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NORMAL FORM FOR SPATIAL DYNAMICS IN THE SWIFT-HOHENBERG EQUATION

John Burke and Edgar Knobloch

University of California
Department of Physics
Berkeley, CA 94720, USA

Abstract. The reversible Hopf bifurcation with 1:1 resonance holds the key to the presence of spatially localized steady states in many partial differential equations on the real line. Two different techniques for computing the normal form for this bifurcation are described and applied to the Swift-Hohenberg equation with cubic/quintic and quadratic/cubic nonlinearities.

1. **Introduction.** In recent years there has been a great deal of interest in the origin and properties of spatially localized structures in partial differential equations on the real line [2, 4, 11]. It is now understood that steady localized states can bifurcate from the trivial state, and that in bistable systems the branches of localized states that result often undergo homoclinic snaking as they approach a spatially periodic structure [8, 14]. The theory is most complete for reversible variational systems [8]. In this theory spatially localized states are viewed as homoclinic orbits to the trivial state. The presence of such states requires that the trivial state be hyperbolic; reversibility implies that for each unstable direction there is a stable direction, and is used to prove the presence of homoclinic orbits and their persistence with respect to parameters [8]. In variational systems the presence of a so-called Maxwell point [8] provides a simple and intuitive understanding of the observed homoclinic snaking, but the variational property is not otherwise required [4].

The equation that is most studied from this point of view is the generalized Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = ru - \left(\partial_x^2 + q_c^2\right)^2 u + f(u)\,, \qquad -\infty < x < \infty\,, \tag{1}$$

with either $f(u) = f_{35} \equiv b_3u^3 - b_5u^5$, or $f(u) = f_{23} \equiv c_2u^2 - c_3u^3$. Here $b_5, c_3 > 0$ provide large amplitude stabilization, while r is the control parameter. In both cases the time-independent version of this equation forms a fourth order reversible dynamical system in space: the equation is invariant under the spatial reflection $(x \to -x,\ u \to u)$; in the former it is in addition odd with respect to $u \to -u$. It is easy to check that for $r < 0$ the eigenvalues of the state $u_0 = 0$ are $\pm iq_c \pm (\sqrt{-r}/2q_c) + \mathcal{O}(r)$, while for $r > 0$ they are $\pm iq_c \pm i(\sqrt{r}/2q_c) + \mathcal{O}(r)$. Thus for $r < 0$ the eigenvalues form a quartet, and u_0 is hyperbolic with two stable eigenvalues and two unstable eigenvalues. In contrast for $r > 0$ all the eigenvalues lie on the imaginary axis and u_0 is not hyperbolic. At $r = 0$ there is a pair of imaginary

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eigenvalues $\pm iq_c$ of double multiplicity. The bifurcation at $r = 0$ is thus a Hopf bifurcation in a reversible system with 1:1 (spatial) resonance.

The normal form for this bifurcation is [9]:

$$A' = iq_c A + B + iA\,P\,(\mu; y, w) \tag{2a}$$

$$B' = iq_c B + iB\,P\,(\mu; y, w) + A\,Q\,(\mu; y, w)\ , \tag{2b}$$

where $y \equiv |A|^2$, $w \equiv \frac{i}{2}(A\bar{B} - \bar{A}B)$, the overbar refers to complex conjugation, and in the context of spatial dynamics the prime denotes differentiation with respect to x; μ is an unfolding parameter analogous to r. The functions A and B transform under spatial reflection as $(A, B) \to (\bar{A}, -\bar{B})$, and P and Q are polynomials with real coefficients, which we truncate to include only the first few terms:

$$\begin{aligned} P(\mu; y, w) &= p_1\mu + p_2 y + p_3 w + p_4 y^2 + p_5 wy + p_6 w^2 \\ Q(\mu; y, w) &= -q_1\mu + q_2 y + q_3 w + q_4 y^2 + q_5 wy + q_6 w^2\,. \end{aligned} \tag{3}$$

The 1:1 Hopf bifurcation from the trivial state $A = B = 0$ occurs at $\mu = 0$; this state is hyperbolic in the region $\mu < 0$ provided $q_1 > 0$. The fact that the normal form is completely integrable is of great assistance in its analysis [9]. One finds that there are two possible types of behavior depending on the sign of q_4 at $q_2 = 0$. When $q_4 < 0$ homoclinic solutions are present in the whole half-space $\mu < 0$. In contrast, when $q_4 > 0$ homoclinic solutions are present only for $q_2 < 0$ and then only in $\mu_D < \mu < 0$, where $\mu_D = -3q_2^2/16q_1q_4$. At μ_D homoclinic solutions terminate in a heteroclinic connection between the flat state ($A = 0$) and a nontrivial state ($A \neq 0$) with the same 'energy'. To classify the solutions it is therefore particularly important to determine the normal form coefficients q_1, q_2 and q_4. For the Swift-Hohenberg equation with f_{23} an attempt to compute these coefficients was made by Woods [13, 14], but his results differ from those obtained below.

The purpose of this note is to describe two distinct methods to calculate the normal form coefficients $\{p_i, q_i\}$ for the reversible 1:1 Hopf bifurcation in the Swift-Hohenberg equation (1), and to correct the results in [8, 13, 14]. The methods follow [7] but need to be taken to higher order thereby revealing certain subtleties which we believe are of general interest; however, when correctly applied both lead to the same expressions. In the following we present detailed expressions only for the f_{35} case; the case f_{23} is more involved and only the results are stated.

2. **Normal form theory.**

We begin by writing the time-independent Eq. (1) as a system of four first order equations in $u_n \equiv \partial_x^n u$ for $n = 0, 1, 2, 3$:

$$\frac{d}{dx}U = \mathcal{L}U + \mathcal{N}\,. \tag{4}$$

At the bifurcation point $r = 0$ these matrices are:

$$U = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q_c^4 & 0 & -2q_c^2 & 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_3 u_0^3 - b_5 u_0^5 \end{pmatrix}. \tag{5}$$

In general a coordinate transformation from the U basis to the $\{A, B, \bar{A}, \bar{B}\}$ basis can be written as

$$\begin{aligned} A = A(U) &= \sum_{n_i \geq 0} A_{n_0 n_1 n_2 n_3} u_0^{n_0} u_1^{n_1} u_2^{n_2} u_3^{n_3} \\ B = B(U) &= \sum_{n_i \geq 0} B_{n_0 n_1 n_2 n_3} u_0^{n_0} u_1^{n_1} u_2^{n_2} u_3^{n_3} \,. \end{aligned} \tag{6}$$

The coefficients of $\bar{A}$ and $\bar{B}$ are determined by the requirement of reversibility imposed on A and B:

$$\begin{aligned} \bar{A} = \bar{A}(U) &= \sum_{n_i \geq 0} A_{n_0 n_1 n_2 n_3} u_0^{n_0} (-u_1)^{n_1} u_2^{n_2} (-u_3)^{n_3} \\ \bar{B} = \bar{B}(U) &= \sum_{n_i \geq 0} B_{n_0 n_1 n_2 n_3} u_0^{n_0} (-u_1)^{n_1} u_2^{n_2} (-u_3)^{n_3} \,. \end{aligned} \tag{7}$$

We define the 'order' of any term $u_0^{n_0} u_1^{n_1} u_2^{n_2} u_3^{n_3}$ to be the sum $n = n_0 + n_1 + n_2 + n_3$ of the exponents in the monomial. At fixed order n there are $N = (n+3)(n+2)(n+1)/6$ unique monomials, so at this order each of the two expansions in (6) contains N terms. We refer to the sets of order n transformation coefficients as $A_{(n)}$ and $B_{(n)}$.

We wish to determine the coordinate transformation such that the *dynamics* described by Eqs. (4)-(5) are identical to those described by Eqs. (2)-(3). To do so we apply the chain rule to A' and B' in Eqs. (2):

$$\frac{\partial A}{\partial u_0}\frac{du_0}{dx} + \frac{\partial A}{\partial u_1}\frac{du_1}{dx} + \frac{\partial A}{\partial u_2}\frac{du_2}{dx} + \frac{\partial A}{\partial u_3}\frac{du_3}{dx} = iq_c A + B + iAP\,(y,w) \tag{8a}$$

$$\frac{\partial B}{\partial u_0}\frac{du_0}{dx} + \frac{\partial B}{\partial u_1}\frac{du_1}{dx} + \frac{\partial B}{\partial u_2}\frac{du_2}{dx} + \frac{\partial B}{\partial u_3}\frac{du_3}{dx} = iq_c B + iBP(y,w) + AQ(y,w)\,. \tag{8b}$$

Having already imposed reversibility in (7) the two additional equations that could be derived from $\bar{A}'$ and $\bar{B}'$ are redundant. Replacing du_i/dx with the known values from Eq. (4) and rearranging terms gives

$$\begin{aligned} &\left[\frac{\partial A}{\partial u_0}u_1 + \frac{\partial A}{\partial u_1}u_2 + \frac{\partial A}{\partial u_2}u_3 + \frac{\partial A}{\partial u_3}\left(-q_c^4 u_0 - 2q_c^2 u_2\right) - iq_c A - B\right] \\ &\qquad + \Big[-iAP\,(y,w)\Big] + \left[\frac{\partial A}{\partial u_3}\left(b_3 u_0^3 - b_5 u_0^5\right)\right] = 0 \end{aligned} \tag{9a}$$

$$\begin{aligned} &\left[\frac{\partial B}{\partial u_0}u_1 + \frac{\partial B}{\partial u_1}u_2 + \frac{\partial B}{\partial u_2}u_3 + \frac{\partial B}{\partial u_3}\left(-q_c^4 u_0 - 2q_c^2 u_2\right) - iq_c B\right] \\ &\qquad + \Big[-iBP\,(y,w) - AQ\,(y,w)\Big] + \left[\frac{\partial B}{\partial u_3}\left(b_3 u_0^3 - b_5 u^5\right)\right] = 0\,. \end{aligned} \tag{9b}$$

On replacing $A, B, \bar{A}, \bar{B}$ with their expansions (6)-(7), each of the above equations becomes an infinite sum of terms in U. As each of the four components of U can be treated as an independent function for the purpose of matching terms, these two equations represent an infinite system of equations which determine the transformation *and* the normal form coefficients.

The terms in Eqs. (9) are bracketed in groups based on their dependence on the unknown coefficients. At order n, the first bracketed term only depends linearly on the unknown transformation coefficients in $A_{(n)}, B_{(n)}$. The third bracketed term depends linearly on coefficients from $A_{(n-2)}, B_{(n-2)}$ (multiplied by b_3) and $A_{(n-4)}, B_{(n-4)}$ (multiplied by b_5). The second bracketed term in Eqs. (9)

contains the normal form coefficients. We denote by $\phi_{(n)}$ the normal form coefficients that first appear in this term at order n. So $\phi_{(n)}$ is empty if $n = 1$ or n is even, and the first two nonvanishing sets are $\phi_{(3)} = \{p_2, q_2, p_3, q_3\}$ and $\phi_{(5)} = \{p_4, q_4, p_5, q_5, p_6, q_6\}$. The second bracketed term therefore depends linearly on the normal form coefficients in $\phi_{(m)}$ for all $m \leq n$, and nonlinearly on the transformation coefficients in $A_{(m)}, B_{(m)}$ for $m < n$.

Matching all terms at a fixed order n produces N equations from each of the two equations in (9). If the coefficients for all orders $m < n$ are known, then the only unknowns in this system of $2N$ equations are the $2N$ coefficients $A_{(n)}, B_{(n)}$, as well as the normal form coefficients $\phi_{(n)}$. All dependence on these unknowns is linear so the system of $2N$ equations can easily be formulated as a matrix equation. As these $2N$ equations are not necessarily linearly independent, and the number of unknowns may exceed the number of equations, it follows that the solution is not necessarily unique.

Owing to the $u \to -u$ symmetry of Eq. (1), the even-order terms in Eqs. (9) are determined by setting all $A_{(n)}$ and $B_{(n)}$ to zero for all even values of n. The transformation (6) therefore involves only odd-order terms. At order $n = 1$ the system of $2N = 8$ equations arising from the matching requirement (9) for the unknowns $A_{(1)} = \{A_{1000}, A_{0100}, A_{0010}, A_{0001}\}$, $B_{(1)} = \{B_{1000}, B_{0100}, B_{0010}, B_{0001}\}$ has a two-parameter family of solutions. The order $n = 1$ transformation in Eq. (6) is

$$A = \xi_1 \left(\frac{1}{2} u_0 - \frac{3i}{4q_c} u_1 - \frac{i}{4q_c^3} u_3 \right) + \xi_2 \left(\frac{1}{4} u_0 - \frac{i}{4q_c} u_1 + \frac{1}{4q_c^2} u_2 - \frac{i}{4q_c^3} u_3 \right) \tag{10a}$$

$$B = \xi_1 \left(-\frac{iq_c}{4} u_0 - \frac{1}{4} u_1 - \frac{i}{4q_c} u_2 - \frac{1}{4q_c^2} u_3 \right), \tag{10b}$$

where $\xi_{1,2} \in \mathbb{C}$, $\xi_1 \neq 0$. This linear transformation puts the linear part of the Swift-Hohenberg system as written in Eq. (4) into the Jordan normal form in Eqs. (2). In the following we choose $\xi_1 = 1$ and $\xi_2 = 0$, so

$$A = \frac{1}{2} u_0 - \frac{3i}{4q_c} u_1 - \frac{i}{4q_c^3} u_3, \quad B = -\frac{iq_c}{4} u_0 - \frac{1}{4} u_1 - \frac{i}{4q_c} u_2 - \frac{1}{4q_c^2} u_3 \,. \tag{11}$$

At order $n = 3$ the system of $2N = 40$ equations from Eqs. (9) for the unknowns $A_{(3)}, B_{(3)}, \phi_{(3)}$ has a four parameter family of solutions, parametrized by $\xi_3, \xi_4, \xi_5, \xi_6$. For a particular choice of these parameters

$$\begin{aligned} A =& \frac{1}{2} u_0 - \frac{3i}{4q_c} u_1 - \frac{i}{4q_c^3} u_3 + \frac{b_3}{1024} \Bigg[- \frac{425}{4q_c^4} u_0^3 - \frac{177i}{q_c^5} u_0^2 u_1 - \frac{246}{q_c^6} u_0 u_1^2 \\ & - \frac{114i}{q_c^7} u_1^3 + \frac{147}{4q_c^6} u_0^2 u_2 + \frac{327i}{2q_c^7} u_0 u_1 u_2 - \frac{825}{4q_c^8} u_1^2 u_2 + \frac{405}{4q_c^8} u_0 u_2^2 \\ & + \frac{15i}{q_c^9} u_1 u_2^2 + \frac{129}{4q_c^{10}} u_2^3 - \frac{165i}{2q_c^7} u_0^2 u_3 - \frac{183}{4q_c^8} u_0 u_1 u_3 - \frac{60i}{q_c^9} u_1^2 u_3 \\ & + \frac{60i}{q_c^9} u_0 u_2 u_3 - \frac{591}{4q_c^{10}} u_1 u_2 u_3 + \frac{3i}{q_c^{11}} u_2^2 u_3 + \frac{267}{4q_c^{10}} u_0 u_3^2 - \frac{9}{q_c^{12}} u_2 u_3^2 \Bigg] + \ldots \end{aligned} \tag{12a}$$

$$B = -\frac{iq_c}{4}u_0 - \frac{1}{4}u_1 - \frac{i}{4q_c}u_2 - \frac{1}{4q_c^2}u_3 + \frac{b_3}{1024}\left[\frac{19i}{4q_c^3}u_0^3 - \frac{282}{q_c^4}u_0^2u_1 \right. \tag{12b}$$
$$+ \frac{234i}{q_c^5}u_0u_1^2 - \frac{117}{q_c^6}u_1^3 - \frac{195i}{4q_c^5}u_0^2u_2 - \frac{63}{4q_c^6}u_0u_1u_2 + \frac{591i}{4q_c^7}u_1^2u_2$$
$$- \frac{243i}{4q_c^7}u_0u_2^2 - \frac{3}{4q_c^8}u_1u_2^2 - \frac{93i}{4q_c^9}u_2^3 - \frac{573}{4q_c^6}u_0^2u_3 + \frac{609i}{4q_c^7}u_0u_1u_3 - \frac{69}{q_c^8}u_1^2u_3$$
$$\left. - \frac{129}{4q_c^8}u_0u_2u_3 + \frac{471i}{4q_c^9}u_1u_2u_3 - \frac{12}{q_c^{10}}u_2^2u_3 + \frac{45i}{4q_c^9}u_0u_3^2 + \frac{15i}{q_c^{11}}u_2u_3^2\right] + \dots .$$

However, the solution for $\phi_{(3)}$,

$$\phi_{(3)} = \{p_2, q_2, p_3, q_3\} = \left\{-\frac{9b_3}{16q_c^3}, -\frac{3b_3}{4q_c^2}, 0, -\frac{3b_3}{8q_c^3}\right\}, \tag{13}$$

is independent of the choice of $\xi_3, \xi_4, \xi_5, \xi_6$, although it does depend on the choice of ξ_1, ξ_2. More generally,

$$\phi_{(3)} = \frac{1}{\xi_1^2}\left\{-\frac{9b_3}{16q_c^3}, -\frac{3b_3}{4q_c^2}, 0, -\frac{3b_3}{8q_c^3}\right\} + \frac{\xi_2}{\xi_1^3}\left\{-\frac{3b_3}{8q_c^3}, 0, -\frac{b_3}{q_c^4}, -\frac{9b_3}{4q_c^3}\right\}. \tag{14}$$

In fact ξ_1 corresponds to an overall amplitude scaling of the fields [akin to the transformation $(A, B) \to \xi_1(A, B)$] while ξ_2 is related to an additional symmetry of the normal form [6].

At order $n = 5$ the system of $2N = 112$ equations from (9) for the unknowns $A_{(5)}, B_{(5)}, \phi_{(5)}$ has a six parameter family of solutions. While the fifth order transformation is too lengthy to give here, the solution for the normal form coefficients is

$$\phi_{(5)} = \{p_4, q_4, p_5, q_5, p_6, q_6\} = \left\{-\frac{1243b_3^2}{1024q_c^7} + \frac{25b_5}{12q_c^3}, -\frac{177b_3^2}{128q_c^6} + \frac{5b_5}{2q_c^2}, \right. \tag{15}$$
$$\left. \frac{2601b_3^2}{5120q_c^8}, \frac{89b_3^2}{128q_c^7} + \frac{5b_5}{3q_c^3}, \frac{1107b_3^2}{2048q_c^9}, -\frac{4941b_3^2}{5120q_c^8}\right\}.$$

Having fixed the transformation coefficients at lower orders the coefficients in $\phi_{(5)}$ are once again uniquely determined. Note that q_4 at $q_2 = 0$ (i.e., $b_3 = 0$) is always equal to $5b_5/2q_c^2$, provided only that $\xi_1 = 1$.

Hamiltonian normal form theory that takes advantage of the Hamiltonian structure (in x) of the time-independent Eq. (1) leads to identical results.

3. **Normal form via asymptotics.**

In this section we describe a method that appears to be simpler, and that can be carried out at least to low orders without computer algebra. The method is based on reducing both Eq. (1) and Eqs. (2) to the *same* Ginzburg-Landau equation.

3.1. **Scaling of the Swift-Hohenberg equation.**

We introduce a small parameter $\epsilon \ll 1$ and define a large spatial scale $X = \epsilon x$, and look for stationary solutions of Eq. (1) of the form

$$u_s(x) = \epsilon u_1(x, X) + \epsilon^2 u_2(x, X) + \dots . \tag{16}$$

With this scaling Eq. (1) at $r = 0$ becomes

$$
\begin{aligned}
&(\partial_x^2 + q_c^2)^2 \left(\epsilon u_1 + \epsilon^2 u_2 + \ldots\right) \qquad (17)\\
&= - \Big[4\epsilon \partial_{xX}(\partial_x^2 + q_c^2) + 4\epsilon^2 \partial_{xxXX} + 2\epsilon^2 \partial_{XX}(\partial_x^2 + q_c^2) + 4\epsilon^3 \partial_{xXXX} \\
&\quad + \epsilon^4 \partial_X^4\Big] \left(\epsilon u_1 + \epsilon^2 u_2 + \ldots\right) + b_3 \left(\epsilon u_1 + \ldots\right)^3 - b_5 \left(\epsilon u_1 + \ldots\right)^5 .
\end{aligned}
$$

Matching terms order by order in ϵ gives:

$$\mathcal{O}(\epsilon): \quad \left(\partial_x^2 + q_c^2\right)^2 u_1 = 0 \qquad (18a)$$

$$\mathcal{O}(\epsilon^2): \quad \left(\partial_x^2 + q_c^2\right)^2 u_2 = -4\partial_{xX} \left(\partial_x^2 + q_c^2\right) u_1 \qquad (18b)$$

$$
\begin{aligned}
\mathcal{O}(\epsilon^3): \quad \left(\partial_x^2 + q_c^2\right)^2 u_3 &= -4\partial_{xX} \left(\partial_x^2 + q_c^2\right) u_2 - 4\partial_{xxXX} u_1 \qquad (18c)\\
&\quad - 2\partial_{XX} \left(\partial_x^2 + q_c^2\right) u_1 + b_3 u_1^3
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}(\epsilon^4): \quad \left(\partial_x^2 + q_c^2\right)^2 u_4 &= -4\partial_{xX} \left(\partial_x^2 + q_c^2\right) u_3 - 4\partial_{xxXX} u_2 \qquad (18d)\\
&\quad - 2\partial_{XX} \left(\partial_x^2 + q_c^2\right) u_2 - 4\partial_{xXXX} u_1 + 3b_3 u_1^2 u_2
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}(\epsilon^5): \quad \left(\partial_x^2 + q_c^2\right)^2 u_5 &= -4\partial_{xX} \left(\partial_x^2 + q_c^2\right) u_4 - 4\partial_{xxXX} u_3 \qquad (18e)\\
&\quad - 2\partial_{XX} \left(\partial_x^2 + q_c^2\right) u_3 - 4\partial_{xXXX} u_2 - \partial_X^4 u_1 \\
&\quad + 3b_3 \left(u_1 u_2^2 + u_1^2 u_3\right) - b_5 u_1^5 .
\end{aligned}
$$

The $\mathcal{O}(\epsilon, \epsilon^2)$ equations are solved by

$$u_1(x, X) = A_1(X) e^{iq_c x} + c.c., \qquad u_2(x, X) = A_2(X) e^{iq_c x} + c.c., \qquad (19)$$

where $A_{1,2}(X)$ are as yet undetermined and *c.c.* denotes a complex conjugate.

With the Ansatz

$$u_3(x, X) = A_3(X) e^{iq_c x} + C_3(X) e^{3iq_c x} + c.c. \qquad (20)$$

the $\mathcal{O}(\epsilon^3)$ equation in (18) can be solved by matching terms with common $e^{niq_c x}$ dependence. The $n = 1$ terms give

$$4q_c^2 A_1'' = -3b_3 A_1 |A_1|^2 , \qquad (21)$$

while the $n = 3$ terms give

$$C_3 = \frac{b_3}{64 q_c^4} A_1^3 . \qquad (22)$$

The Ansatz

$$u_4(x, X) = A_4(X) e^{iq_c x} + C_4(X) e^{3iq_c x} + c.c. \qquad (23)$$

in the $\mathcal{O}(\epsilon^4)$ equation likewise leads to

$$4q_c^2 A_2'' = 4iq_c A_1''' - 3b_3 \left(2|A_1|^2 A_2 + A_1^2 \bar{A}_2\right) , \qquad (24)$$

obtained from the $n = 1$ terms; the $n = 3$ terms, which determine C_4 in terms of $A_{1,2}$, are not needed in what follows. Finally, the $\mathcal{O}(\epsilon^5)$ equation with the Ansatz

$$u_5(x, X) = A_5(X) e^{iq_c x} + C_5(X) e^{3iq_c x} + E_5(X) e^{5iq_c x} + c.c. \qquad (25)$$

yields

$$
\begin{aligned}
4q_c^2 A_3'' &= 4iq_c A_2''' + A_1'''' \qquad (26)\\
&\quad - 3b_3 \left(2A_1 |A_2|^2 + \bar{A}_1 A_2^2 + 2|A_1|^2 A_3 + A_1^2 \bar{A}_3 + \bar{A}_1^2 C_3\right) + 10 b_5 A_1 |A_1|^4 ,
\end{aligned}
$$

again from the $n = 1$ terms. Eliminating C_3 we have

$$4q_c^2 A_3'' = 4iq_c A_2''' + A_1'''' - 3b_3 \left(2A_1|A_2|^2 + \bar{A}_1 A_2^2 + 2|A_1|^2 A_3 + A_1^2 \bar{A}_3\right) + \left(-\frac{3b_3^2}{64q_c^4} + 10b_5\right) A_1|A_1|^4 . \tag{27}$$

It is easy to show that Eqs. (21), (24) and (27) can now be assembled into a *single* equation for the amplitude $Z(X, \epsilon) \equiv A_1(X) + \epsilon A_2(X) + \epsilon^2 A_3(X) + \dots$ of $e^{iq_c x}$:

$$4q_c^2 Z'' = -3b_3 Z|Z|^2 + 4iq_c \epsilon Z''' + \epsilon^2 \left[Z'''' + \left(-\frac{3b_3^2}{64q_c^4} + 10b_5\right) Z|Z|^4\right] + \mathcal{O}(\epsilon^3). \tag{28}$$

This represents the Ginzburg-Landau approximation to the Swift-Hohenberg equation at $r = 0$. If $r \neq 0$ is introduced into this equation (see below) and all derivatives with respect to X are set to zero one recovers the correct equation for steady spatially periodic states with wavenumber q_c.

In the following it is necessary to eliminate all second and higher derivatives from the right side of this equation by iteratively replacing Z'' with its power series expansion. Eliminating Z''' and Z'''' in this way, Eq. (28) becomes

$$4q_c^2 Z'' = -3b_3 Z|Z|^2 - \frac{3i\epsilon b_3}{q_c}(2Z'|Z|^2 + Z^2 \bar{Z}') + \epsilon^2 \left[\frac{9b_3}{2q_c^2}\left(2ZZ'\bar{Z}' + (Z')^2 \bar{Z}\right) + \left(-\frac{327b_3^2}{64q_c^4} + 10b_5\right) Z|Z|^4\right] + \mathcal{O}(\epsilon^3) . \tag{29}$$

Note that setting derivatives to zero now gives incorrect values of the coefficients required to describe spatially periodic states.

3.2. **Scaling of the normal form equation.** To match the scaling of the previous section to the normal form equation (2) we set $\mu = 0$ and write $(A, B) = (\epsilon\tilde{A}(X), \epsilon^2 \tilde{B}(X))e^{iq_c x}$, where $X = \epsilon x$ as before. Dropping the tildes, the polynomials P and Q in Eqs. (3) become

$$\begin{aligned} P &= \epsilon^2 p_2 |A|^2 + \epsilon^3 p_3 \frac{i}{2}(A\bar{B} - \bar{A}B) + \mathcal{O}(\epsilon^4) \\ Q &= \epsilon^2 q_2 |A|^2 + \epsilon^3 q_3 \frac{i}{2}(A\bar{B} - \bar{A}B) + \epsilon^4 q_4 |A|^4 + \mathcal{O}(\epsilon^5) , \end{aligned} \tag{30}$$

and the normal form (2) takes the form

$$\epsilon^2 A' = \epsilon^2 B + i\epsilon A\left[\epsilon^2 p_2|A|^2 + \epsilon^3 p_3 \frac{i}{2}(A\bar{B} - \bar{A}B)\right] + \mathcal{O}(\epsilon^5) \tag{31a}$$

$$\epsilon^3 B' = i\epsilon^2 B\left[\epsilon^2 p_2|A|^2 + \epsilon^3 p_3 \frac{i}{2}(A\bar{B} - \bar{A}B)\right] + \epsilon A\left[\epsilon^2 q_2|A|^2 + \epsilon^3 q_3 \frac{i}{2}(A\bar{B} - \bar{A}B) + \epsilon^4 q_4|A|^4\right] + \mathcal{O}(\epsilon^6) . \tag{31b}$$

Equation (31a) yields a power series expansion for B in terms of A:

$$\begin{aligned} B &= A' - i\epsilon p_2 A|A|^2 + \epsilon^2 \frac{p_3}{2} A(A\bar{B} - \bar{A}B) + \mathcal{O}(\epsilon^3) \\ &= A' - i\epsilon p_2 A|A|^2 + \epsilon^2 \frac{p_3}{2} A(A\bar{A}' - \bar{A}A') + \mathcal{O}(\epsilon^3). \end{aligned} \tag{32}$$

Differentiation with respect to X together with the leading order approximation from Eq. (31b) yields

$$B' = A'' - i\epsilon p_2(2AA'\bar{A} + A^2\bar{A}') + \epsilon^2\frac{p_3}{2}(AA'\bar{A}' - (A')^2\bar{A}) + \mathcal{O}(\epsilon^3)\,. \tag{33}$$

Moreover, Eq. (31b) gives an alternative power series expansion for B':

$$\begin{aligned} B' &= q_2A|A|^2 + i\epsilon p_2|A|^2B + i\epsilon\frac{q_3}{2}A(A\bar{B} - \bar{A}B) \\ &\quad + \epsilon^2 q_4 A|A|^4 - \epsilon^2\frac{p_3}{2}(A\bar{B} - \bar{A}B)B + \mathcal{O}(\epsilon^3) \\ &= q_2A|A|^2 + i\epsilon p_2|A|^2(A' - i\epsilon p_2A|A|^2) + i\epsilon\frac{q_3}{2}A^2(\bar{A}' + i\epsilon p_2\bar{A}|A|^2) \\ &\quad - i\epsilon\frac{q_3}{2}|A|^2(A' - i\epsilon p_2A|A|^2) + \epsilon^2 q_4A|A|^4 - \epsilon^2\frac{p_3}{2}(A\bar{A}' - \bar{A}A')A' + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{34}$$

Equating Eqs. (33) and (34) now leads to

$$\begin{aligned} A'' = q_2A|A|^2 &+ i\epsilon\left[\left(3p_2 - \frac{1}{2}q_3\right)A'|A|^2 + \left(p_2 + \frac{1}{2}q_3\right)A^2\bar{A}'\right] \\ &+ \epsilon^2\left[p_3((A')^2\bar{A} - AA'\bar{A}') + (q_4 - q_3p_2 + p_2^2)A|A|^4\right] + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{35}$$

Notice that neither Eq. (29) nor Eq. (35) contain second or higher derivatives on the right side. A comparison of these two equations permits us to identify the coefficients in the normal form (35) in terms of the parameters of the Swift-Hohenberg equation. To match the two expressions one final transformation is required: $Z = A + \epsilon^2\rho A|A|^2 + \mathcal{O}(\epsilon^4)$, where ρ is to be determined. This transformation converts Eq. (29) into

$$\begin{aligned} 4q_c^2A'' =& -3b_3A|A|^2 + i\epsilon\left[-\frac{3b_3}{q_c}(2A'|A|^2 + A^2\bar{A}')\right] \\ &+ \epsilon^2\Bigg[\left(\frac{9b_3}{q_c^2} - 16q_c^2\rho\right)AA'\bar{A}' + \left(\frac{9b_3}{2q_c^2} - 8q_c^2\rho\right)(A')^2\bar{A} \\ &+ \left(-\frac{327b_3^2}{64q_c^4} + 10b_5\right)A|A|^4\Bigg] + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{36}$$

Matching terms with Eq. (35) gives:

$$\begin{gathered} \rho = \frac{9b_3}{16q_c^4}\,, \quad p_2 = -\frac{9b_3}{16q_c^3}\,, \quad q_2 = -\frac{3b_3}{4q_c^2}\,, \quad p_3 = 0\,, \quad q_3 = -\frac{3b_3}{8q_c^3}\,, \\ q_4 = -\frac{177b_3^2}{128q_c^6} + \frac{5b_5}{2q_c^2}\,, \end{gathered} \tag{37}$$

in agreement with the results (13) and (15) in the previous section. Matching at higher orders in ϵ involves higher order terms omitted from the expansion (3) of the polynomials P, Q.

As in the previous section these normal form coefficients are not unique. For example, to reproduce the $\xi_{1,2}$-dependent versions of q_2, p_2, q_3 in Eq. (14) we would need to take the following relation between Z and A:

$$Z = {\xi_1}^{-1}\,(A - i\epsilon\xi_2A') + \mathcal{O}(\epsilon^2)\,. \tag{38}$$

4. **Unfolding.** Recall that Eq. (1) is hyperbolic in $r < 0$, while Eqs. (2) is hyperbolic in $\mu < 0$. Although it is possible to stretch μ relative to r we choose for the sake of simplicity to identify μ with r. In principle it is possible to perform the calculations of sections 2 and 3 away from the bifurcation point and in this way determine p_1, q_1. In section 2 this involves the standard unfolding procedure: extend Eq. (4) to a five-dimensional system in $U = (u_0, u_1, u_2, u_3, r)^T$, where r obeys the equation $\frac{d}{dx}r = 0$. In section 3 this involves choosing the correct scaling for the bifurcation parameter: $r \equiv \mu = -\epsilon^2\mu_2$, where $\mu_2 = \mathcal{O}(1)$. In both cases one must be careful to include in Eqs. (3) all the necessary coefficients p_i, q_i as determined by the desired order in ϵ.

As written, Eqs. (3) include the leading order behavior in μ only. The required coefficients p_1 and q_1 are most easily computed directly from the eigenvalues of the linearization of the two systems. Assuming a solution of the form $u \sim \varepsilon e^{\lambda x}$, Eq. (1) gives

$$0 = (r - {q_c}^4) - 2q_c^2\lambda^2 - \lambda^4 \,. \tag{39}$$

Near the bifurcation point the eigenvalues are therefore

$$\lambda_{\mathrm{SH}} = \pm\left(iq_c \pm \frac{1}{2q_c}\sqrt{-r} + \frac{i}{8q_c^3}(-r)\right) + \mathcal{O}(r^{3/2}) \,. \tag{40}$$

The corresponding eigenvalues from the linearization of (2) are

$$\lambda_{\mathrm{NF}} = iq_c \pm \sqrt{q_1}\sqrt{-\mu} - ip_1(-\mu) + \mathcal{O}(\mu^{3/2}) \,, \tag{41}$$

along with two complex conjugate eigenvalues from the $(\bar{A}, \bar{B})$ system. It follows that

$$p_1 = -\frac{1}{8q_c^3} \,, \qquad q_1 = \frac{1}{4q_c^2} \,. \tag{42}$$

5. **Quadratic/cubic Swift-Hohenberg equation.** If instead we choose the nonlinearity f_{23} in Eq. (1) the $u \to -u$ symmetry is lost, and even order terms are now present at various stages in the computation. Nevertheless, the two procedures still match. The result is [2]

$$\begin{gathered} p_1 = -\frac{1}{8q_c^3} \,, \quad p_2 = \frac{9c_3}{16q_c^3} - \frac{187c_2^2}{216q_c^7} \,, \quad p_3 = -\frac{8c_2^2}{9q_c^8} \,, \\ q_1 = \frac{1}{4q_c^2} \,, \quad q_2 = \frac{3c_3}{4q_c^2} - \frac{19c_2^2}{18q_c^6} \,, \quad q_3 = \frac{3c_3}{8q_c^3} - \frac{41c_2^2}{108q_c^7} \,, \\ q_4 = -\frac{177c_3^2}{128q_c^6} + \frac{5089c_2^2c_3}{288q_c^{10}} - \frac{78131c_2^4}{7776q_c^{14}} \,, \end{gathered} \tag{43}$$

while the coefficient ρ in section 3 is

$$\rho = -\frac{9c_3}{16q_c^4} + \frac{355c_2^2}{216q_c^8} \,. \tag{44}$$

These results were independently confirmed by D. Lloyd (private communication); no choice of the parameters $\xi_{1,2}$ reproduces the results in [8, 13, 14].

6. **Conclusion.** In this note we have presented two quite distinct methods for calculating the coefficients in the normal form for the 1:1 reversible Hopf bifurcation, and applied the results to two different cases of the Swift-Hohenberg equation, both with competing nonlinearities. In both cases our results for these coefficients correspond to a particular solution from a multi-parameter family parametrized by $\xi_{1,2}$, etc. This non-uniqueness is ultimately a consequence of normal form symmetry [6]. From these results we can obtain the quantity q_4 evaluated at $q_2 = 0$ that distinguishes between different regimes in the unfolding [9]; this quantity is independent of ξ_{2-6}. In addition μ_D, the location of the heteroclinic connection, is independent of all the transformation parameters ξ.

Our results apply to the primary bifurcation to steady spatially localized states in the Swift-Hohenberg equation on the real line. Within normal form theory these localized states come in a one parameter family, with an arbitrary spatial phase $\phi \in S^1$ that determines the phase of the periodic inclusion $\exp iq_c x$ within the (slowly varying) envelope $Z(X)$. This property of the solutions is a consequence of normal form symmetry. If this symmetry is broken (beyond all orders in ϵ) an even number of solutions is in general selected. For f_{23} this number is 2; for f_{35} it is 4. Consequently 2 (resp. 4) branches of localized solutions bifurcate toward $r < 0$ simultaneously with the subcritical bifurcation to spatially periodic states. The beyond-all-orders terms also lead to the intersection of the two-dimensional stable and unstable manifolds of the origin, and hence to (homoclinic) snaking [5, 8, 10, 14]. It turns out that the 2 (resp. 4) snaking branches are connected by segments of asymmetric states (i.e., states that do not lie in the fixed point subspace of the reversibility symmetry), resulting in a characteristic 'snakes-and-ladders' structure [2, 3].

In the context of Eq. (1) the stability of these states can be computed [2, 3]. Other physical systems lead to versions of this equation where the time dependence takes a different form (as in water wave problems [1, 15]), or time-independent versions where the issue of stability does not arise (as in buckling problems [8, 12]).

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E-mail address: knobloch@px1.Berkeley.EDU