \mathcal{A}(M, M)$.

The only topology on $\mathcal{C}(M, N)$ that will be used in both Part I and Part II is the compact-open topology, defined as follows, (Kelley [19], Chapter 7): let C and O be compact and open sets in M and N respectively, and let

$$W(C, O) = \{f \in \mathcal{C}(M, N) : f(C) \subseteq O\}.$$

The open sets of the compact-open topology are then the unions of finite intersections of such $W(C\ O)$; i.e. $\{W(C, O)\}$ is a subbase. A basic fact is that, since M, N are both second countable, so is $\mathcal{C}(M, N)$ in this topology. The Cauchy integral formula implies in a standard way that $\mathcal{A}(M, N)$ is closed in $\mathcal{C}(M, N)$. This entials the pleasant consequence that a subset $\mathfrak{F} \subseteq \mathcal{A}(M, N)$ is compact or closed in $\mathcal{C}(M, N)$ iff it is so in $\mathcal{A}(M, N)$. From now on, *the compact-open topology on* $\mathcal{C}(M, N)$ *will be used without explicit reference.*

It is well-known that such complex manifolds under consideration are metrizable. A customary and useful device is to metrize these by imposing on them a hermitian metric h, from which one derives a distance function $d(\ ,\ )\equiv d_h(\ ,\ )$ which converts the manifold into a metric space. At any rate, once a distance function is chosen on N, a sequence $\{f_i\}\subseteq \mathcal{C}(M, N)$ converges to an $f\in\mathcal{C}(M, N)$ iff f_i converges to f uniformly on compact sets. (Kelley [19], p. 229.) The above remark on the closedness of $\mathcal{A}(M, N)$ in $\mathcal{C}(M, N)$ can easily be derived from this fact. Because of the metrizability of such complex manifolds, it is convenient to relax M and N to be just *locally compact connected metric spaces*. The space $\mathcal{C}(M, N)$ and its (compact-open) topology can be defined in the same fashion. It is known that all locally compact connected metric spaces are second countable (Kobayashi–Nomizu [23], p. 269) so that $\mathcal{C}(M, N)$ is itself second countable. In this context, a sequence $\{f_i\}\subseteq\mathcal{C}(M, N)$ is called *compactly divergent* iff given any compact K in M, and compact K' in N, there exists an i_0 such that $f_i(K)\cap K'=\emptyset$ for all $i\geqslant i_0$.

Definition 1.1. A subset $\mathfrak{F}$ of $\mathcal{C}(M, N)$ is called *normal* iff every sequence of $\mathfrak{F}$ contains a subsequence which is either relatively compact in $\mathcal{C}(M, N)$ or compactly divergent.

As usual, a set is *relatively compact* in a space X iff it has compact closure in X. Since $\mathcal{C}(M, N)$ is second countable, compactness is equivalent to sequential compactness. This accounts for the use of sequence in the definition of normality and is a useful thing to keep in mind. Two related concepts which will play an important role are *local compactness* and *equicontinuity*. The latter will now be defined. Let the distance function of N be d_N. Then $\mathfrak{F}\subseteq\mathcal{C}(M, N)$ is called an *equicontinuous family* iff given any $\varepsilon>0$ and any $m\in M$, there exists a neighborhood U of m such that $m'\in U$ implies $d_N(f(m), f(m'))<\varepsilon$ for all $f\in\mathfrak{F}$. The relationship among these concepts is clarified by the following two lemmas.

LEMMA 1.1. *Let $\mathfrak{F}\subseteq\mathcal{C}(M, N)$, where M, N are connected, locally compact metric spaces. Then*

(i) *If $\mathfrak{F}$ is compact, then $\mathfrak{F}$ is normal.*

(ii) *If $\mathfrak{F}$ is normal, then its closure is locally compact.*

(iii) *If $\mathfrak{F}$ is equicontinuous and if each bounded subset of N is relatively compact, then $\mathfrak{F}$ is normal.*

Remark. It is obvious that the converse of (i) and (ii) is false, and that normality does not imply equicontinuity. The condition on N in (iii) implies, but is stronger than, the completeness of N; however, they coincide for hermitian manifolds (cf. § 8, Lemma 8.1). The next lemma is the wellknown Ascoli theorem, (Kelley [19], p. 233–6). Because this theorem will be invoked so often for the rest of this paper, it is good to have an explicit statement of it suitable for our purpose.

LEMMA 1.2 (Ascoli theorem). *$\mathfrak{F} \subseteq \mathcal{C}(M, N)$ is compact iff:*

(a) *$\mathfrak{F}$ is closed in $\mathcal{C}(M, N)$.*

(b) *$\mathfrak{F}(m)$ ($\equiv\{n \in N: n = f(m)$ for some $f \in \mathfrak{F}\}$) is relatively compact in N for every $m \in M$.*

(c) *$\mathfrak{F}$ is equicontinuous.*

Remark. Thus, if N is compact, closed + equicontinuous = compact = closed + normal = locally compact.

Proof of Lemma 1.1. (i) is obvious. For the proof of (ii), consider any $f \in \mathfrak{F}$ and take any point $m \in M$. Let O be any relatively compact open neighborhood of $f(m)$. The neighborhood $W(\{m\}, O)$ of f is then relatively compact, for the following reason: no sequence in $W(\{m\}, O)$ can be compactly divergent and so normality implies that every sequence in $W(\{m\}, O)$ is relatively compact. Consequently, the closure of $W(\{m\}, O)$ in $\mathcal{C}(M, N)$ furnishes a compact neighborhood of f.

It remains to prove (iii). First a notation: if K is a subset of M, $\mathfrak{F}(K)$ will denote the set of all $f(m)$, where $f \in \mathfrak{F}$ and $m \in K$. The proof of (iii) is broken into three steps; the first two make no use of the assumption on N.

Step 1: If F is a sequence in $\mathfrak{F}$ and $F(K)$ is relatively compact for every compact $K \subseteq M$, then F is relatively compact in $\mathcal{C}(M, N)$.

Reason: Consider the closure $\bar{F}$ in $\mathcal{C}(M, N)$. Since $\mathfrak{F}$ is by assumption equicontinuous, conditions (a)–(c) of Lemma 1.2 are fulfilled and $\bar{F}$ is compact.

Step 2: If $F = \{f_i\}$ is a sequence in $\mathfrak{F}$ and if for one compact set K_0 in M and one compact set K_1 in N, $f_i(K_0) \cap K_1 \neq \emptyset$, for all $f_i \in F$, then $F(K)$ is a bounded subset of N for all compact $K \subseteq M$.

Reason: Since M is connected, it will be sufficient to prove this assertion for all connected compact K containing K_0. Thus, take such a K and we know that $f_i(K) \cap K_1 \neq \emptyset$. Choose an arbitrary $\varepsilon > 0$, then associate with each $m \in K$ a neighborhood U such that the diameter of $f_i(U)$ is less than ε for all $f_i \in F$. K then admits a finite covering by such neighborhoods $U_1, \ldots, U_l$. It is clear that the diameter of the connected $f_i(K)$ cannot exceed $2l\varepsilon$ for all i. Let the diameter of K_1 be η, ($\eta < \infty$ because K_1 compact) and let n_0 be a fixed point in K_1. Because $f_i(K) \cap K_1 \neq \emptyset$, $f_i(K)$ is necessarily contained in the ball about n_0 of radius $\eta + 2l\varepsilon$. Since i is arbitrary, $F(K)$ is then a subset of this ball also.

Step 3: Let F be a sequence in $\mathcal{J}$. Unless F is compactly divergent, F contains a relatively compact subsequence. (This clearly proves (iii).)

Reason: Assume F *not* compactly divergent, then there is some compact K_0 in M and some compact K_1 in N such that for infinitely many g_i of F, $g_i(K_0) \cap K_1 \neq \emptyset$. By Step 2, $\{g_i\}$ carries each compact K of M into a bounded set in N, and the latter must then be relatively compact by hypothesis on N. Now Step 1 implies that $\{g_i\}$ is a relatively compact subset of $C(M, N)$. Q.E.D.

It will be convenient later on to introduce the following two notions.

Definition 1.2. A complex manifold N is called *taut* iff for every complex manifold M, the set of all holomorphic mappings $\mathcal{A}(M, N)$ is normal.

Definition 1.3. Let N be a complex manifold and d be a metric on N inducing its topology. Then (N, d) (or if no confusion is possible, just N) is *tight* iff for every complex manifold M, the set of all holomorphic mappings $\mathcal{A}(M, N)$ is equicontinuous.

Remark. It should be emphasized that while tautness is an intrinsic property of the complex structure of N, tightness is dependent on the metric d used. It can happen that two metrics d and d' both induce the same topology of N, but (N, d) is tight while (N, d') is not. In all applications which we shall consider, d will be the distance function associated with a hermitian metric on N. See § 2 as well as § 8 and § 9.

Lemma 1.1 gives a link between tightness and tautness. If N is compact, the two concepts coincide, by Lemma 1.2. Clearly both concepts are entirely local questions, so the following is clear.

LEMMA 1.3. *N is taut if $\mathcal{A}(D^n, N)$ is normal for all n, where D^n is the unit ball in $\mathbf{C}^n$. (N, d) is tight if $\mathcal{A}(D^n, N)$ is equicontinuous for all n.*

To conclude this section, let us examine the most popular compact family of holomorphic mappings. If M is a complex manifold as usual, then $\mathcal{J} \subseteq \mathcal{A}(M, \mathbf{C}^n)$ is called *uniformly bounded* iff there is a constant C such that $\sup_{m \in M} \|f(m)\| < C$ for all $f \in \mathcal{J}$. Here $\| \ \|$ denotes the ordinary euclidean norm of $\mathbf{C}^n$. The following is classical, (Montel [25], p. 241).

LEMMA 1.4 (Theorem of Montel). *A uniformly bounded family of holomorphic mappings from M into $\mathbf{C}^n$ is equicontinuous and hence relatively compact in $\mathcal{A}(M, \mathbf{C}^n)$.*

This immediately implies:

LEMMA 1.5. *Every bounded domain in $\mathbf{C}^n$ is a tight manifold in its usual metric.*

It is much harder for a domain to be taut. See § 6.

2. Theorems A and B

Let us set up the notation. If N is an hermitian manifold of dimension n (without further notice, dimension = complex dimension) with hermitian metric h, we shall agree to denote the volume element of h by Ω. h gives rise to a distance function on N which is denoted by d or d_h. By the distance between two points in N, we shall always mean the distance relative to this d_h. Suppose p_0 is a fixed point of another complex manifold M also of dimension n, and μ is a fixed nonzero real co-vector of degree $2n$ at p_0, i.e. $\mu \in \Lambda^{2n} M^*_{p_0}$, where $M^*_{p_0}$ denotes the real cotangent space of M at p_0 and Λ denotes exterior power. If $f \in \mathcal{A}(M, N)$, then $(f^*\Omega)_{p_0} = c\mu$ for some real number c. *We shall denote* c *by* $(f^*\Omega/\mu)$ *for convenience.* Consider now a holomorphic family $\mathfrak{F}_a \subseteq \{f \in (M, N): |f^*\Omega/\mu| \geqslant a > 0\}$. Roughly speaking, this simply means that the Jacobian of $f \in \mathfrak{F}_a$ at p_0 is bounded away from zero by a in absolute value.

Definition 2.1. If $f: M \to N$ is holomorphic and N is hermitian, then a *univalent ball (for f)* is an open ball in the image of f onto which f maps an open set biholomorphically.

By *biholomorphic*, we mean as usual that the holomorphic mapping is one-one and has everywhere nonsingular differential. (One may note that the second requirement is redundant in the case of holomorphic mappings; see Bochner–Martin [6], p. 179. But this definition will serve us better in Part III, § 10.)

THEOREM A. *If $\mathfrak{F}_a$ is a relatively compact family, then there exists a positive constant α such that every $f \in \mathfrak{F}_a$ possesses a univalent ball of radius α around $f(p_0)$.*

THEOREM B. *Suppose M also hermitian. Hypothesis as above, then there is a positive constant λ such that for every $f \in \mathfrak{F}_a$, f is biholomorphic on the open ball of radius λ about p_0.*

Proof of Theorem A. Let $\mathcal{E}_a$ denote $\{f \in \mathcal{A}(M, N): |f^*\Omega/\mu| \geqslant a > 0\}$. We first show that $\mathcal{E}_a$ is closed in $\mathcal{A}(M, N)$. Let f belong to the closure of $\mathcal{E}_a$, then there is a sequence $\{f_i\}$ in $\mathcal{E}_a$ such that $f_i \to f$. This means f_i converges to f uniformly on all compact subsets of M and so, by a well-known property of holomorphic functions which states that uniform convergence entails uniform convergence of all corresponding partial derivatives of all orders, $f_i^*\Omega \to f^*\Omega$ uniformly on compact subsets of M. (This last can be taken to mean "with respect to any hermitian metric on M".) As a result, $f_i^*\Omega/\mu \to f^*\Omega/\mu$ which implies $|f^*\Omega/\mu| \geqslant a > 0$. Thus $f \in \mathcal{E}_a$, and $\mathcal{E}_a$ is closed in $\mathcal{A}(M, N)$. In particular, the compact closure of $\mathfrak{F}_a$ will still reside in $\mathcal{E}_a$. We will *therefore assume $\mathfrak{F}_a \subseteq \mathcal{E}_a$ and is compact.* Furthermore, a second reduction is possible: we will assume that *M is the unit open ball D^n in $\mathbf{C}^n$ and p_0 is the origin O.* Indeed, we may choose complex coordinates $z^1, \ldots, z^n$ around p_0 so that $z^\sigma(p_0) = 0$, $\sigma = 1, \ldots, n$, and such that $D^n \equiv \{(\Sigma_\sigma |z^\sigma|^2)^{\frac{1}{2}} < 1\}$ is contained in this coordinate neighborhood.

Then we restrict each f of $\mathfrak{F}_a$ to $\{(\Sigma_\sigma|z^\sigma|^2)^{\frac{1}{2}}<1\}$ and work entirely within it. Assume this done. We will now prove Theorem A in this form:

For a compact family of holomorphic mappings $\mathfrak{F}_a \subseteq \mathcal{A}(D^n, N)$, with $|f^*\Omega/\mu| \geqslant a > 0$ for each $f \in \mathfrak{F}_a$ (where μ is a nonzero real co-vector of degree $2n$ at O), there is a positive constant α such that every $f \in \mathfrak{F}_a$ has a univalent ball of radius α around $f(O)$.

Assume this assertion false, then there is a sequence $\{f_i^{\#}\} \subseteq \mathfrak{F}_a$ such that the radius of the *maximal* univalent ball for $f_i^{\#}$ around $f_i^{\#}(O)$ approaches zero as $i \to \infty$. By compactness of $\mathfrak{F}_a$, there is a subsequence $\{f_i\}$ which converges (in the compact open topology of course) to an $f: D^n \to N$, and $f \in \mathfrak{F}_a$. Since $|f^*\Omega/\mu| \geqslant a > 0$, df is nonsingular at the origin O. Hence there is a univalent ball of radius $r > 0$ for f about $f(O)$. Call this B. Let A be the open set in D^n which contains O and which f maps biholomorphically onto B. (Although not needed in the sequel, we would like to clarify the situation with this remark: A is the component of $f^{-1}(B)$ containing O. This can be established by a completely elementary reasoning or by a dimension argument.) Now let B' (resp. B'') be the open ball of radius $r-2\varepsilon$ (resp. $r-4\varepsilon$) around $f(O)$, where ε is any sufficiently small positive number. Let A' (resp. A'') be the unique subset of A which f carries onto B' (resp. B''). It is clear that the closure $\bar{A}'$ of A' (resp. $\bar{A}''$ of A'') is compact and contains no boundary point of D^n.

Now let B_i be the *maximal* univalent ball of radius r_i, say, for f_i around $f_i(O)$. By our hypothesis, $r_i \to 0$ as $i \to \infty$. Let A_i be the open set in D^n containing 0 which f_i maps biholomorphically onto B_i. The crucial property of A_i is this: *the boundary ∂A_i of A_i must contain a boundary point of D^n or a point p_i at which df_i is singular.* For if ∂A_i contains neither, then a simple application of the inverse function theorem will enable us to construct a univalent ball containing $f_i(O)$ with radius strictly larger than r_i, and B_i will not be maximal.

Now select an integer i_0 so large that $B_i \subseteq B''$ if $i \geqslant i_0$. This is possible because $f_i(O) \to f(O)$ and $r_i \to 0$ as $i \to \infty$. Since $\bar{A}'$ is compact, $f_i \to f$ uniformly on $\bar{A}'$ so that we may select an integer i_1 with the property: $\max_{p \in \bar{A}'} d_h(f_i(p), f(p)) < \varepsilon$ if $i \geqslant i_1$. Let $i_2 \geqslant \{i_0, i_1\}$. We claim:

$$A_i \subseteq A' \quad \text{if} \quad i \geqslant i_2.$$

For if not, there will be a point $p \in A_i \cap \partial A'$ for an $i \geqslant i_2$. ($\partial A' \equiv$ boundary of A' in D^n.) Since $p \in A_i$, and $f_i(A_i) = B_i \subseteq B''$, $f_i(p)$ is of distance at least 2ε away from $\partial B'$. On the other hand, $p \in \partial A'$. Then the two facts $d_h(f(p), f_i(p)) < \varepsilon$ and $f(\partial A') = \partial B'$ imply that $f_i(p)$ is of distance at most ε from $\partial B'$. A contradiction.

Thus, $\{\bar{A}_i\}$ for all $i \geqslant i_2$ are contained in the compact set $\bar{A}'$. As $\bar{A}'$ contains no boundary point of D^n, nor does any of the $\bar{A}_i$ for $i \geqslant i_2$. By the above remark, there must be a point $p_i \in \partial A_i$ such that $(df_i)_{p_i}$ is singular. Let p be an accumulation point of $\{p_i\}$;

$p \in \bar{A}'$. Note again that $\{f_i\}$ are *holomorphic* mappings converging uniformly to f on $\bar{A}'$, so df_i converges *uniformly* to df on $\bar{A}'$ relative to the hermitian metric h of N and, say, the flat metric on D^n. Therefore if $\{p_{i'}\}$ is a subsequence of $\{p_i\}$ such that $\{p_{i'}\} \to p$ as $i' \to \infty$, it is clear that

$$(df_{i'})_{p_{i'}} \to (df)_p \quad \text{as} \quad i' \to \infty.$$

Consequently, $(df)_p$ is singular. But $p \in \bar{A}' \subseteq A$, and df is nowhere singular on A because f is biholomorphic on A. This contradiction completes the proof.

Proof of Theorem B. Let us choose a coordinate system $z^1, \ldots, z^n$ around p_0 so that $z^\sigma(p_0) = 0$ for all σ and so that the unit ball $\{\Sigma_\sigma |z^\sigma|^2 < 1\}$ is contained in this coordinate neighborhood. It is well known (Kobayashi–Nomizu [23], p. 166) that the distaince function of the hermitian metric on M and the ordinary euclidean metric induced by $z^1, \ldots, z^n$ both define the same topology in this coordinate neighborhood. Therefore, as in the proof of Theorem A, it suffices to prove the following equivalent statement:

For a compact family $\mathfrak{F}_a \subseteq \mathcal{A}(D^n, N)$ such that $|f^*\Omega/\mu| \geqslant a > 0$ for all $f \in \mathfrak{F}_a$, (μ being a nonzero real co-vector of degree $2n$ at the origin O), there is a positive constant λ such that every $f \in \mathfrak{F}_a$ is biholomorphic on an open ball of radius λ around O.

In view of Theorem A, we can choose a positive constant α^* so that if $f \in \mathfrak{F}_a$, then there is a univalent ball B_f of f of radius α^* about $f(O)$ with the property: there exists an open set A_f containing O with compact closure $\bar{A}_f$ which f maps biholomorphically onto $\bar{B}_f$. Indeed, it suffices to take α^* to be any positive constant smaller than the α of Theorem A. Next, we claim:

$$\bigcap_{f \in \mathfrak{F}_a} A_f \text{ contains an open ball around } O.$$

This clearly implies the theorem. In turn, to prove this claim, we will prove the following equivalent statement:

$$\inf_{f \in \mathfrak{F}_a} \operatorname{dist}(O, \partial A_f) > 0,$$

where dist(,) means the ordinary euclidean distance of $\mathbf{C}^n$. If this were false, then there exists a sequence $\{f_i^\#\}$ such that $\operatorname{dist}(0, \partial A_i^\#) \to 0$ as $i \to \infty$, where we have denoted $\partial A_{f_i^\#}$ by $\partial A_i^\#$ for simplicity. Again, the compactness of $\mathfrak{F}_a$ implies that a subsequence $\{f_i\}$ converges to some $f \in \mathfrak{F}_a \subseteq \mathcal{A}(D^n, N)$. Then there is an open set A of D^n containing O with compact closure $\bar{A}$ which f maps biholomorphically onto the closure $\bar{B}$ of an open ball of radius α^* around $f(O)$.

Let $p_i \in \partial A_i$ ($\equiv \partial A_{f_i}$) realize the distance from O to ∂A_i, i.e. let $\operatorname{dist}(O, p_i) = \operatorname{dist}(O, \partial A_i)$. This is possible because ∂A_i is compact. Since $\operatorname{dist}(O, \partial A_i) \to 0$ as $i \to \infty$, $p_i \to 0$ as $i \to \infty$. Consequently, $d_h(f(p_i), f(O)) \to 0$ as $i \to \infty$. On the other hand, f_i converges uniformly on the compact set $\tilde{A}$. Using this, the usual argument gives $d_h(f_i(p_i), f(O)) \to 0$ as $i \to \infty$. We will see that this is not possible. Indeed, an integer i_0 can be chosen so that $d_h(f_i(O), f(O)) < \frac{1}{2}\alpha^*$ if $i \geqslant i_0$. But for all i, $d_h(f_i(O), f_i(p_i)) = \alpha^*$ because $p_i \in \partial A_i$, and hence $f_i(p_i) \in \partial B_i$. Thus, $i \geqslant i_0$ implies $d_h(f_i(p_i), f(O)) > \frac{1}{2}\alpha^* > 0$. This is a contradiction. Q.E.D.

Combining Lemma 1.4 with Theorems A and B, we deduce trivially:

COROLLARY 2.1. *Let X be a domain (=open connected set) of $\mathbf{C}^n$, and let E be any bounded domain of $\mathbf{C}^n$. If $\mathfrak{J}_a \subseteq \mathcal{A}(X, E)$ has the property that at some fixed point $x_0 \in X$, $|Jf(x_0)| \geqslant a > 0$ for all $f \in \mathfrak{J}_a$, where Jf denotes the Jacobian determinant, then:*

(i) *There is a positive constant α such that every $f \in \mathfrak{J}_a$ possesses a univalent ball of radius α around $f(x_0)$.*

(ii) *There is a positive constant λ such that every $f \in \mathfrak{J}_a$ is biholomorphic on the open ball of radius λ around x_0.*

The explicit determination of α and λ for $X = E =$ (a symmetric bounded domain in $\mathbf{C}^n$) should be an interesting problem. In particular, a lower bound for α in the case of the unit ball D^n in $\mathbf{C}^n$ will yield a lower bound for the Bloch constant β in the general Bloch theorem of § 11. But such a lower bound for β will be very poor in general.

3. Two examples

First let us recall that the classical theorem of Bloch deals with holomorphic mappings from the unit disc D^1 into $\mathbf{C}$. (i) of Corollary 2.1 is the embryonic form of the extension of this result to n variables; it is unsatisfactory because it was necessary to restrict the receiving space to be a bounded domain rather than all of $\mathbf{C}^n$ itself. The relaxation of this restriction will be carried out in Part III. (See particularly the Appendix.) But in the process of doing this, a restriction of a different nature will creep in, namely, the family of such mappings will have to be quasi-conformal. See Appendix for definition. It is the purpose of this section to show that some kind of restriction is to be expected once we go outside of one complex variable. The first example is naïve but has the virtue of being convincing, while the second is more sophisticated but possibly less instructive.

Example One. Define $\varphi_i : D^2 \to \mathbf{C}^2$ (D^2 as usual denotes the open unit ball in $\mathbf{C}^2$) by: $\varphi_i(z_1, z_2) = (iz_1, 1/i\, z_2)$, where i is an integer. Clearly $|J\varphi_i(O)| = 1$ for all i and the univalent balls for φ_i are of radius at most $1/i$. Therefore Corollary 2.1 (i) breaks down for the family $\{\varphi_i\}$ $(i = 1, 2, \ldots)$ in the strong sense that the supremum of the radii of the univalent balls *anywhere* when all members of $\{\varphi_i\}$ are taken into account is zero.

The trouble lies in the fact that, as $i \to \infty$, φ_i expands infinitely in the z_1-direction and contracts infinitely in the z_2-direction. This points to the necessity of looking at those mappings whose infinitesimal dilatation is bounded, i.e. the ratio of the infinitesimal expansion over infinitesimal contraction is bounded by an absolute constant. These quasi-conformal holomorphic mappings will have applications not only in extending the Bloch theorem, but also in n-dimensional value distribution theory. The latter will be pursued in a separate publication.

Example Two. This time, a series of $\psi_i : D^1 \times D^1 \to C^2$ will be defined (D^1 is the unit disc in $\mathbf{C}$), where Image ψ_i for each ψ_i will be all of $\mathbf{C}^2$ and $|J\psi_i(0)| = 1$. Yet, Corollary 2.1 (i) will break down for the family $\{\psi_i\}$ $(i = 1, 2, ...)$ in the same strong sense as above. The improvement over the previous example is that the image of each of the φ_i in Example One is only a bounded subset of $\mathbf{C}^2$.

First we define a holomorphic $f: D^1 \to \mathbf{C}$. So let η be an entire function with double zeroes at *exactly* all the lattice points of $\mathbf{C}$ except at the origin, i.e. at $\mathcal{L} \setminus \{0\}$ where $\mathcal{L}$ is the set of all Gaussian integers in $\mathbf{C}$. Take the Riemann surface of $\sqrt{\eta}$; call it S, and let ϱ: $S \to \mathbf{C}$ be the natural projection of this ramified covering. We claim that D^1 is the universal covering surface of S. To show this, it suffices to show that the universal covering surface of S cannot be $\mathbf{C}$. If it were, then let $\pi' : \mathbf{C} \to S$ be the natural projection, then $\varrho \circ \pi'$: $\mathbf{C} \to \mathbf{C}$ is an entire function, and by our construction, no univalent ball of $\varrho \circ \pi'$ can exceed 2 in radius. On the other hand, using the original Bloch theorem, one can verify that for an entire function, there exist univalent balls of arbitrarily large radius. This contradicts the above. Hence D^1 is the universal covering surface of S. So let π: $D^1 \to S$ be the natural projection and define f to be $\varrho \circ \pi$. Clearly f is onto and df/dz vanishes at $f^{-1}(\mathcal{L} \setminus \{0\})$.

Now by definition, ψ_i: $D^1 \times D^1 \to \mathbf{C}^2$ is $\psi_i(z_1, z_2) = \left(\frac{1}{ic} f(z_1), \frac{i}{c} f(z_2)\right)$, where $c \equiv (df/dz)(0) \neq 0$. Consequently, $|J\psi_i(0)| = 1$ and ψ_i maps onto $\mathbf{C}^2$. Also, if $T = f^{-1}(\mathcal{L} - \{0\})$, then $\psi_i(T \times T) = \left\{\left(\frac{a}{ic}, \frac{ib}{c}\right) : a, b \text{ are integers}\right\}$ and $J\psi_i(0) = 0$ if $p \in \{T \times \mathbf{C}\} \cup \{\mathbf{C} \times T\}$. Therefore each ψ_i is not one-one in any neighborhood of $\{T \times \mathbf{C}\} \cup \{\mathbf{C} \times T\}$, and hence any ball containing a point of $\psi_i(T \times T)$ is not univalent. From this, one sees readily that the size of the univalent balls anywhere for ψ_i shrinks down to zero as $i \to \infty$.

4. Theorem C

We begin with the statements of the theorem itself and some of its consequences.

THEOREM C. *Let M be a relatively compact open submanifold of a complex manifold N. Let d be some metric on N such that (M, d) is tight (Definition 1.3). If f: $M \to M$ is holomor-*

phic and $f(p)=p$, consider df_p: $M_p \to M_p$, where M_p denotes the tangent space of M at p. Then:

(1) $|\det df_p| \leq 1$.

(2) *If df_p is the identity linear transformation, then f is the identity mapping of M.*

(3) $|\det df_p| = 1$ *iff f is an automorphism.*

By an *automorphism* of a complex manifold, we mean a biholomorphic mapping of the manifold onto itself. The following should be obvious:

COROLLARY 4.1. *Hypothesis on M and N as above, let f, g be automorphisms of M. If $f(q)=g(q)$ and $df_q = dg_q$ for some $q \in M$, then $f=g$.*

COROLLARY 4.2. *Suppose M, N as above. For two f, g in $\mathcal{A}(M)$ both leaving a point p fixed, df_p is the inverse of dg_p iff f and g are both automorphisms and f is the inverse of g.*

From the proof of Theorem C to be given presently, it will be clear that the following also holds.

THEOREM C′. *Let M be a taut complex manifold and let $f \in \mathcal{A}(M)$ be such that $f(p)=p$. Then* (1)–(3) *of Theorem C are true.*

COROLLARY 4.2′. *Suppose M is taut and $f, g \in \mathcal{A}(M)$ such that $f(p)=g(p)=p$. Then $df_p = (dg_p)^{-1}$ iff f, g are both automorphisms and $f = g^{-1}$.*

Finally, let us note an immediate consequence of (3) of Theorem C and Lemma 1.5; this the *Theorem of H. Cartan–Caratheodory:* If E is a bounded domain in $\mathbf{C}^n$ and f: $E \to E$ leaves $p \in E$ fixed and is holomorphic, then $|Jf(p)| = 1$ implies f is an automorphism. (Jf = Jacobian determinant.)

Proof of Theorem C. Consider the iterates $\{f^i\}$ of f. Since (M, d) is tight, $\{f^i\}$ is an equicontinuous family. As M is relatively compact in N, it follows from Lemma 1.2 that, regarded as a subset of $\mathcal{A}(M, N)$, $\{f^i\}$ is relatively compact. Let $\{f^{i(s)}\}$ ($s = 1, 2, \ldots$) be a subsequence of $\{f^i\}$ which converges to an $h \in \mathcal{A}(M, N)$. An argument which is familiar to us by now yields $df_p^{i(s)} \to dh_p$ as $s \to \infty$. In particular, $(\det df_p)^{i(s)} \to \det dh_p$ as $s \to \infty$.

Now if $|\det df_p| > 1$, the left side goes off to infinity as $s \to \infty$ while the right side is finite. Impossible. Thus $|\det df_p| \leq 1$, which proves (1). Suppose f is an automorphism of M, then its inverse satisfies, by (1), $|\det df_p^{-1}| \leq 1$, which is the same as saying $|\det df_p| \geq 1$. Thus, f is an automorphism implies $|\det df_p| = 1$. This proves one part of (3).

We now begin the proof of (2). Thus $f(p) = p$ and df_p is the identity linear map of M_p. Let U be a coordinate neighborhood of p with coordinate functions $z^1, \ldots, z^n$ such that $z^\sigma(p) = 0$, $\sigma = 1, \ldots, n$. Since $\{f^i\}$ is an equicontinuous family and $f^i(p) = p$, we may assume that $\{f^i\}$ carries the closure of the unit ball $D^n = \{\Sigma_\sigma |z^\sigma|^2 < 1\}$ into U. Considering $\{f^i\}$ as a subset of $\mathcal{A}(D^n, U)$, we can decompose each f^i into coordinate functions relative to $z^1, ..., z^n$: $f^i = (f_1^i, ..., f_n^i)$. (Of course, $f^1 = f$). By hypothesis, we have:

$$\frac{\partial f_\sigma}{\partial z^\varrho}(O) = \delta_\varrho^\sigma, \quad 1 \leqslant \sigma, \varrho \leqslant n, \tag{4.1}$$

where δ_ϱ^σ is the Kronecker delta. We will show that all higher derivatives of the f_σ's vanish at O. As before, consider the subsequence $\{f^{i(s)}\}$ which converges to $h: M \to N$. Clearly, h also carries the closure of D^n into U and so will be regarded as in $\mathcal{A}(D^n, U)$. Let $h = (h_1, ..., h_n)$ relative to $z^1, ..., z^n$. Then as $s \to \infty$, $f^{i(s)}$ together with all its derivatives converge uniformly to h_σ on D^n, $1 \leqslant \sigma \leqslant n$. Now, by the chain rule:

$$\frac{\partial f_\sigma^i}{\partial z^\varrho} = \sum_{\sigma_1, \dots, \sigma_{i-1}} \frac{\partial f_\sigma}{\partial z^{\sigma_1}} \cdot \frac{\partial f_{\sigma_1}}{\partial z^{\sigma_2}} \cdot \dots \cdot \frac{\partial f_{\sigma_{i-1}}}{\partial z^\varrho}. \tag{4.2}$$

(Note that f_σ^i is *not* the ith power of f_σ.) From (4.1) and (4.2), we see that:

$$\frac{\partial^2 f_\sigma^i}{\partial z^\varrho \partial z^\tau}(O) = i \cdot \frac{\partial^2 f_\sigma}{\partial z^\varrho \partial z^\tau}(O)$$

for $1 \leqslant \varrho, \sigma, \tau \leqslant n$ and all i. Hence it is clear that $\dfrac{\partial^2 f_\sigma^{i(s)}}{\partial z^\varrho \partial z^\tau}(O)$ cannot converge to $\dfrac{\partial^2 h_\sigma}{\partial z^\varrho \partial z^\tau}(O)$ as $s \to \infty$ unless $\dfrac{\partial^2 f_\sigma}{\partial z^\varrho \partial z^\tau}(O) = 0$ for all ϱ, σ, τ. We now induct on the order of the partials of f_σ. Suppose we know that the partials of f_σ of order $2, 3, \dots, r-1$ all vanish at O, then a computation based on (4.2) reveals that:

$$\frac{\partial^r f_\sigma^i}{\partial z^{\sigma_1} \dots \partial z^{\sigma_r}}(O) = i \cdot \frac{\partial^r f_\sigma}{\partial z^{\sigma_1} \dots \partial z^{\sigma_r}}(O).$$

Then again, in order that $\dfrac{\partial^r f_\sigma^{i(s)}}{\partial z^{\sigma_1} \dots \partial z^{\sigma_r}}(O)$ converge to $\dfrac{\partial^r h_\sigma}{\partial z^{\sigma_1} \dots \partial z^{\sigma_r}}(O)$, it is necessary that $\dfrac{\partial^r f_\sigma}{\partial z^{\sigma_1} \dots \partial z^{\sigma_r}}(O) = 0$ for all $\sigma_1, \dots, \sigma_r, \sigma$. We have therefore proved that all the derivatives of the f_σ, $1 \leqslant \sigma \leqslant n$, must vanish at O with the sole exception of (4.1). If we develop f_σ into a power series in D^n, this means $f_\sigma(z^1, \dots, z^n) = z^\sigma$. Thus the mapping $f = (f_1, \dots, f_n)$ of D^n into U is just the inclusion map $D^n \subseteq U$. As f is holomorphic, its behavior on M is completely determined by its behavior on any open set. Thus f is the identity of M onto itself, which proves (2).

Finally, we turn to the proof of the second half of (3): If $f(p)=p$ and $|\det df_p|=1$, then f is an automorphism of M. Pick D^n, U, $\{z^1, \ldots, z^n\}$ as before, and let $f^{i(s)} \xrightarrow{s} h$ as before. Consider the Jordan canonical form Γ of the matrix of df_p relative to the basis $\left\{\frac{\partial}{\partial z^1}(O), \ldots, \frac{\partial}{\partial z^n}(O)\right\}$ of M_p. (Recall that p corresponds to the origin O of this coordinate neighborhood.) We claim:

Γ is a unitary diagonal matrix.

To prove this, note as before that $(df_p)^{i(s)} \xrightarrow{s} (dh_p)$ as linear transformations, so that $\Gamma^{i(s)} \xrightarrow{s} \Delta$, where Δ is the matrix of dh_p relative to the same basis as Γ. In particular, all entries of $\Gamma^{i(s)}$ are bounded by a constant independent of s. Hence the diagonal entries of Γ all have absolute value at most one because of the fact that if the (k, k)-th entry of Γ is λ, then the (k, k)-th entry of $\Gamma^{i(s)}$ is $\lambda^{i(s)}$. Now if one diagonal entry of Γ in fact has absolute value strictly less than one, then the hypothesis of $|\det df_p|=1$ $(=|\det \Gamma|)$ will force some other diagonal entry to have absolute value strictly bigger than one. This last is forbidden. Hence, if λ is in the diagonal of Γ, $|\lambda|=1$.

Suppose Γ is not in diagonal form, then it must have a diagonal block of this form:

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

It is trivial to see that the corresponding diagonal block of Γ^l is then of the form:

$$\begin{bmatrix} \lambda^l & l\lambda^{l-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & l\lambda^{l-1} \\ 0 & & & \lambda^l \end{bmatrix}$$

This means that the entries immediately above the diagonal of $\Gamma^{i(s)}$ will diverge to infinity as $s\to\infty$. Then again, this contradicts an observation made above. Thus, Γ has diagonal entries all of the form $e^{i\theta}$, $\theta \in \mathbf{R}$, and has zero entries off the diagonal. Our claim is established.

It is now clear that we can pick a subsequence of $\{\Gamma^i\}$ so that it converges to the identity matrix. Equivalently, there is a subsequence of $\{df_p^i\}$ which converges to the identity transformation of $M_p \to M_p$. To save symbols, let us assume that this is the subsequence $\{df^{i(s)}\}$ above. Since $f^{i(s)}\to h$, we see that $dh_p=$identity map of $M_p\to M_p$. By (2),

h is the identity map of $M \to M$. An immediate consequence of this is that f is one–one. For, if $f(p_1) = f(p_2)$, then $f^{i(s)}(p_1) = f^{i(s)}(p_2)$. As $s \to \infty$, we have $p_1 = p_2$. To conclude the proof of (3) and the theorem, let us show f is onto. Suppose not, then there is a $q \in M \setminus f(M)$. Clearly then, $q \in M \setminus f^{i(s)}(M)$ for all s. Now equip M with any hermitian metric h whose global distance function and volume form we denote by d_h and Ω as usual. Since $df_q^{i(s)} \xrightarrow{s}$ identity map of $M_q \to M_q$, it is clear that $(f^{i(s)})^*\Omega/\Omega(q) \to 1$ as $s \to \infty$. Thus for $s \geqslant s_0$, say, $|(f^{i(s)})^*\Omega/\Omega(q)| \geqslant 1 - \varepsilon > 0$. By Theorem A, there is a univalent ball of radius α (relative to d_h) around $f^{i(s)}(q)$ for $s \geqslant s_0$. But $f^{i(s)}(q) \xrightarrow{s} q$, so q belongs to all these univalent balls as soon as s is so large that $d_h(f^{i(s)}(q), q) < \frac{1}{2}\alpha$, say. Hence $f^{i(s)}(M)$ contains q for all sufficiently large s, contradicting q is in $M \setminus f^{i(s)}(M)$. Q.E.D.

5. Theorems D and E

We will consider the group $\mathcal{H}(N)$ of automorphisms ($\equiv$biholomorphic, onto self-mappings) of a complex manifold N in this section. Note that $\mathcal{H}(N)$ is a closed subset of $\mathcal{A}(N)$, so it is also closed in $\mathcal{C}(N)$. The dominating notion here will be the tightness of N (always with respect to some pre-assigned metric d on N). See Definition 1.3.

THEOREM D. *If (N, d) is a tight complex manifold, then its automorphism group $\mathcal{H}(N)$ is a (not necessarily connected) Lie group, and the isotropy subgroup of $\mathcal{H}(N)$ at a point is a compact Lie group.*

By the *isotropy group* of $\mathcal{H}(N)$ at a point p, we mean the set of all elements in $\mathcal{H}(N)$ leaving p fixed. Theorem D will be a consequence of a general lemma about transformation groups of a metric space, Lemma 5.1 below. A related theorem, which however needs a separate proof, is the following.

THEOREM D′. *If N is a taut complex manifold, then its automorphism group $\mathcal{H}(N)$ is a (not necessarily connected) Lie group, and the isotropy subgroup of $\mathcal{H}(N)$ at a point is a compact Lie group.*

The proof of both will depend on the basic *Theorem of Bochner–Montgomery* (Chu–Kobayashi [9], Theorem C) which we quote: A locally compact transformation group of diffeomorphisms of a differentiable manifold is a (not necessarily connected) Lie transformation group of the manifold. With this theorem at hand, Theorem D is clearly a corollary of the following lemma.

LEMMA 5.1. *Let N be a connected locally compact metric space and G a group of homeomorphisms of N which is* (i) *equicontinuous, and* (ii) *closed in $\mathcal{C}(N)$. Then G is a locally compact transformation group of N.*

Proof. Let the metric of N be d. Define a new distance function $\delta \equiv \delta_d$ on N as follows:

$$\delta(x, y) = \sup_{\varphi \in G} d(\varphi(x), \varphi(y))$$

We will prove that δ is a metric on N. Since the identity transformation belongs to G, clearly $\delta(x, y) = 0$ iff $x = y$. The triangular inequality and the symmetry condition $\delta(x, y) = \delta(y, x)$ are both trivial. What remains unproved is that $\delta(x, y) < \infty$ for all $x, y \in N$. For this, we shall need assumption (i) of equicontinuity. Fix an $x \in N$, and define $\Phi(x)$ to be the set of all $y \in N$ such that $\delta(x, y) < \infty$. Since $x \in \Phi(x)$, $\Phi(x)$ is nonempty. Suppose $y \in \Phi(x)$, we will show that so does a neighborhood of y. For let $0 < \varepsilon < \infty$ be given, then the equicontinuity of G implies the existence of a neighborhood U of y such that $d(\varphi(y), \varphi(y')) < \varepsilon$ for all $y' \in U$ and all $\varphi \in G$. Thus $\delta(y, y') < \varepsilon$ for all $y' \in U$ and consequently $\delta(y', x) \leqslant \delta(y', y) + \delta(y, x) < \infty$. So $U \subseteq \Phi(x)$, and $\Phi(x)$ is open. Now, $\Phi(x)$ is also closed by a very similar argument. So the connectedness of N implies $\Phi(x) = N$, and $\delta(x, y) < \infty$ for all $x, y \in N$. Thus δ is a metric on N.

Next we observe that the topology induced by δ on N coincides with the original one induced by d. This also needs equicontinuity, as follows. First observe that, since the identity map of $N \to N$ is in G, $d \leqslant \delta$; so every ε-ball of d contains an ε-ball of δ. Conversely, let an ε-ball of δ be given. Call it B' and let it be centered at x_0. By equicontinuity of G, we can choose an ε'-ball of d (call it B) centered at x_0 such that $x \in B$ implies $d(\varphi(x_0), \varphi(x)) < \frac{1}{2}\varepsilon$ for all $\varphi \in G$. We claim: $B \subseteq B'$. For if $x \in B \setminus B'$, then $\delta(x, x_0) \geqslant \varepsilon$, and there will be a sequence $\varphi_i \in G$ such that $d(\varphi_i(x), \varphi_i(x_0)) \to \varepsilon$ for this $x \in B$. This contradicts the choice of B. Thus the d-topology and the δ-topology of N coincide. In the following we may therefore replace d by δ.

Since G is a group, it is obvious that it becomes a group of *isometries* of (N, δ). By a well-known theorem of van Dantzig – van der Waerden (Kobayashi–Nomizu [23], p. 46), the group of all isometries of a connected locally compact metric space is a locally compact transformation group. Now, G is closed in $\mathcal{C}(N)$ by assumption (i), and hence closed in the set of all δ-isometries of N. Therefore G is a locally compact transformation group of N. The compactness assertion concerning the isotropy group is a corollary of the proof of the van Dantzig – van der Waerden theorem, (Kobayashi–Nomizu [23], p. 49). Q.E.D.

Proof of Theorem D'. Recall that $\mathcal{C}(N)$ is second countable, so we shall employ sequences exclusively to deal with all questions concerning continuity. Also, to simplify matters, we impose a metric d on N.

We first show that the inverse operation is continuous in $\mathcal{H}(N)$. Let $f_i \to f$, and we must show $f_i^{-1} \to f^{-1}$. It suffices to show: if K is a compact set and $\varepsilon > 0$ is given, then there is some i_0 such that $d(f_i^{-1}(x), f^{-1}(x)) < \varepsilon$ for all $x \in K$ and all $i \geq i_0$. To this end, let $K' = f^{-1}(K)$, and let $y = f^{-1}(x) \in K'$. Thus:

$$d(f_i^{-1}(x), f^{-1}(x)) = d(f_i^{-1}(f(y)), y) = d(f_i^{-1}(f(y)), f_i^{-1}(f_i(y)))$$

We claim: There exist a $\delta > 0$ and an i_1 such that if $z, z' \in K$ and if $d(z, z') < \delta$, then $d(f_i^{-1}(z), f_i^{-1}(z')) < \varepsilon$ for all $i \geq i_1$.

Granting this for a moment, choose i_2 so that $d(f(y), f_i(y)) < \delta$ if $i \geq i_2$ and if $y \in K'$. This is possible since $f_i \to f$ and $K' = f^{-1}(K)$ is compact. Let $i_0 \geq \{i_1, i_2\}$, then the above claim implies:

$$d(f_i^{-1}(x), f^{-1}(x)) < \varepsilon \quad \text{if} \quad y \in K' \quad \text{and} \quad i \geq i_0.$$

In other words, this holds for all $x \in K$ and all $i \geq i_0$, as desired.

We now turn to the proof of the claim. Let $y_0 \in K'$ and let $x_0 = f(y_0) \in K$. Let U be any relatively compact neighborhood of x_0. Since $f_i \to f$, for a large enough i_1, $i \geq i_1$, will imply that $f_i(y_0) \in U$. Thus the sequence $F = \{f_i^{-1}\}_{i \geq i_1}$ has the property: the compact set $\{y_0\}$ is contained in all $f_i^{-1}(\bar{U})$, where $\bar{U}$ denotes the compact closure of U. Thus no subsequence of F can be compactly divergent, and the tautness of N then implies that F is a relatively compact set in $\mathcal{H}(N)$. By Lemma 1.2, F is an equicontinuous family. Therefore, for each $z \in K$, there is an open neighborhood U_z such that $z' \in U_z$ implies

$$d(f_i^{-1}(z), f_i^{-1}(z')) < \tfrac{1}{2}\varepsilon \quad \text{if} \quad i \geq i_1.$$

Let this compact K be covered by finitely many such U_z and let δ be the Lebesgue covering number, i.e. $d(z_1, z_2) < \delta$ implies z_1, z_2 belongs to the same U_{z^*} for some z^*. The claim clearly follows from this choice of δ.

The fact that multiplication in $\mathcal{H}(N)$ is continuous is provable in a similar fashion. To show that the action on N: $\mathcal{H}(N) \times N \to N$ with $(f, x) \to f(x)$ is continuous, one proves $f_i \to f$ and $x_j \to x$ imply $f_i(x_j) \to f(x)$. But this is a standard consequence of the uniformity of the convergence of f_i to f (on some compact neighborhood of x). Thus $\mathcal{H}(N)$ is a topological group acting effectively on N. $\mathcal{H}(N)$ is locally compact in view of Lemma 1.1 (ii). Hence the above theorem of Bochner–Montgomery applies and $\mathcal{H}(N)$ is a Lie group. Now clearly the isotropy subgroup of $\mathcal{H}(N)$ at p, by its very definition, can contain no compactly divergent subsequences. Then the tautness of N implies that it is compact in $\mathcal{H}(N)$. In particular, it is closed in $\mathcal{H}(N)$ and so a Lie group also. Q.E.D.

It should be remarked that there are no known examples of a taut manifold which is not tight relative to some suitable metric. See Part II. So in applications, only Theorem D is ever needed. Note also that Theorem D together with Lemma 1.5 imply the famous

Theorem of H. Cartan to the effect that the automorphism group of a bounded domain in $\mathbf{C}^n$ is a Lie group.

The special case of Theorems D and D′ for a compact N is particularly interesting. (Both theorems coincide in this case.) But we first need a basic result.

THEOREM E. *If (N, d) is a tight complex manifold, then there is no nonconstant holomorphic map of $\mathbf{C}^n$ into N.*

Proof. Clearly it suffices to prove that every holomorphic map $f: \mathbf{C} \to N$ reduces to a constant. Let $f(0) = p$, and let $z^1, \ldots, z^n$ be a coordinate system around p such that $z^\sigma(p) = 0$ $(\sigma = 1, \ldots, n)$ and the unit ball $D^n = \{\sum_\sigma |z^\sigma|^2 < 1\}$ is contained in this neighborhood. Consider the sequence of mappings φ_i: $D^1 \to \mathbf{C}$ such that $\varphi_i(z) = iz$, where i is a nonnegative integer. Then $\psi_i \equiv f \circ \varphi_i$ is a sequence of holomorphic mappings from D^1 into N. By the tightness of N, there is a $\delta > 0$ so that if $\xi \in D_\delta \equiv \{|z| < \delta\}$, then $\psi_i(\xi) \in D^n$ for all i. We now show that this implies $f(\mathbf{C}) \subseteq D^n$. For let $\zeta \in C$ and let i_0 be so large that $i_0\delta > |\zeta|$. Thus $\zeta \in \varphi_{i_0}(D_\delta)$. Let $\xi \in D_\delta$ be such that $\varphi_{i_0}(\xi) = \zeta$. Then $f(\zeta) = f(\varphi_{i_0}(\xi)) = \psi_{i_0}(\xi) \in D^n$. Since ζ is arbitrary, $f(\mathbf{C}) \subseteq D^n$ as desired. By Liouville's Theorem, $f(\mathbf{C}) = \{p\}$. Q.E.D.

COROLLARY 5.2. *If N is compact and taut, then $\mathcal{H}(N)$ is finite.*

Proof. Since N is compact, its tautness implies that $\mathcal{H}(N)$ is a compact Lie group. Another theorem of Bochner–Montgomery states: the automorphism group $\mathcal{H}(N)$ of a compact complex manifold N is a complex Lie group and the action ϱ: $\mathcal{H}(N) \times N \to N$, $\varrho(g, p) = g(p)$, is *holomorphic*. (Bochner–Montgomery [5].) So $\mathcal{H}(N)$ is a compact complex Lie group and therefore its identity component is a torus T. If T consists of more than one element, then it admits $\mathbf{C}^r$ as universal covering group, where $r > 0$. Let π: $\mathbf{C}^r \to T$ be the covering map. Take a $g \in T$ distinct from the identity and let p be some point of N such that $g(p) \neq p$. Then the mapping of $\mathbf{C}^r \to N$ such that $\zeta \to \varrho(\pi(\zeta), p)$ is holomorphic and nonconstant. This contradicts Theorem E. Hence T is just the identity element, and the finiteness of $\mathcal{H}(N)$ follows from its compactness. Q.E.D.

6. Digression into domains in $\mathbf{C}^n$

THEOREM F. *If a domain E (= open connected set) in $\mathbf{C}^n$ is taut and $E \neq \mathbf{C}^n$, then E is pseudo-convex and hence a domain of holomorphy.*

We have to recall a definition of pseudo-convexity which is most suitable for our purpose. It is the classical Kontinuitätsatz, as found in L. Bers [2], p. 30, or Katznelson [18], p. 2.8. First, the notation $A \subset\subset B$ is employed to denote "A has compact closure in

B." By definition, an *analytic disc* in $\mathbf{C}^n$ is a holomorphic mapping of the closed unit disc $\bar{D}^1=\{|z|\leqslant 1\}\subseteq \mathbf{C}$ into $\mathbf{C}^n$. If s: $\bar{D}^1\rightarrow \mathbf{C}^n$ is an analytic disc in $\mathbf{C}^n$, by abuse of language, we sometimes identify s with its image $s(\bar{D}^1)\subseteq C^n$. The boundary ∂s of s denotes either the restriction of s to the unit circle $\partial\bar{D}^1$ or the image set $s(\partial\bar{D}^1)$. A domain E of $\mathbf{C}^n$ $(E\neq\mathbf{C}^n)$ is called *pseudo-convex* iff every sequence of analytic discs $\{s_i\}$ in E with $\bigcup_i \partial s_i \subset\subset E$ has the property that $\bigcup_i s_i \subset\subset E$.

Proof of Theorem F. Let $\{s_i\}$ be a sequence of analytic discs in E with $\bigcup_i \partial s_i \subset\subset E$ as above. $\bigcup_i \partial s_i$ has compact closure and hence bounded. By the maximum modulus principle, $\bigcup_i s_i$ is also a bounded subset of $\mathbf{C}^n$ and hence by Montel's Theorem (Lemma 1.4), $\{s_i\}$ is a relatively compact subset of $\mathcal{A}(D^1, \mathbf{C}^n)$. It is then clear that there is a compact neighborhood K of $\bigcup_i \partial s_i$ in E such that $\partial s_i \cap K\neq\emptyset$ for all i. This means that the sequence $\{s_i\}\subseteq\mathcal{A}(D^1, E)$ can contain no compactly divergent subsequence. By tautness of E, $\{s_i\}$ is a relatively compact subset of $\mathcal{A}(D^1, E)$. It follows that $\bigcup_i s_i \subset\subset E$, and E is pseudo-convex. By the celebrated Oka solution of Levi's Problem, E is a domain of holomorphy. Q.E.D.

Remark. It follows from the classical uniformization theorem as well as Corollary 8.4 in § 8 that a domain E in $\mathbf{C}$ is taut iff E is neither $\mathbf{C}$ nor $\mathbf{C}$-{point}. However, the converse to Theorem F is unknown already for $n=2$.

PART II

7. Hermitian geometry

In this section are to be found the basic facts about hermitian geometry; in particular, we present in some detail the passage from a hermitian connection on the unitary bundle to a connection on the associated orthogonal bundle. The reason for doing this known (but generally inaccessible) material is that there are grievous errors as well as confusion in signs in the literature. Since the *negativeness* of the holomorphic curvature will be of the utmost importance to us, it is imperative that we do this carefully and correctly.

There are two conventions in sign concerning the structure equations of a connection. The one adopted here coincides with the one in Kobayashi–Nomizu [23] and is the more natural of the two. Namely, we regard Lie algebras as *left* invariant vector fields and the structure group acts on the *right* in the principal bundle. The connection form is a one-form on the bundle which should be the *identity* on the Lie algebras of each fibre. Let G be a subgroup of $\mathrm{Gl}(n, \mathbf{R})$ or $\mathrm{Gl}(n, \mathbf{C})$. Then the structure equations read:

$$\left.\begin{aligned} d\xi^\alpha &= -\Sigma_\beta \vartheta^\alpha_\beta \wedge \xi^\beta + \Xi^\alpha \\ d\vartheta^\alpha_\beta &= -\Sigma_\gamma \vartheta^\alpha_\gamma \wedge \vartheta^\gamma_\beta + \Theta^\alpha_\beta \end{aligned}\right\}, \tag{7.1}$$

where $1 \leqslant \alpha, \beta, \gamma \leqslant n$; (ϑ^α_β) is the connection form, (Ξ^α) the torsion form and (Θ^α_β) the curvature form. Finally, (ξ^α) is the canonical one-form attached to every G-structure independent of the connection. See [23], p. 121.

Now let h $(\equiv \langle\,,\rangle)$ be a hermitian metric on a given n-dimensional complex manifold M. It gives rise to a bundle of unitary frames, $F^{\#}(M) = \{(m, e_1, ..., e_n)\colon m \in M,\ e_\alpha \in M_m,\ \langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}\}$. The structure group of $F^{\#}(M)$ is of course the n-dimensional unitary group $U(n)$. The canonical hermitian connection of h is the unique connection on $F^{\#}(M)$ whose torsion form is of type (2, 0) (Chern [8], Griffiths [11].) In this case, we rewrite the structure equations as follows:

$$\left.\begin{aligned} d\psi^\alpha &= -\Sigma_\beta \omega^\alpha_\beta \wedge \psi^\beta + \Psi^\alpha \\ d\omega^\alpha_\beta &= -\Sigma_\gamma \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta \end{aligned}\right\}, \tag{7.2}$$

where $1 \leqslant \alpha, \beta, \gamma \leqslant n$. The connection and curvature forms $\psi = (\psi^\alpha_\beta)$ and $\Omega = (\Omega^\alpha_\beta)$ now take value in the skew-hermitian matrices so that, $\psi^\alpha_\beta = -\bar{\psi}^\beta_\alpha$, $\Omega^\alpha_\beta = -\Omega^\beta_\alpha$. The torsion form Ψ^α is of type (2, 0) as noted above and Ω is of type (1, 1). The hermitian connection can be alternatively characterized by: when the connection on $F^{\#}(M)$ is extended to the full $\mathrm{Gl}(n, \mathbf{C})$-bundle of all complex bases then its connection form is of type (1, 0). (Chern [8]. There is also a very elegant discussion of this in Singer's paper in the Pacific J. Math. 1959.)

Now decompose h into real and imaginary parts: $h = g + (-\sqrt{-1})\,k$. Then g is a riemannian metric on M and k is the real (1,1) Kähler form of h. If $dk = 0$, h is called Kähler; this happens iff $\Psi = 0$ in (7.2). (Chern [8].) Let the complex structure tensor of M be J, then $g(Jx, Jy) = g(x, y)$. Associated with $F^{\#}(M)$ is the bundle

$$F^*(M) = \{(m, e_1, \ldots, e_n, Je_1, \ldots, Je_n)\colon\ m \in M,\ \langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta},\ 1 \leqslant \alpha, \beta \leqslant n\}.$$

Clearly $F^*(M)$ is a subbundle of the $\mathrm{SO}(2n)$-bundle of oriented orthonormal frames (with respect to g). The structure group of $F^*(M)$ is the subgroup $\mathrm{SO}^*(2n)$ of $\mathrm{SO}(2n)$ consisting of all matrices of the form $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ where $(A + \sqrt{-1}\,B) \in U(n)$. Now, $U \to \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is a group isomorphism from $U(n)$ onto $\mathrm{SO}^*(2n)$ and this induces a map $(m, e_1, \ldots, e_n) \to (m, e_1, \ldots, e_n, Je_1, \ldots, Je_n)$ of $F^{\#}(M)$ onto $F^*(M)$. This is a bundle isomorphism and *we identify the two bundles* via this isomorphism. The hermitian connection on $F^{\#}(M)$ induces a unique connection on $F^*(M)$, and we want an explicit form of the latter directly in terms of the Lie algebra of $SO^*(2n)$. So decompose ω and Ω into real and imaginary parts: $\omega = \omega' + \sqrt{-1}\,\omega''$, $\Omega = \Omega' + \sqrt{-1}\,\Omega''$. Consider the following $(2n \times 2n)$ matrices of differential forms:

$$\vartheta = (\vartheta^i_j) = \begin{bmatrix} \omega' & -\omega'' \\ \omega'' & \omega' \end{bmatrix}, \quad \Theta = (\Theta^i_j) = \begin{bmatrix} \Omega' & -\Omega'' \\ \Omega'' & \Omega' \end{bmatrix}, \tag{7.4}$$

where $1 \leqslant i, j, \leqslant 2n$. Explicitly: $\vartheta^\alpha_\beta = \omega'^\alpha_\beta$, $\vartheta^\alpha_{n+\beta} = -\omega''^\alpha_\beta$, $\vartheta^{n+\alpha}_\beta = \omega''^\alpha_\beta$, $\vartheta^{n+\alpha}_{n+\beta} = \omega'^\alpha_\beta$, where $1 \leqslant \alpha, \beta \leqslant n$. Same for Θ, Ω' and Ω''. Similarly, decompose (ψ^α) and (Ψ^α) into real and imaginary parts: $\psi = \psi' + \sqrt{-1}\,\psi''$, $\Psi = \Psi' + \sqrt{-1}\,\Psi''$, and consider also the following $(1 \times 2n)$ matrices:

$$\varphi = (\varphi^i) = [\psi' \; \psi''], \quad \Phi = (\Phi^i) = [\Psi' \; \Psi''], \tag{7.5}$$

where $1 \leqslant i \leqslant 2n$. Explicitly: $\varphi^\alpha = \psi'^\alpha$, $\varphi^{n+\alpha} = \psi''^\alpha$, where $1 \leqslant \alpha \leqslant n$. Same for Φ, Ψ', and Ψ''. Then by virtue of (7.2), (7.3) and (7.4), a simple computation gives:

$$\left.\begin{aligned} d\varphi^i &= -\textstyle\sum_j \vartheta^i_j \wedge \varphi^j + \Phi^i \\ d\vartheta^i_j &= -\textstyle\sum_k \vartheta^i_k \wedge \vartheta^k_j + \Theta^i_j \end{aligned}\right\}, \tag{7.6}$$

where $1 \leqslant i, j, k \leqslant 2n$.

Now, it is easy to show that $\varphi = (\varphi^i)$ is the canonical one-form attached to the $SO^*(2n)$-bundle $F^*(M)$ and that $\vartheta = (\vartheta^i_j)$ is a connection form on $F^*(M)$. Comparison of (7.1) with (7.6) therefore yields this fact: Φ is the torsion and Θ *is the curvature of the connection* ϑ. We pause to remark that, because $SO^*(2n) \subseteq SO(2n)$, ϑ is the unique connection with Φ as torsion form and that if the hermitian metric h is Kähler, then $\Psi = \Phi = 0$, and consequently ϑ would be the unique riemannian connection of g.

We shall be particularly interested in the sectional curvature of the holomorphic planes, span $\{e_\alpha, Je_\alpha\}$, $(\alpha = 1, \ldots, n)$ of the hermitian connection ω. This will be defined via ϑ, and instead of staying up in the bundle $F^{\#}(M)$ and $F^*(M)$ all the time, we shall go down to M—not out of necessity, but purely as a matter of convenience. So let $e^0_\alpha \in M_{m^0}$ be a fixed vector and let s be a C^∞ section in a neighborhood of m^0 into $F^*(M)$, such that if $s(m) = (m, e_1(m), \ldots, e_n(m), Je_1(m), \ldots, Je_n(m))$, then $e^0_\alpha = e_\alpha(m^0)$. s^* pulls down the structure equations (7.2) and (7.6) into M, but once this is understood, we shall omit s^* in front of each term of (7.2) and (7.6) for the sake of brevity. Now, if ω were a Kähler connection, then ϑ would be torsionfree and hence the riemannian connection of g. In this case the sectional curvature (in the usual sense) of the complex line span $\{e^0_\alpha, Je^0_\alpha\}$ is known to be:

$$+\,\Theta^\alpha_{n+\alpha}(e^0_\alpha, Je^0_\alpha). \tag{7.7}$$

We now *define* in general: *the holomorphic curvature in the direction e^0_α of the hermitian connection ω is* (7.7) regardless of whether ω is Kähler or not. As usual, (7.7) so defined is independent of the section s used. By (7.4), (7.7) equals $-\Omega''^{\alpha}_{\alpha}(e^0_\alpha, Je^0_\alpha)$. Since Ω takes value in skew hermitian matrices, the diagonal entries of Ω are purely imaginary so that $\Omega^\alpha_\alpha = \sqrt{-1}\,\Omega''^{\alpha}_{\alpha}$. Thus (7.7) in fact equals $\sqrt{-1}\,\Omega^\alpha_\alpha(e^0_\alpha, Je^0_\alpha)$. Now let $\Omega^\alpha_\beta = \sum_{\gamma,\delta} R^\alpha_{\beta\gamma\delta}\,\psi^\gamma \wedge \bar{\psi}^\delta$, $1 \leqslant \gamma, \delta \leqslant n$. Then via the section s, this becomes an equation on M, and from $\psi^\alpha(e_\beta) = \delta^\alpha_\beta$, $\bar{\psi}^\alpha(Je_\beta) = -\sqrt{-1}\,\delta^\alpha_\beta$, $\psi^\alpha(Je_\beta) = \sqrt{-1}\,\delta^\alpha_\beta$, and $\bar{\psi}^\alpha(e_\beta) = \delta^\alpha_\beta$, one deduces that (7.7) equals $2R^\alpha_{\alpha\alpha\alpha}(m^0)$. Now, recalling the identification between $F^{\#}(M)$ and $F^*(M)$, we have clearly proved:

LEMMA 7.1. *Let Ω be the curvature form of the hermitian connection ω in (7.2) and let $\Omega^\alpha_\beta = \sum_{\gamma,\delta} R^\alpha_{\beta\gamma\delta}\,\psi^\gamma \wedge \bar{\psi}^\delta$. Then via a section $s(m) = (m, e_1(m), \dots, e_n(m))$ into $F^{\#}(M)$, the functions $(+2R^\alpha_{\alpha\alpha\alpha})$ give the holomorphic curvature of the complex lines* span $\{e_\alpha, Je_\alpha\}$ *with respect to ω.*

Now let M' be a complex submanifold of M of dimension r. Choose a local C^∞ section s of M' into $F^{\#}(M)$ so that the first r vectors are everywhere tangent to M'. Let us agree on these ranges of indices:

$$1 \leqslant a, b, c, d \leqslant r, \quad r+1 \leqslant \sigma, \varrho, \tau \leqslant n, \quad 1 \leqslant \alpha, \beta, \gamma \leqslant n.$$

Hence with respect to this section s, ψ^σ $(\equiv s^*\psi^\sigma) = 0$, and the first equation of (7.2) (*via* s^*, as is everything else here) implies that:

$$0 = -\sum_a \omega^\sigma_a \wedge \psi^a + \Psi'^\sigma$$

or

$$\sum_a \omega^\sigma_a \wedge \psi^a = \Psi'^\sigma.$$

Now locally in M', $\{\psi^a, \bar{\psi}^b\}$ is a basis of C^∞ one-forms. Since ψ^a is of type $(1,0)$ and Ψ' of type $(2,0)$, there will be some C^∞ functions $\{S^\sigma_{ab}\}$ defined locally in M' such that:

$$\omega^\sigma_a = \sum_b S^\sigma_{ab}\,\psi^b \,(= -\bar{\omega}^a_\sigma)$$

$\{S^\sigma_{ab}\}$ constitutes the complex second fundamental form of M' in M; for a more thorough discussion of this concept in hermitian geometry, see Griffiths [11] and [12]. The second equation of (7.2) then implies that

$$\begin{aligned} d\omega^a_b &= -\sum_\gamma \omega^a_\gamma \wedge \omega^\gamma_b + \Omega^a_b \\ &= -\sum_c \omega^a_c \wedge \omega^c_b + (-\sum_\sigma \omega^a_\sigma \wedge \omega^\sigma_b) + \sum_{c,d} R^a_{bcd}\,\psi^c \wedge \bar{\psi}^d \\ &= -\sum_c \omega^a_c \wedge \omega^c_b + \sum_{c,e} (R^a_{bcd} - \sum_\sigma S^\sigma_{bc}\,\bar{S}^\sigma_{ad})\,\psi^c \wedge \bar{\psi}^d. \end{aligned}$$

In other words, if we let $T^a_{bcd} = (R^a_{bcd} - \sum_\sigma S^\sigma_{bc}\,\bar{S}^\sigma_{ad})$, and let $\Lambda^a_b = \sum_{c,d} T^a_{bcd}\psi^c \wedge \bar{\psi}^d$, then

$$d\omega_b^a = -\sum_c \omega_c^a \wedge \omega_b^c + \Lambda_b^a. \tag{7.8}$$

By the reasoning familiar in riemannian geometry, since (ω_b^a) is obviously a connection in $F^{\#}(M')$ and since the torsion form Ψ'^a of $d\psi^a = -\sum_b \omega_b^a \wedge \psi^b + \Psi'^a$ is of type $(2, 0)$ the uniqueness of the hermitian connection implies that (ω_b^a) is in fact the hermitian connection in M' of the induced metric in M'. By (7.8) (Λ_b^a) is then the curvature form of this induced connection. But $T_{aaa}^a = R_{aaa}^a - \sum_\sigma |S_{aa}^\sigma|^2$, and so via Lemma 7.1, we have proved:

LEMMA 7.2 (Kobayashi). *Let M' be a complex submanifold of a hermitian manifold M and let l be a complex line tangent to M'. Then the holomorphic curvature of l in M' (with respect to the induced metric) is not greater than the holomorphic curvature of l in M.*

The most important special case of this lemma is when M' is a holomorphically imbedded disc. In this case, the induced metric on the disc is always a Kähler metric (for purely dimensional reasons) and the holomorphic curvature of such an l (which is then necessarily the tangent space to the disc at some point) is exactly the Gaussian curvature (in the usual sense) of the induced Kähler metric.

8. The basic theorem

Thus far, we have assigned arbitrary metrics (in the sense of point-set topology) to complex manifolds. Suppose now N is hermitian, with hermitian metric h, then h gives rise to a global distance function d_h (or just d if there can be no confusion) which of course makes N a metric space. *Henceforth, when we speak of a hermitian manifold N as a metric space, we shall mean with respect to d_h, and if we say the hermitian N is tight, we simply mean (N, d_h) is tight,* in the terminology of Definition 1.3. We recall this well-known fact:

LEMMA 8.1 (Hopf–Rinow). *A hermitian manifold is a complete metric space iff every bounded subset is compact.*

For discs in $\mathbf{C}$, or more generally, for Riemann surfaces, we must consider a more general class of hermitian metrics.

Definition 8.1. A *pseudo-hermitian metric* h^0 on a Riemann surface M is a C^∞ covariant tensor of rank two such that, (1) h^0 is a hermitian metric on $M \setminus S$, where S is a subset of M consisting only of isolated points, and (2) h^0 is zero on S.

Such objects arise naturally because if we take a nonconstant holomorphic mapping φ: $M \to N$, where M is a Riemann surface and N has hermitian metric h, then $\varphi^* h$ is a pseudo-hermitian metric on M, the set S in question being simply the zeroes of $d\varphi$. Now in general, since h^0 is a hermitian metric on $M \setminus S$, it makes sense to speak of the (Gaussian) curvature of h^0 on $M \setminus S$. In the sequel, it will be understood that *by the curvature of h^0 on M, we mean the curvature of h^0 on $M \setminus S$.* In this connection, we recall another famous theorem:

LEMMA 8.2 (Ahlfors). *If a pseudo-hermitian metric g^0 on the unit disc $D^1 \subseteq \mathbf{C}$ has curvature bounded above by $-k_0 < 0$, then $k_0 g^0 \leqslant H$, where H is the Poincaré hyperbolic metric on D^1 with constant curvature -1.*

Definition 8.2. Let $\mathfrak{F}$: $M \to N$ be a family of holomorphic mappings from a complex manifold M into a hermitian manifold N with hermitian metric h. $\mathfrak{F}$ is called a *strongly negatively curved family* (*of order* $-k_0 < 0$) iff for any $f \in \mathfrak{F}$ and for any holomorphically imbedded disc D in M the curvature of the pseudo-hermitian metric $(f|D)^*h$ is bounded above by $-k_0 < 0$.

The main result of this section, which entails as a corollary the main theorem of Grauert–Reckziegel [10], Satz 1, is the following.

BASIC THEOREM. *A strongly negatively curved family $\mathfrak{F}$ of holomorphic mappings from a complex manifold M into a hermitian manifold N is equicontinuous. If N is complete, then $\mathfrak{F}$ is normal.*

Proof. Suppose we know $\mathfrak{F}$ is equicontinuous and N is complete, then by Lemma 8.1 and Lemma 1.1 (iii), $\mathfrak{F}$ is normal. Hence it suffices to prove the equicontinuity of $\mathfrak{F}$, assuming that it is strongly negatively curved. This is an entirely local question, so we fix an arbitrary point $p \in M$, and take p-centered coordinate functions $\{z^1, ..., z^n\}$ so that the unit ball $D^n = \{\Sigma_\sigma |z^\sigma|^2 < 1\}$ is well-defined. To facilitate matters, though not strictly necessary, we assume $\mathfrak{F}$ is strongly negatively curved of order -1, and that the Bergman metric b on D^n (which is a Kähler metric of constant *holomorphic* curvature -1, see Helgason [15]) has been imposed. The former can be achieved through multiplication of the hermitian metric h on N by a suitable scalar factor. Assume both done, and we restrict $\mathfrak{F}$ to D^n. For our purpose, it suffices to deduce a simple-minded distance-decreasing property of $\mathfrak{F}$ with respect to d_b on D^n and d_h on N. (For full generality, see Kobayashi [21]). Let $q \in D^n$ be given, pass a linearly and holomorphically imbedded unit disc D through q and the origin O of D^n. Now D is a totally geodesic submanifold of (D^n, b), so that if b' is the induced metric on D, $d_{b'}(q, O) = d_b(q, O)$. By assumption on $\mathfrak{F}$, if $f \in \mathfrak{F}$, then the pseudo-hermitian metric $(f|D)^*h$ on D has curvature less than -1. Lemma 8.2 then implies that $d_{b'}(q, O) \geqslant d_h(f(q), f(O))$. Thus $d_b(q, O) \geqslant d_h(f(q), f(O))$, for all $q \in D^n$ and all $f \in \mathfrak{F}$. Equicontinuity is now obvious: if ε is given, choose a ball B of radius $\frac{1}{2}\varepsilon$ relative to d_b around the origin of D^n so that if q, q' are in B, and $f \in \mathfrak{F}$, we have

$$d_h(f(q), f(q')) \leqslant d_h(f(q), f(O)) + d_h(f(q'), f(O)) \leqslant d_b(q, O) + d_b(q', O) \leqslant \varepsilon. \quad \text{Q.E.D.}$$

A first consequence of the basic theorem is the following corollary suggested to me by Phil Griffiths. In his work on the moduli problem of algebraic manifolds, he found that the situation described below presents itself naturally and that in all the specific cases he has computed, the hypothesis of the corollary is always fulfilled. See his long paper [13], especially the initial section. For the terms that enter into the following statement, recall that a *complex distribution* on a complex manifold N is a C^∞ subbundle of the tangent bundle of N (*which we denote by* $T(N)$) whose fibres are complex subspaces of the tangent spaces of N.

COROLLARY 8.3. *Let N be a hermitian manifold which admits a complex distribution S with the property that the holomorphic curvature of every complex line in S is bounded above by a fixed negative constant $-k_0<0$. Let $\mathfrak{F}$: $M\to N$ be a holomorphic family with the property that each $f\in\mathfrak{F}$ maps $T(M)$ into S; i.e. the image of each tangent space under df, for all $f\in\mathfrak{F}$, lies in S. Then $\mathfrak{F}$ is equicontinuous. If N is complete, $\mathfrak{F}$ is normal.*

Proof. Let $D\subseteq M$ be a holomorphically imbedded disc of M and let $f\in\mathfrak{F}$. Then lemma 7.2 implies that the curvature of the pseudo-hermitian metric $(f|D)^*h$ on D is bounded above by $-k_0<0$. Hence $\mathfrak{F}$ is a strongly negatively curved family and the corollary follows from the basic theorem.

If S happens to be all of $T(N)$, then the above yields (essentially) the main theorem of Grauert–Reckziegel [10], Satz 1. However, it will be convenient to first introduce this definition: a hermitian manifold is *strongly negatively curved* (*of order* $-k_0<0$) iff its holomorphic curvature of all complex lines is bounded above by $-k_0<0$.

COROLLARY 8.4 (Grauert–Reckziegel). *If N is a strongly negatively curved hermitian manifold, then it is tight. If N is furthermore complete, then it is taut.* (*See Definition 1.2, 1.3.*)

We mention in passing that direct products and complex submanifolds of strongly negatively curved hermitian manifolds are also strongly negatively curved; so are the covering manifolds of such. In this way, quite a few tight and taut manifolds can be generated. It is an open question whether there are tight manifolds (resp. taut manifolds) which are not strongly negatively curved (resp. strongly negatively curved and complete) in some suitable hermitian metric.

Note finally that, for Riemann surfaces, the existence of a strongly negatively curved *pseudo*-hermitian metric is sufficient to guarantee the validity of Corollary 8.4. One can see that the concept of completeness is well-defined in that case, and the proof carries over verbatim.

9. Applications

In this section, we apply Corollary 8.4 to the main results of Part I. In so doing, we have tried as far as possible to avoid invoking the completeness assumption on N as it is difficult to verify this condition in practice.

THEOREM α. *Let M be complex, N strongly negatively curved hermitian and* $\dim M = \dim N = n$. *Let $\mathfrak{F}_a$: $M \to N$ be a family of holomorphic mappings with these properties:*

(1) *At a fixed point m_0 of M, $|f^*\Omega/\mu| \geq a > 0$ for all $f \in \mathfrak{F}_a$, where Ω = volume element of N and μ is a fixed nonzero real covector at m_0 of degree $2n$.*

(2) *m_0 gets carried by each $f \in \mathfrak{F}_a$ into some fixed compact set K in N.*

Then there is a positive constant α such that each $f \in \mathfrak{F}_a$ possesses a univalent ball of radius α around $f(m_0)$.

THEOREM β. *Hypothesis as above, let M be hermitian also. Then there is a positive constant λ such that each $f \in \mathfrak{F}_a$ is biholomorphic on an open ball of radius λ around m_0.*

Proof of Theorems α and β. We first prove Theorem α. By Corollary 8.4, $\mathfrak{F}_a$ is equicontinuous. Let V be a relatively compact, open neighborhood of the compact K of (2) in N and let the distance from K to the complement of V be ε. Choose a neighborhood U of m_0 such that $m \in U$ implies $d(m_0, m) < \varepsilon$. Now if $\mathfrak{F}_a$ is considered as a subset of $\mathcal{A}(U, N)$, it is relatively compact in virtue of Lemma 1.2. Theorem A applies to conclude the proof.

For Theorem β, we also consider $\mathfrak{F}_a$ as a subset of $\mathcal{A}(U, N)$ and apply Theorem B. Q.E.D.

The following special case of Theorems C and C′ is almost a direct translation and needs no comment.

THEOREM γ. *Let M be a complex manifold which is either an open relatively compact submanifold of a strongly negatively curved hermitian manifold N, or a complete strongly negatively curved hermitian manifold itself. Then for a holomorphic f: $M \to M$ which leaves a point p fixed:*

(i) $|\det df_p| \leq 1$.

(ii) *If df_p: $M_p \to M_p$ is the identity linear map, then f is the identity mapping of $M \to M$.*

(iii) $|\det df_p| = 1$ *iff f is an automorphism of N.*

Here we interpose a remark about bounded domains in $\mathbf{C}^n$. Since every bounded domain can be enclosed in a large enough open ball in $\mathbf{C}^n$, and since each open ball can be equipped with the Bergman metric which is complete and has holomorphic curvature equal to -1, every bounded domain is strongly negatively curved in the induced metric. Then Corollary 8.4 implies Montel's theorem, and Theorem γ implies the H. Cartan–Caratheodory theorem without recourse to Montel's theorem. The same comment applies to theorem δ and H. Cartan's theorem.

THEOREM δ. *If a hermitian manifold is strongly negatively curved, then its automorphism group $\mathcal{H}(N)$ is a (not necessarily connected) Lie group and the isotropy group of $\mathcal{H}(N)$ at a point is a compact Lie group. If N is compact, then $\mathcal{H}(N)$ is finite.*

THEOREM ε. *Every holomorphic mapping from $\mathbf{C}^n$ into a strongly negatively curved hermitian manifold reduces to a constant.*

These are immediate from Theorems D, E and Corollary 5.2 of § 5. The following consequences of Theorem ε are of particular interest:

COROLLARY 9.1. *$\mathbf{C}^n$ cannot be equipped with a hermitian metric whose holomorphic curvature is bounded above by a negative constant $-k_0<0$.*

COROLLARY 9.2. *A Riemann surface can be given a strongly negatively curved pseudo-hermitian metric iff it is not the plane, the punctured plane, the torus or the sphere.*

The last assertion makes use of the remarks at the end of § 8. The following is implied by Theorem F.

THEOREM ζ. *If a domain E in $\mathbf{C}^n$ ($E \neq \mathbf{C}^n$) can be given a complete and strongly negatively curved hermitian metric, then it is a domain of holomorphy.*

It is well known that every bounded symmetric domain (in particular the open balls) of $\mathbf{C}^n$ is strongly negatively curved and complete in its Bergman metric. (See, for instance, Helgason [15].) We have therefore retrieved the known result that every bounded symmetric domain ($\equiv$every hermitian symmetric space of noncompact type) is a Stein manifold. Cf. [28].

PART III

10. A more general setting

We shall reexamine a bit the proofs of Theorems A and B and, to a lesser extent, (1) of Theorem C in Part I. It is quite obvious that the only property of holomorphic mappings we made use of was the fact that $f_i \to f$ uniformly implies the convergence of the corresponding first partial derivatives (with the understanding that we operate within a compact coordinate neighborhood). From this, we concluded that $f_i^* \to f^*$ in the usual topology. If now instead of postulating that $\mathcal{J}_a$ be a family of holomorphic mapping, we *require* that $\mathcal{J}_a$ has the property that a sequence $\{f_i\} \subseteq \mathcal{J}_a$ converges iff their corresponding first partial derivatives also converge (uniformly on compact sets), then clearly Theorems A, B, C(1) would extend to this class of mappings. To formalize this, we have to introduce the *weak* (*or coarse*) C^r-*topology* $(0 \leqslant r \leqslant \infty)$ in the space of differentiable mappings between real differentiable manifolds, (see Munkres [26], p. 25 ff.).

First of all, suppose $U \subseteq \mathbf{R}^p$, $V \subseteq \mathbf{R}^q$ are open sets and $f\colon U \to V$ is of class C^r, $0 \leqslant r < \infty$. Note that we leave out the case $r = \infty$ for the moment. Let C be a compact subset of U, and let O be an open neighborhood of $f(C)$ in V. Then define

$$W(f, C, \varepsilon) = \{g\colon U \to V \text{ is a } C^r\text{-mapping such that } g(C) \subseteq 0 \text{ and } \max_{\substack{p \in C \\ |\alpha| \leqslant r}} \| (D_\alpha f - D_\alpha g)(p) \| < \varepsilon\},$$

where we have used the notation of multi-indices: $\alpha = (\alpha_1, \ldots, \alpha_p)$ with each α_i being a non-negative integer, $|\alpha| \equiv \sum_i \alpha_i$, and

$$D_\alpha f = \left(\frac{\partial^{|\alpha|} f_1}{\partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}}, \ldots, \frac{\partial^{|\alpha|} f_q}{\partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}} \right),$$

where $f = (f_1, \ldots, f_q)$ as usual. $\| \ \|$ denotes the euclidean norm of $\mathbf{R}^q$. The weak C^r-topology of $\mathcal{C}^r(U, V)$ (where $r < \infty$, and $\mathcal{C}^r(U, V)$ denotes the set of C^r mappings from U to V) is by definition the collection of open sets obtained by arbitrary unions of finite intersections of such $W(f, C, \varepsilon)$.

Suppose $f\colon M \to N$ is a C^r mapping between differentiable manifolds. If C is a compact subset of a coordinate neighborhood in M such that $f(C)$ also lies in a coordinate neighborhood in N, then again $W(f, C, \varepsilon)$ makes sense via the coordinate systems.

Definition 10.1. The *weak* C^r-*topology* $(r < \infty)$ for the set of all C^r mappings $\mathcal{C}^r(M, N)$ between differentiable manifolds M and N is the collection of open sets generated by arbitrary unions of finite intersections of all such $W(f, C, \varepsilon)$, where C and $f(C)$ both lie in some coordinate neighborhood. The *weak* C^∞-*topology* for the set of C^∞ mappings $\mathcal{C}^\infty(M, N)$ is the union of all C^r-topologies on $\mathcal{C}^\infty(M, N)$.

Note as usual that the weak C^r-topology is independent of the particular coordinate systems chosen for C and $f(C)$. Also, the weak C^0-topology clearly coincides with the compact-open topology, i.e. $\mathcal{C}^0(M, N) = \mathcal{C}(M, N)$ in the notation of Parts I and II. Of course, the beautiful thing about holomorphic mappings (and one that is so heavily exploited in this paper) is that on $\mathcal{A}(M, N)$ (M, N are now complex manifolds), the compact-open topology coincides with the weak C^∞-topology. We shall see that this property is shared by a wider class of functions, Lemma 11.3. At any rate, making use of the observation made above, we can now state the generalizations of Theorems A, B and C (1).

LEMMA 10.1. *Let M, N be differentiable manifolds of the same real dimension n, with N riemannian. Let $\mathfrak{F}_a \subseteq C^1(M, N)$ be relatively compact in the weak C^1-topology. Suppose each $f \in \mathfrak{F}_a$ has the property that $|f^*\Omega/\mu| \geqslant a > 0$ where Ω is the volume element of the riemannian metric in N and μ is a fixed nonzero covector of degree n at a fixed point $p_0 \in M$. Then there exists a fixed positive constant α such that each $f \in \mathfrak{F}_a$ possesses a univalent ball of radius α around $f(p_0)$.*

Here, by analogy with § 2, a *univalent ball* for f is an open ball in the image of f onto which f maps an open set one–to–one and with everywhere nonsingular differential.

LEMMA 10.2. *Let $\mathfrak{F}_a$: $M \to N$ be as above, and let M be riemannian also. Then each $f \in \mathfrak{F}_a$ possesses an open ball of radius λ around p_0 ($\lambda > 0$, independent of f) on which f is one–one and has nowhere singular differential.*

LEMMA 10.3. *Let f: $N \to N$ be C^1 and let f leave a point p fixed. If the sequence of iterates $\{f^i\}$ is relatively compact in the weak C^1-topology of $C^1(N, N)$, then $|\det df_p| \leqslant 1$.*

The following special case of Lemma 10.1 will be of particular importance and so we single it out as

COROLLARY 10.4. *Let B^n be the open unit ball in $\mathbf{R}^n$. If $\mathfrak{F}_a \subseteq C^1(B^n, \mathbf{R}^n)$ is a relatively compact family in the weak C^1-topology and the Jacobian determinant of each $f \in \mathfrak{F}_a$ is equal to a fixed positive constant a at the origin O of B^n, then there exists a positive constant α such that each f has a univalent ball of radius α around $f(O)$.*

11. The general Bloch Theorem

In this section, we continue to consider differentiable mappings between real differentiable manifolds. The main application we have in mind is of course for holomorphic mappings; see Appendix. First of all, we wish to define quasi-conformality of such C^∞ mappings; to make life simple we do it only for f: $U \to \mathbf{R}^n$ where U is any open subset of $\mathbf{R}^n$. It will be understood once and for all that *any open subset of* $\mathbf{R}^n$ *will be automatically endowed with the flat metric* if not otherwise specified, and that tangent spaces at all points of $\mathbf{R}^n$ are identified with $\mathbf{R}^n$ itself. These enter into the following discussion.

Definition 11.1. A subset $\mathfrak{F}$ of $C^1(U, \mathbf{R}^n)$ ($U \subseteq \mathbf{R}^n$) is called a *K-quasiconformal family* iff the following holds: for any point $p \in U$ and any $f \in \mathfrak{F}$, if S_p denotes the unit sphere in the tangent space U_p, the ratio of the length of the maximal axis of the hyperellipsoid $df_p(S_p)$ to the length of its minimal axis is bounded by K.

We need an equivalent analytic formulation of this geometric concept. Denote the canonical coordinate functions on U and $\mathbf{R}^n$ by $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y)_n$ respectively, and let $f = (f_1, \ldots, f_n) \equiv (y_1 \circ f, \ldots, y_n \circ f)$ as usual. Now in self-explanatory notation, if $t = \sum_{i=1}^{i=n} a_i \dfrac{\partial}{\partial x_i}$ is a vector in U_p, then

$$df_p(t) = \sum_j \left(\sum_i a_i \frac{\partial f_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

Denoting the inner product of $\mathbf{R}^n$ by $\langle \, , \rangle$ we see that

$$\| df_p(t) \|^2 \equiv \langle df_p(t), df_p(t) \rangle = \sum_{i,j} a_i a_j \left(\sum_k \frac{\partial f_k}{\partial x_i} \frac{\partial f_k}{\partial x_j} \right).$$

This defines a quadratic form $Q(a)$ on U_p by

$$Q(a) = \| df_p(t) \|^2 = {}^t a G a,$$

where $a = [a_1 \ldots a_n]$ is a row matrix and ${}^t a$ is its transpose. G is the $n \times n$ matrix with $G_{ij} = \sum_k \dfrac{\partial f_k}{\partial x_i} \dfrac{\partial f_k}{\partial x_j}$. We are interested in the maximum and minimum of $Q(a)$ as a varies through the unit sphere $\sum_i a_i^2 = 1$. Now the extreme (or, critical) values of $Q(a)$ for $\sum_i a_i^2 = 1$ are exactly the eigenvalues of the symmetric, positive semidefinite matrix G, as is well-known, (see, for instance, Halmos (14], p. 181). The following algebraic lemma is then relevant; the proof is a simple exercise which can be left out.

LEMMA 11.1. *Let S be a symmetric positive semi-definite matrix with eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. Then $\lambda_n / \lambda_1 \leqslant K'$ implies* trace $S \leqslant nK'$ (det $S)^{1/n}$. *Conversely, if* trace $S \leqslant K''$ (det $S)^{1/n}$, *then $\lambda_n / \lambda_1 \leqslant (K'')^n$.*

Apply this lemma to our situation: if Jf denotes the Jacobian determinant of f, then obviously $\det G = (\det Jf)^2$. The requirement that $\dfrac{\max_{\|t\|=1} \| df_p(t) \|}{\min_{\|t\|=1} \| df_p(t) \|} \leqslant K$, is by the previous remark equivalent to $(\lambda_n / \lambda_1)^{\frac{1}{2}} \leqslant K$, where λ_n, λ_1 are respectively the maximum and minimum eigenvalues of G at p. We therefore have this fact.

LEMMA 11.2. *$\mathfrak{F} \subseteq C^1(U, R^n)$ is a K-quasiconformal family iff there is a constant K_0 so that*

$$\sum_{i,j}\left(\frac{\partial f_i}{\partial x_j}\right)^2 \leqslant K_0 (Jf)^{2/n}. \tag{11.1}$$

Note that from a purely analytic point of view, this lemma says that K-quasiconformality of $\mathcal{F}$ is essentially the assertion that the *reverse* Hadamard inequality for the Jacobian matrix of each $f \in \mathcal{F}$ should hold everywhere with the same constant. We rephrase the above slightly. Consider the following mappings

$$\frac{\partial f}{\partial x_i} \equiv \left(\frac{\partial f_1}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i}\right), \quad i = 1, \ldots, n.$$

Then (11.1) is clearly equivalent to: there exists a constant K_1 such that in U,

$$\left\|\frac{\partial f}{\partial x_i}\right\| \leqslant K_1 |Jf|^{1/n} \tag{11.2}$$

for all $i = 1, \ldots, n$ and for all $f \in \mathcal{F}$.

For the statement of the general Bloch theorem, we have to consider linear systems of partial differential equations (PDE) defined in a $U \subseteq \mathbf{R}^n$, i.e. expressions of the form $\sum_{j=1}^{j=n} l_{ij}(x, D) f_j = g_i$, $i = 1, \ldots, n$, where each $l_{ij}(x, D)$ is a linear partial differential operator defined in U and f_j, g_i, are distributions in the sense of L. Schwartz. Usually, we rewrite this as follows.

$$\text{Let } f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}, \quad \text{and let } \underline{L}(x, D) = \begin{bmatrix} l_{11}(x, D) \ldots l_{1n}(x, D) \\ \vdots \qquad\qquad \vdots \\ l_{n1}(x, D) \ldots l_{nn}(x, D) \end{bmatrix}.$$

Then the above becomes

$$\underline{L}(x, D) f = g \tag{11.3}$$

Recall that $\underline{L}(x, D)$ is called *hypoelliptic* iff f is C^∞ wherever g is in (11.3). See Hörmander [16]. Note that in the event g is C^∞ on a $U \subseteq \mathbf{R}^n$, we may regard the set of all solutions of (11.3) as C^∞ mappings from U into $\mathbf{R}^n$ in the obvious way. The most common hypoelliptic systems are of course the elliptic ones with C^∞ coefficients. The fundamental fact we need about this class of operators is the following weakened version of a result due to Malgrange ([24], p. 331, Proposition 2).

LEMMA 11.3 (Malgrange lemma). *Suppose $\underline{L}(x, D)$ is hypoelliptic. Then in the space of solutions $\mathcal{S}$ of $\underline{L}(x, D) f = 0$, the compact open topology and the weak C^∞-topology coincide when $\mathcal{S}$ is viewed as a subset of $\mathbf{C}^\infty(U, \mathbf{R}^n)$.*

From the point of view of this lemma, the fact that the compact open topology and the weak C^∞-topology on holomorphic mappings coincide can be readily explained by noting that the latter are exactly the solutions of the strongly elliptic system of Cauchy–Riemann equations. In what follows, we only have occasion to consider a very special class of *PDE*.

Definition 11.3. $\underline{L}(x, D)$ defined on $U \subseteq \mathbf{R}^n$ is said to be *homogeneous* (*of order m*) iff either $l_{ij}(x, D) \equiv 0$ on U or else $l_{ij}(x, D)$ consists of derivatives of order exactly m throughout U.

THEOREM 11.4 (The general Bloch theorem): *Let $\mathfrak{F}$: $\bar{B}^n \to \mathbf{R}^n$ be a family of C^∞ mappings, where $\bar{B}^n$ denotes the closed unit ball $\{\sum_i x_i^2 \leqslant 1\}$. Suppose $\mathfrak{F}$ satisfies the three conditions:*

(a) *$\mathfrak{F}$ is a K-quasiconformal family. Equivalently, there exists a constant K_1 so that (11.2) holds in $\bar{B}^n$; i.e. $\left\|\dfrac{\partial f}{\partial x_i}\right\| \leqslant K_1 |Jf|^{1/n}$ $i = 1, \ldots, n$.*

(b) *Every $f \in \mathfrak{F}$ is a solution of a fixed linear hypoelliptic homogeneous system of PDE: $\underline{L}(x, D) f = 0$ on $\bar{B}^n$.*

(c) *For all $f \in \mathfrak{F}$, $|Jf(O)| = 1$, where O denotes the origin of $\bar{B}^n$.*

Then there is a universal constant $\beta \equiv \beta(\underline{L}(x, D), n, K) > 0$ such that each $f \in \mathfrak{F}$ possesses a univalent ball of radius β.

Some comments before we begin the proof. The result of Bochner [5] alluded to in the introduction corresponds to the case

$$\underline{L}(x, D) = \begin{bmatrix} \Delta & & 0 \\ & \ddots & \\ 0 & & \Delta \end{bmatrix},$$

where Δ is the Laplacian of $\mathbf{R}^n$. This special case of course already implies the theorem of Bloch for quasiconformal holomorphic mappings in n dimensions. Next, S. Takahashi (Ann. of Math. (2) 53, 1951, 464–471) has observed that the condition (a) of quasiconformality in Bochner's theorem can be replaced by the following slightly weaker requirement:

$$\max_{\|p\| \leqslant r} \sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2 (p) \leqslant \max_{\|p\| \leqslant r} (Jf(p))^{2/n} \tag{11.4}$$

for all $r \in [0, 1]$. It will be readily seen that the proof of Theorem 11.4 as given here is also valid if (a) is relaxed to (11.4). We have not done that explicitly because we feel that K-quasiconformality has such a simple geometric meaning that foregoing this extra bit of generality is justified. Perhaps an interpretation of (11.4) is not out of place here. In the presence of condition (c), $\mathfrak{F}$ fulfills (11.4) if $\mathfrak{F}$ has uniformly bounded first derivatives and $\mathfrak{F}$ is K^*-quasiconformal is a neighborhood of the boundary of $\bar{B}^n$.

Now, a word about the connection of this theorem with the *pseudo-analytic functions* of Lipman Bers [4]. We shall therefore consider the special case $\mathfrak{F}$: $\bar{B}^2 \to \mathbf{R}^2$. Let

$$Q(x, D) = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & -\alpha_{11}(x_1, x_2)\dfrac{\partial}{\partial x_1} - \alpha_{12}(x_1, x_2)\dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_2} & -\alpha_{21}(x_1, x_2)\dfrac{\partial}{\partial x_1} - \alpha_{22}(x_1, x_2)\dfrac{\partial}{\partial x_2} \end{bmatrix} \tag{11.5}$$

where $\alpha_{11}, \ldots, \alpha_{22}$ are C^∞ functions defined on the closed unit disc $\bar{B}^2$. $Q(x, D)$ is clearly homogeneous of order one. We require $Q(x, D)$ to be *uniformly* (*or strongly*) *elliptic*; in other words,

$$0 < \alpha_{12}, \quad \text{and} \quad 0 < \frac{(\alpha_{12} + \alpha_{21})^2}{4\alpha_{12}\alpha_{21} - (\alpha_{11} + \alpha_{22})^2} < K \tag{11.6}$$

throughout $\bar{B}^2$, and K is some constant. If $\alpha_{11} = -\alpha_{22}$ and $\alpha_{12} = \alpha_{21} > 0$, then clearly $Q(x, D)$ is uniformly elliptic in $\bar{B}^2$. In this case, the solutions $\omega = \varphi + \sqrt{-1}\,\psi$ of $Q(x, D)\begin{bmatrix}\psi\\ \varphi\end{bmatrix} = 0$ are exactly Bers' *pseudo-analytic functions of the second kind.* On the other hand, it can be shown that the uniform ellipticity of $Q(x, D)$ implies that it can be reduced to the normal form with $\alpha_{11} = \alpha_{22}$, $\alpha_{12} = \alpha_{21} > 0$ as above. See Bers [3]. Now, it is quite easy to verify that the family of all solutions of $Q(x, D)\begin{bmatrix}f_1\\ f_2\end{bmatrix} = 0$ is K_0-quasiconformal, where K_0 is dependent only on K of (11.6), That is, if $f = (f_1, f_2)$, and $Q(x, D)\begin{bmatrix}f_1\\ f_2\end{bmatrix} = 0$, then f: $\bar{B}^2 \to \mathbf{R}^2$ satisfies $\sum_{ij}\left(\dfrac{\partial f_i}{\partial x_j}\right)^2 \leqslant K_0\,|Jf|$. Thus Theorem 11.4 implies:

COROLLARY 11.5. *If $Q(x, D)$ of (11.5) is uniformly elliptic on $\bar{B}^2$ and $\alpha_{11}, \ldots, \alpha_{22}$ are C^∞ there, then for the family of solutions $f = (f_1, f_2)$ of $Q(x, D)\begin{bmatrix}f_1\\ f_2\end{bmatrix} = 0$ with $|Jf(0)| = 1$, there is a constant $\beta > 0$ such that each such f possesses a univalent ball of radius β. In other words, the classical theorem of Bloch holds also for pseudoanalytic functions of the second kind whose defining PDE has C^∞ coefficients in $\bar{B}^2$.*

Remark. It is presumably all right to reduce the C^∞ assumption on the coefficients of $Q(x, D)$ to C^1. One simply has to modify Malgrange's lemma accordingly.

Proof of Theorem 11.4. We briefly outline the argument. The main point is to reduce the theorem to Corollary 10.4. However, we cannot hope to prove that $\mathfrak{F}$ is itself relatively compact in the weak C^1-topology. Instead, take $b \in B^n$ and consider a linear map $\psi: \bar{B}^n \to \bar{B}^n$ with $\psi(p) = b + \frac{1}{2}(1 - \|b\|)\,p$. Then define $F = f \circ \psi$. Thus, ψ maps onto a closed ball of radius $\frac{1}{2}(1 - \|b\|)$ around b and F is f restricted to this ball. (The $\frac{1}{2}$ is purely a safety factor.) Let $\mathfrak{F}^{\#}$ denote the family of all C^∞ mappings $F: \bar{B}^n \to \mathbf{R}^n$ so obtained; clearly a univalent ball for F is automatically one for f. With the help of hypotheses (a)–(c), we shall be able to locate a suitable b for each f so that $\mathfrak{F}^{\#}$ is weak-C^1-relatively-compact, and so Corollary 10.4 applies to conclude the proof. The strategic location of such a b depends on a trick going back to Edmund Landau (Hille [17], p. 386–7) which consists of considering the function $C(s)$ to be defined presently.

The formal proof now begins. Let $f \in \mathfrak{F}$ and let $M(s) = \max\limits_{\|p\| \leqslant s} |Jf(p)|^{1/n}$. Define

$$C(s) = sM(1-s).$$

Now (c) implies that $C(0) = 0$, $C(1) = 1$. Let r be the *smallest* value of s for which $C(s) = 1$. Clearly $0 < r \leqslant 1$ and:

$$M(1-r) = \frac{1}{r}. \tag{11.7}$$

$$r' < r \text{ implies } M(1-r') < \frac{1}{r'}. \tag{11.8}$$

Let b be any point inside the ball of radius $(1-r)$ such that $|Jf(b)|^{1/n} = M(1-r)$. Define then $F: \bar{B}^n \to \mathbf{R}^n$ by:

$$F(p) = f\left(b + \frac{r}{2}\,p\right) - f(b).$$

The summand "$-f(b)$" is merely to insure that

$$F(0) = 0. \tag{11.9}$$

We have yet to show F is well-defined. This is so because

$$\left\| b + \frac{r}{2}\,p \right\| \leqslant \|b\| + \frac{r}{2}\,\|p\| \leqslant (1-r) + \frac{r}{2} = 1 - \frac{r}{2},$$

so that $\left\| b + \frac{r}{2}\,p \right\| < 1$. Hence $f\left(b + \frac{r}{2}\,p\right)$ makes sense, and so does F. Furthermore,

$$|JF(0)| = \left(\frac{r}{2}\right)^n |Jf(b)| = \left(\frac{r}{2}\right)^n (M(1-r))^n = \left(\frac{1}{2}\right)^n,$$

where the last step is by virtue of the definition of b and (11.7). Therefore, for every such F,

$$|JF(0)| = \left(\frac{1}{2}\right)^n. \tag{11.10}$$

Now, for any $p \in \bar{B}^n$ and for $i = 1, \ldots, n$,

$$\left\|\frac{\partial F}{\partial x_i}(p)\right\| = \frac{r}{2} \cdot \left\|\frac{\partial f}{\partial x_i}\left(b + \frac{r}{2}p\right)\right\| \leq K_1 \cdot \frac{r}{2} \cdot \left|Jf\left(b + \frac{r}{2}p\right)\right|^{1/n} \quad \text{by (a)}$$

$$\leq K_1 \cdot \frac{r}{2} \cdot M\left(1 - \frac{r}{2}\right) \quad \text{because } \left\|\alpha + \frac{r}{2}p\right\| < 1 - \frac{r}{2}.$$

So by (11.8)

$$\left\|\frac{\partial F}{\partial x_i}(p)\right\| \leq K_1.$$

This inequality together with the mean value theorem for several variables very easily leads to this fact: if p, q are any two points in $\bar{B}^n$, then $\|F(p) - F(q)\| \leq nK_1\|p - q\|$. In particular, $\|F(p)\| = \|F(p) - F(0)\| \leq nK_1\|p\| \leq K_1$, where we have used (11.9). Thus, these two statements imply that, if $\mathfrak{F}^\#: \bar{B}^n \to \mathbf{R}^n$ is the family of C^∞ mappings consisting of all such F's, $\mathfrak{F}^\#$ is a uniformly bounded and equicontinuous family. By Ascoli's theorem (Lemma 1.2), $\mathfrak{F}^\#$ is relatively compact in the compact-open topology.

Here, assumption (b) enters. $\mathfrak{F}$ consists of solutions of the hypoelliptic system $\underline{L}(x, D)f = 0$, and $\underline{L}(x, D)$ is homogeneous of order m, say. Then,

$$(\underline{L}(x, D)F)(p) = \left(\frac{r}{2}\right)^m (\underline{L}(x, D)f)\left(b + \frac{r}{2}p\right) = 0$$

for all $p \in \bar{B}^n$. Hence $\mathfrak{F}^\#$ is also a family of solutions of $\underline{L}(x, D)F = 0$. By Lemma 11.3, $\mathfrak{F}^\#$ must then be relatively compact in the weak C^∞-topology. By (11.10), Corollary 10.4 can be applied and consequently, there is a $\beta > 0$ such that each $F \in \mathfrak{F}^\#$ has a univalent ball of radius β around $F(0)$. In the above notation, this means that each $f \in \mathfrak{F}$ has a univalent ball of radius β around $f(b)$ (although this b depends on f). The proof is thereby concluded.

Appendix

This appendix proves the Bloch theorem directly for a K-quasiconformal family of holomorphic mappings. With the exception of two casual references to the preceding sections, the presentation here is intended to be self-contained. The only tools we need are Montel's theorem (Lemma 1.4 of § 1) and the fact that uniform convergence of holomorphic functions entails uniform convergence of derivatives of all orders. These enter into the lemma below, which is in fact a restatement of (i) of Corollary 2.1. We shall operate directly with $\mathbf{C}^n$ and holomorphic functions, in contradistinction to § 11 where real variables were used.

Let $\bar{D}^n$ be the closed unit ball $\{\Sigma_\sigma |z^\sigma|^2 \leq 1\}$ of $\mathbf{C}^n$ and let $\mathfrak{F}: \bar{D}^n \to \mathbf{C}^n$ be a family of holomorphic mappings. We say $\mathfrak{F}$ is *K-quasiconformal* iff there exists a constant K so that, for each $f = (f_1, ..., f_n)$ of $\mathfrak{F}$, the following holds throughout $\bar{D}^n$,

$$\left\| \frac{\partial f}{\partial z^\sigma} \right\| \leq K |\det Jf|^{1/n} \tag{A.1}$$

for $\sigma = 1, \ldots, n$. Here, $\| \ \|$ denotes the usual norm of $\mathbf{C}^n$, $\frac{\partial f}{\partial z^\sigma} = \left(\frac{\partial f_1}{\partial z^\sigma}, \ldots, \frac{\partial f_n}{\partial z^\sigma} \right)$ and Jf is the complex Jacobian determinant $\left\{ \frac{\partial f_\sigma}{\partial z^\varrho} \right\}$ of f. Of course, (A.1) is automatically satisfied for the case $n = 1$ by letting $K \geq 1$.

(The reason for this name is given by the following fact which will not be needed in the sequel: Let $\bar{D}^n$ and $\mathbf{C}^n$ be both given the flat hermitian metric. Then $\mathfrak{F}$ is K quasiconformal iff there exists a constant K_0 such that for each $p \in \bar{D}^n$ and for each $f \in \mathfrak{F}$, the ratio of the lengths of the longest axis to the shortest axis of $df_p(S_p)$ is bounded by K_0 (df_p: $\bar{D}^n_p \to \mathbf{C}^n_p$, and S_p is the unit sphere in the tangent space $\bar{D}^n_p$.)

The proof is almost identical with that of Lemma 11.2. The obvious differences are: symmetric matrices are replaced by hermitian matrices, real tangent spaces are replaced by holomorphic tangent spaces, etc.)

The crucial fact needed is given by the following

LEMMA. *Let $\mathfrak{F}_a: \bar{D}^n \to C^n$ be a uniformly bounded family of holomorphic mappings with $|Jf(0)| = a > 0$ for any $f \in \mathfrak{F}_a$. Then there is an $\alpha > 0$ so that each $f \in \mathfrak{F}_a$ has a univalent ball of radius α around $f(0)$, i.e. a ball of radius α onto which f maps some open set biholomorphically.*

Outline of proof. By Montel's theorem, $\mathfrak{F}_a$ is relatively compact in the compact open topology. So if the lemma were false, there would exist a sequence $\{f_i\}$ in $\mathfrak{F}_a$ such that $f_i \to f$ (where $f: \bar{D}^n \to \mathbf{C}^n$) and the *maximal* univalent of f_i has radius $r_i \to 0$ as $i \to \infty$. Now, $|Jf(0)| = \lim_{i\to\infty} |Jf_i(0)| = a > 0$ and so f has a univalent ball of positive radius r about $f(0)$. Call the latter B. By shrinking B a little if necessary, we may assume f maps a compact set A containing the origin O biholomorphically onto $\bar{B}$. Thus $f_i \to f$ uniformly on A and for large i, $|Jf_i|$ is nowhere zero on A because $|Jf|$ is nowhere zero on A. From this one can deduce without much difficulty that for i sufficiently large, f_i possesses a univalent ball of radius at least $\frac{1}{4}r$, say, inside B. This contradicts $r_i \to 0$. (For full details, see the proof of Theorem A in § 2, starting with the third paragraph and replace "$f^*\Omega/\mu$" everywhere by "$|Jf(O)|$.") Q.E.D.

The following is the main result we are after.

BLOCH'S THEOREM. *Let $\mathfrak{J}$: $\bar{D}^n \to \mathbf{C}^n$ be a K-quasiconformal family of holomorphic mappings such that $|Jf(0)| = 1$ for all $f \in \mathfrak{J}$. Then there is a $\beta > 0$ such that every $f \in \mathfrak{J}$ possesses a univalent ball of radius β.*

Proof. We wish to apply the preceding lemma, but not directly to $\mathfrak{J}$ because $\mathfrak{J}$ is not known to be uniformly bounded. Instead, we consider for each $f \in \mathfrak{J}$ the restriction $F = f|B$, where B is some closed ball inside $\bar{D}^n$ with center b. If we can choose this b properly for each f, roughly speaking, the set of all such F will form a uniformly bounded family $\mathfrak{J}^\#$ to which the Lemma would be applicable. The initial section of this proof is concerned with this correct choice of b for each f.

So let $f \in \mathfrak{J}$ and define $M(s) = \max_{\|z\| \leqslant s} |Jf(z)|^{1/n}$. Let $C(s) = sM(1-s)$. By hypothesis, $C(0) = 0$ and $C(1) = 1$. Let r be the *smallest* value of s for which $C(s) = 1$. Clearly $0 < r \leqslant 1$ and r satisfies:

$$r' < r \text{ implies } M(1-r') < \frac{1}{r'} \tag{A.2}$$

$$M(1-r) = \frac{1}{r}. \tag{A.3}$$

Let b the point inside the ball of radius $(1-r)$ such that

$$|Jf(b)|^{1/n} = M(1-r) = \frac{1}{r}.$$

Define F: $\bar{D}^n \to \mathbf{C}^n$ by:

$$F(z) = f\left(b + \frac{r}{2} z\right) - f(b).$$

This is the restriction map of f referred to above; the subtraction of $f(b)$ from F is merely to normalize F so that

$$F(0) = 0. \tag{A.4}$$

We have yet to show F well-defined. Since $\left\| b + \frac{r}{2} z \right\| \leqslant \|b\| + \frac{r}{2} \|z\| \leqslant (1-r) + \frac{r}{2} = 1 - \frac{r}{2}$, $\left\| b + \frac{r}{2} z \right\| < 1$. In particular $f\left(b + \frac{r}{2} z\right)$ makes sense, and so does F. Now,

$$\begin{aligned}\left\|\frac{\partial F}{\partial z^\sigma}(z)\right\| &= \frac{r}{2}\cdot\left\|\frac{\partial f}{\partial z^\sigma}\left(b+\frac{r}{2}z\right)\right\| \\ &\leqslant \frac{r}{2}\cdot K\cdot\left|Jf\left(b+\frac{r}{2}b\right)\right|^{1/n} && \text{by (A.1)} \\ &\leqslant \frac{r}{2}\cdot K\cdot M\left(1-\frac{r}{2}\right) && \text{because } \left\|b+\frac{r}{2}z\right\| < 1-\frac{r}{2} \\ &< \frac{r}{2}\cdot K\cdot\left(\frac{2}{r}\right) && \text{by (A.2)} \\ &= K.\end{aligned}$$

Thus, throughout $\bar{D}^n$ and for $\sigma = 1, \ldots, n$, we have:

$$\left\|\frac{\partial F}{\partial z^\sigma}\right\| \leqslant K. \tag{A.5}$$

Together with (A.4) we shall deduce the uniform boundedness of all such F. First a remark about holomorphic functions of one variable. Let $g(z)$ be defined in a disc around z_0, then for each z' in the disc, we can find a z^* in the line segment joining z_0 and z' so that $|g(z') - g(z_0)| \leqslant |z' - z_0| \left|\frac{dg}{dz}(z^*)\right|$. This is a consequence of the Cauchy theorem because if we integrate along this line segment:

$$g(z') - g(z_0) = \int_{z_0}^{z'} \frac{dg}{dz}\, dz$$

so that $$|g(z') - g(z_0)| \leqslant \int_{z_0}^{z'} \left|\frac{dg}{dz}\right| |dz| = \left|\frac{dg}{dz}(z^*)\right| \int_{z_0}^{z'} |dz| = \left|\frac{dg}{dz}(z^*)\right| |z' - z_0|,$$

where we have used the mean value theorem for integrals to obtain z^*. Now, if $F(z^1, \ldots, z^n)$ is a holomorphic function of n variables, we may derive, by a standard procedure, the following from the preceding result: if F is defined in a ball around p_0, then for each p' in this ball, we can find n points $p_1, \ldots, p_n$ in this ball so that:

$$\|F(p') - F(p_0)\| \leqslant \left\{\left(\left\|\frac{\partial F}{\partial z_1}(p_1)\right\| + \ldots + \left\|\frac{\partial F}{\partial z^n}(p_n)\right\|\right)\cdot\|p' - p_0\|\right\}. \tag{A.6}$$

Returning to our original F, (A.4)–(A.6) imply that $\|F(z)\| = \|F(z) - F(0)\| \leqslant nK\|z\| \leqslant nK$. Thus if we denote by $\mathfrak{F}^{\#}$ the set of all such F so obtained from the

f of $\mathfrak{F}$, $\mathfrak{F}^{\#}$ is uniformly bounded. It is obvious from (A.3) that $|JF(0)| = \left(\frac{r}{2}\right)^n |Jf(b)|^n = \left(\frac{1}{2}\right)^n$. Thus the Lemma is applicable and each $F \in \mathfrak{F}^{\#}$ possesses a univalent ball of radius β around $F(0)$, where $\beta > 0$ is independent of F. This means that each f has a univalent ball of radius β around $f(b)$ (b depending on f). Q.E.D.

It is an easy consequence of the preceding theorem that if $\mathfrak{F}$ is a K-quasiconformal family of holomorphic mappings from the *entire* space $\mathbf{C}^n$ into $\mathbf{C}^n$, and if $|Jf(0)| = 1$ for each $f \in \mathfrak{F}$, then the radii of the univalent balls for each $f \in \mathfrak{F}$ are unbounded. We close this paper with the *conjecture* that, in this case, the assumption of K-quasiconformality is unnecessary.

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