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Normal Gorenstein Surfaces with Ample Anti-canonical Divisor

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Introduction

Let k be an algebraically closed field of arbitrary characteristic and X be a normal projective surface over k. Since every normal surface is Cohen-Macaulay, there exists the dualizing sheaf ω_X on X and X is a Gorenstein surface if and only if ω_X is invertible. (We use the words "dualizing sheaf" and "canonical divisor" interchangeably.) A typical example of a Gorenstein surface is an effective divisor on a non-singular threefold. In this article, we determine the structure of normal Gorenstein surfaces with ample anti-canonical sheaf ω_X^{-1}. If X is non-singular, such a surface is called a Del Pezzo surface and the structure of Del Pezzo surfaces is fairly well-known.

Our first result is that such a surface is either rational or a cone over an elliptic curve and that the singularities on such a surface are rational double points or the unique simple elliptic singular point, according as X is rational or is a cone over an elliptic curve. These results are proved in § 2. In the case k is the complex number field, the same result is obtained by Brenton in [2] using topological properties of ruled surfaces. Our proof uses the theory of resolution of normal surface singularities. An advantage of our proof is that we can treat the problem independent of the characteristic of the base field k.

In § 3, we will study more closely the case that X is rational. We will show that the minimal resolution $\tilde{X}$ of such a surface X can be obtained from $\boldsymbol{P}^2$ by blowing up the points in "almost general position", except for the case when X is a normal quadric surface in $\boldsymbol{P}^3$. Such surfaces are the ones studied by Demazure in [4].

In § 4, we will study the anti-canonical model and the configuration of the singular points on such a surface. If we put $d=\deg X=\omega_X\cdot\omega_X$, we can show that X is a subvariety of degree d in $\boldsymbol{P}^d$ if $d\geqq 3$, a hyper-

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surface of degree 4 in the weighted projective space $P(1, 1, 1, 2)$ if $d=2$ and a hypersurface of degree 6 in $P(1, 1, 2, 3)$ if $d=1$. As for the singular points on X, if $d\geqq 3$, all the possible configurations of the singular points on X are known (cf. [11]). We treat the case $d=2$ in this article and show that the configuration of the singular points on X is a proper subgraph of the extended Dynkin diagram $(\tilde{E}_7)$ except for three cases. Also, we will show that the maximal number of singular points on X is 6 if $d=1$ or 2.

The authors would like to express their gratitude to Professor M. Miyanishi for showing them the article [4].

In this article, we use the following notations. $P(n_0, \cdots, n_d)=\mathrm{Proj}(k[T_0, \cdots, T_d])$ is the weighted projective space (where we put $\deg(T_i)=n_i$ for $i=0, 1, \cdots, d$). We write $P^d=P(1, \cdots, 1)$ as usual. F_n is the P^1-bundle over P^1 defined by $F_n=P(\mathcal{O}\oplus\mathcal{O}(n))$. $h^i(X, *)$ means $\dim_k H^i(X, *)$.

§ 1. The exceptional divisors and the geometric genus of a singularity.

Let X be a normal Gorenstein surface and x be a singular point of X. We denote by $\pi: \tilde{X}\to X$ a minimal resolution of x and by A the exceptional set $\pi^{-1}(x)$. Let $A=\bigcup_{i=1}^n A_i$ be the irreducible decomposition of A. Recall that the fundamental cycle Z_0 of A is the minimum among the effective cycles Z with $Z\cdot A_i\leqq 0$ for every $i=1, \cdots, n$. We know that the support of Z_0 coincides with A.

Lemma 1.1. *The canonical divisor $K_{\tilde{X}}$ of $\tilde{X}$ is linearly equivalent to $\pi^*(\omega_X)-\sum_{i=1}^n r_iA_i$ with $r_i\geqq 0$ for all $i=1, \cdots, n$. Moreover, $\sum_{i=1}^n r_iA_i\geqq Z_0$ unless $r_i=0$ for all $i=1, \cdots, n$.*

Proof. This follows from the facts that $p_a(A_i)=(A_i^2+K_{\tilde{X}}\cdot A_i)/2+1\geqq 0$ and that $A_i^2\leqq -2$ if $p_a(A_i)=0$.

Let us denote the divisor $\sum_{i=1}^n r_iA_i$ by W. The following propositions are the key results for this article.

Proposition 1.2. ([1], [7]). *Uuder the assumptions as above, the following conditions are equivalent.*

(i) $(R^1\pi_*\mathcal{O}_{\tilde{X}})_x=0$ (*resp.* $\dim_k(R^1\pi_*(\mathcal{O}_{\tilde{X}}))_x=1$).

(ii) *x is a rational double point (resp. a minimally elliptic singular point).*

(iii) $W=0$ (*resp.* $W=Z_0$).

In [7], Laufer proved the above result in the case $k=C$. But it is easy to check that the proposition holds over an algebraically closed field of arbitrary characteristic, using the method of "computation sequence" defined in [7]. Recall that the geometric genus $p_g(\mathcal{O}_x)$ is defined by $p_g(\mathcal{O}_x)=\dim_k R^1\pi_*(\mathcal{O}_{\tilde{X}})_x$.

PROPOSITION 1.3. *If* $p_g(\mathcal{O}_x)\geqq 2$, *then* $p_g(\mathcal{O}_x)\geqq p_a(Z_0)+1$.

PROOF. As $(R^1\pi_*(\mathcal{O}_{\tilde{X}})_x)^\wedge \cong \varprojlim H^1(Z, \mathcal{O}_Z)$ by the comparison theorem and as the mappings of the projective system of the right hand side are surjective, we have only to show that $\dim_k H^1(W, \mathcal{O}_W)\geqq p_a(Z_0)+1$. By Proposition 1.2 and by our assumption, we have $W>Z_0$. If we put $Y=W-Z_0$, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_Y(-Z_0) \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

and the exact sequence of cohomology groups

$$H^0(\mathcal{O}_W) \longrightarrow H^0(\mathcal{O}_{Z_0}) \longrightarrow H^1(\mathcal{O}_Y(-Z_0)) \longrightarrow H^1(\mathcal{O}_W) \longrightarrow H^1(\mathcal{O}_{Z_0}) \longrightarrow 0 .$$

We have $\dim_k H^0(\mathcal{O}_{Z_0})=1$ ([7], (2.6)). So, the mapping $H^0(\mathcal{O}_W)\to H^0(\mathcal{O}_{Z_0})$ is surjective. On the other hand, by the adjunction formula, we have $\omega_Y=\mathcal{O}_Y(-W+Y)=\mathcal{O}_Y(-Z_0)$. So, $\dim_k H^1(\mathcal{O}_Y(-Z_0))=\dim_k H^0(\mathcal{O}_Y)>0$. Thus we have $\dim_k H^1(\mathcal{O}_W)=\dim_k H^1(\mathcal{O}_{Z_0})+\dim_k H^1(\mathcal{O}_Y(-Z_0))\geqq p_a(Z_0)+1$.

§ 2. Structure of the surfaces.

In this section, X is a normal Gorenstein surface with ample anti-canonical sheaf ω_X^{-1} and $\pi\colon \tilde{X}\to X$ is a minimal resolution of X. We call the self-intersection number $d=\omega_X\cdot\omega_X$ the degree of X (the intersection number is taken in the sense of [6]). Since ω_X^{-1} is ample, it is clear that $d\geqq 1$.

PROPOSITION 2.1. *X is birationally equivalent to a ruled surface.*

PROOF. By [10], it suffices to show that $H^0(mK_{\tilde{X}})=0$ for all $m\geqq 1$. Since X is normal, it follows from the isomorphism $\mathcal{O}_X\cong\pi_*(\mathcal{O}_{\tilde{X}})$ that $H^0(\omega_X^m)\cong H^0(\pi^*\omega_X^m)$ for all m. But in the notation of § 1, we have $h^0(\omega_X^m)=h^0(\pi^*\omega_X^m)=h^0(mK_{\tilde{X}}+mW)\geqq h^0(mK_{\tilde{X}})$ and $h^0(\omega_X^m)=0$ since ω_X^{-1} is ample.

THEOREM 2.2. *If X is a normal projective Gorenstein surface with ω_X^{-1} ample and if $\pi\colon\tilde{X}\to X$ is a minimal resolution of X, then one of the following two cases occurs.*

(1) *X is rational.*

(2) *$\tilde{X}$ is a $\boldsymbol{P}^1$-bundle over an elliptic curve C and $\tilde{X}\cong\boldsymbol{P}(\mathcal{O}_C\oplus\mathcal{L})$,*

where $\mathscr{L}$ is a line bundle on C with $\deg \mathscr{L}>0$. *X is obtained by contracting the minimal section of $\tilde{X}$.*

PROOF. We have the spectral sequence

$$E_2^{p,q}=H^p(X, R^q\pi_*(\mathscr{O}_{\tilde{X}})) \Longrightarrow H^{p+q}(X, \mathscr{O}_{\tilde{X}}) ,$$

which induces an exact sequence

$$(2.3) \qquad 0 \longrightarrow H^1(\mathscr{O}_X) \longrightarrow H^1(\mathscr{O}_{\tilde{X}}) \longrightarrow H^0(R^1\pi_*(\mathscr{O}_{\tilde{X}})) \longrightarrow H^2(\mathscr{O}_X) .$$

By Serre duality, $H^2(\mathscr{O}_X)=H^0(\omega_X)=0$ since ω_X^{-1} is ample. We put $q=h^1(\mathscr{O}_{\tilde{X}})$. We can divide the situation into three cases.

Case 1. $R^1\pi_*(\mathscr{O}_{\tilde{X}})=0$.
In this case, the singular points of X are rational double points. Hence $K_{\tilde{X}}\cong\pi^*(\omega_X)$. If X is obtained by n-times blowing up from a P^1-bundle, then $0<d=(\omega_X\cdot\omega_X)=K_{\tilde{X}}^2=8-8q-n$. Thus $q=0$ and X is a rational surface.

Case 2. $\dim_k H^0(R^1\pi_*(\mathscr{O}_{\tilde{X}}))=1$.
Let $f: \tilde{X}\to C$ be a fibre space structure over a nonsingular curve C with general fibre P^1. Note that the genus of C is q. From the exact sequence (2.3), we have $q\geqq 1$. By the assumption, there exists only one minimally elliptic singular point x on X. Let Z_0 be the fundamental cycle of $\pi^{-1}(x)$. Then, by Proposition 1.2, $K_{\tilde{X}}=\pi^*(\omega_X)-Z_0$, $p_a(Z_0)=1$ and therefore Z_0 is not contained in a fibre of f.

Claim. Z_0 is reduced and is a section of the morphism f.
To prove this, we need the following

LEMMA 2.4. *Under the assumption of Case 2, let B be an irreducible curve on $\tilde{X}$ which is a component of a fibre of f. If $B\cdot Z_0>0$ and $\pi(B)$ is not a point, then $B^2=0$.*

PROOF OF LEMMA. Since B is a rational curve, $K_{\tilde{X}}\cdot B+B^2=-2$. Hence $B^2=-2+\pi^*(\omega_X^{-1})\cdot B+Z_0\cdot B\geqq 0$. Thus $B^2=0$.

PROOF OF CLAIM. The cycle Z_0 does not contain any fibre component. For, if it did, there exists a component of the fibre which is not a component of Z_0 and meets Z_0. But this is impossible by Lemma 2.4. Let D be a general fibre of f. Then $K_{\tilde{X}}\cdot D+D^2=-2$. As $K_{\tilde{X}}=\pi^*(\omega_X)-Z_0$, $Z_0\cdot D=2+\pi^*(\omega_X)D\leqq 1$ and so $Z_0\cdot D=1$. This shows the Claim.

As $p_a(Z_0)=1$ and Z_0 is a section of f, we have shown $q=1$. Also, Lemma 2.4 shows that X is a P^1-bundle. So, there exists a locally free sheaf $\mathscr{E}$ of rank 2 on C such that $X=P(\mathscr{E})$. It remains to show that

$\mathscr{E}$ is decomposable. But if $\mathscr{E}$ is indecomposable, the self-intersection number of the minimal section is 0 or 1 by [18], which contradicts the fact $Z_0^2<0$. Note that if $\mathscr{E}=\mathscr{O}_C\oplus\mathscr{L}$ with $\deg(\mathscr{L})>0$, then $\deg(\mathscr{L})=$ $d=(\omega_X\cdot\omega_X)$.

Case 3. $\dim_k H^0(R^1\pi_*(\mathscr{O}_{\tilde{X}}))\geqq 2$.
We will show that there is no X belonging to this case. If there exists a point on X with $p_g(\mathscr{O}_x)\geqq 2$, let Z_0 be the fundamental cycle of $\pi^{-1}(x)$. Then $p_a(Z_0)>0$ by [1], Theorem 3. So Z_0 is not contained in a fiber of f. Then a component of Z_0 dominates the base curve C and we have $p_a(Z_0)\geqq p_a(C)$. But by Proposition 1.3, we have $p_g(\mathscr{O}_x)>p_a(Z_0)$ and this contradicts the fact $q\geqq\dim_k(H^0(R^1\pi_*(\mathscr{O}_{\tilde{X}}))$.

If all singularities of X are rational or minimally elliptic, let x_i ($i=1,\cdots,m$, $m\geqq 2$) be the set of elliptic singularities of X. If Z_0^i is the fundamental cycle of $\pi^{-1}(x_i)$, we have $K_{\tilde{X}}=\pi^*(\omega_X)-\sum_{i=1}^m Z_0^i$ by Proposition 1.2. As in the proof of Case 2, each Z_0^i is a section of f. Let D be a general fibre of f. Then $-2=D\cdot K_X=D\cdot\pi^*(\omega_X)-\sum_{i=1}^m D\cdot Z_0^i\leqq -1-m$, which is a contradiction.

COROLLARY 2.5. *If X is a normal projective Gorenstein surface with ω_X^{-1} ample, then $H^1(X,\mathscr{O}_X)=0$.*

PROOF. This follows immediately from Theorem 2.2 and (2.3).

REMARK. In the case $\operatorname{ch}(k)=0$, we can prove Theorem 2.2 without using Proposition 1.3. In fact, consider the spectral sequence

$$E_2^{p,q}=H^p(X,R^q\pi_*\pi^*\omega_X)\Longrightarrow H^{p+q}(X,\pi^*\omega_X)\ ,$$

which induces an exact sequence

$$(*)\qquad 0\longrightarrow H^1(\omega_X)\longrightarrow H^1(\pi^*\omega_X)\longrightarrow H^0(R^1\pi_*(\mathscr{O}_{\tilde{X}}))\longrightarrow H^2(\omega_X)\ .$$

By Serre duality, $H^2(\omega_X)\cong k$ and by [9] it follows that $H^1(\mathscr{O}_X)=H^1(\omega_X)=0$ and $H^1(\pi^*\omega_X)=0$. So $\dim_k H^0(R^1\pi_*(\mathscr{O}_{\tilde{X}}))$ equals to 0 or 1. Therefore, $q\leqq 1$ by the exact sequence (2.3).

§ 3. The rational case.

Let X, $\tilde{X}$, π be as in § 2. In this section, we treat the case when X is rational more closely. In this case, the singular points of X are rational double points. We will show that the smooth surfaces which can appear as $\tilde{X}$ are the ones studied by Demazure in [4].

3.1. Let $\Sigma=\{P_1,\cdots,P_r\}$ be a finite set of points on the projective plane P^2 (infinitely near points allowed) and assume that $|\Sigma|=r\leqq 8$. Denote by Σ_j the subset $\{P_1, \cdots, P_j\}$ $(1\leqq j\leqq r)$ and let $V(\Sigma_j)\to P^2$ be the blowing up of P^2 with center Σ_j. Then there exists a sequence of blowing-ups

$$V(\Sigma)=V(\Sigma_r)\longrightarrow V(\Sigma_{r-1})\longrightarrow\cdots\longrightarrow V(\Sigma_1)\longrightarrow P^2 .$$

Let E_j be the exceptional set of the j-th step ($E_j\subset V(\Sigma_j)$ and E_j is contracted to the point P_j of $V(\Sigma_{j-1})$).

DEFINITION 3.2. ([4]). The points of Σ are in general (resp. almost general) position if

(i) no three (resp. four) of them are on a line.

(ii) no six (resp. seven) of them are on a conic.

(iii) all the points are distinct (resp. for all j $(1\leqq j\leqq r-1)$, the point P_{j+1} on $V(\Sigma_j)$ does not lie on any proper transform $\hat{E}_i$ of E_i $(1\leqq i\leqq j)$ such that $\hat{E}_i^2=-2$).

(iv) when $|\Sigma|=8$, there exists no singular cubic which passes through all the points of Σ and has one of them as the singular point (no corresponding condition for almost general position).

It is easy to see that the points of Σ are in general position if and only if the anti-canonical divisor $-K_{V(\Sigma)}$ of $V(\Sigma)$ is ample. A non-singular projective surface V is called a Del Pezzo surface if $-K_V$ is ample. It is known that a Del Pezzo surface is isomorphic to P^2 or $P^1\times P^1$ or $V(\Sigma)$ as above, where the points of Σ are in general position.

In [4], Demazure showed the following

THEOREM 3.3 ([4], III, Th. 1). *Let $V=V(\Sigma)$ as above with $|\Sigma|=r\leqq 8$. Then the following conditions are equivalent.*

(a) *The points of Σ are in almost general position.*

(b) *The anti-canonical system $|-K_V|$ of V has no fixed components.*

(c) *The anti-canonical system $|-K_V|$ of V contains a non-singular elliptic curve.*

(d) *$H^1(V, \mathcal{O}_V(nK_V))=0$ for all $n\in Z$.*

(e) *$D\cdot K_V\leqq 0$ for every effective divisor D on V.*

(f) *For an irreducible curve D on V, either $D\cdot K_V<0$ or $D\cdot K_V=0$ and $D^2=-2$.*

Demazure also proved that if the points of Σ are in almost general position, then the complete linear system $|-mK_V|$ gives a birational morphism for some $m>0$ and that its image is a normal Gorenstein surface with ample anti-canonical divisor. Now, we will show the converse.

THEOREM 3.4. *If X is a nomal projective Gorenstein surface with ω_X^{-1} ample and if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})=0$, then*

(i) $1 \leqq d=\omega_X \cdot \omega_X \leqq 9$.

(ii) *X is smooth or the singular points of X are rational double points.*

(iii) *If $d=9$, then $X \cong \boldsymbol{P}^2$.*

(iv) *If $d=8$, then either* (a) $X \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *or* (b) $X \cong \boldsymbol{F}_1$ *or* (c) *X is the cone over a quadric in $\boldsymbol{P}^2$ (in this case, $\tilde{X} \cong \boldsymbol{F}_2$ and the resolution π is given by contracting the minimal section of $\tilde{X}$).*

(v) *If $1 \leqq d \leqq 7$, then there exists a set Σ of points on $\boldsymbol{P}^2$ such that the points of Σ are in almost general position, $|\Sigma|=9-d$ and $\tilde{X} \cong V(\Sigma)$. In this case, the resolution π is the contraction of all curves on $\tilde{X}$ with self-intersection number -2.*

PROOF. The assertion (ii) follows from (2.3). Then $K_{\tilde{X}} \cong \pi^*(\omega_X)$ and the resolution π is obtained by a complete linear system $|-mK_{\tilde{X}}|$ for some $m>0$. We need the following

LEMMA 3.5. *We have $K_{\tilde{X}}^2>0$ and if C is an irreducible curve on $\tilde{X}$, then*

(1) *$C \cdot K_{\tilde{X}} \leqq 0$ and $C \cdot K_{\tilde{X}}=0$ if and only if $\pi(C)$ is a point.*

(2) *$C^2 \geqq -2$ and if $C^2=-2$, then $C \cong \boldsymbol{P}^1$ and $\pi(C)$ is a point.*

(3) *If $C^2=-1$, then $C \cong \boldsymbol{P}^1$ and $C \cdot K_{\tilde{X}}=-1$.*

PROOF. As $K_{\tilde{X}}=\pi^*(\omega_X)$, $K_{\tilde{X}}^2=\omega_X \cdot \omega_X=d>0$. As ω_X^{-1} is ample, (1) is clear. Since $C \cdot K_{\tilde{X}}+C^2 \geqq -2$, $C^2 \geqq -2$ and if $C^2=-2$, then $C \cdot K_{\tilde{X}}=0$ and $p_a(C)=0$. If $C^2=-1$, then $2p_a(C)-2=C \cdot K_{\tilde{X}}-1 \leqq -1$ and we have $p_a(C)=0$.

COROLLARY 3.6. *The relatively minimal model of $\tilde{X}$ is either $\boldsymbol{P}^2$, $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ or $\boldsymbol{F}_2$.*

PROOF. This is clear by Lemma 3.5, (2).

PROOF OF THEOREM 3.4 continued. Since $\tilde{X}$ is rational, we have rank(Pic$(\tilde{X})$)$+K_{\tilde{X}}^2=10$. Hence $1 \leqq d \leqq 9$. The statements (iii) and (iv) are immediate from Lemma 3.5 and Corollary 3.6. If $1 \leqq d \leqq 7$, then $\tilde{X}$ has $\boldsymbol{P}^2$ as its relatively minimal model. Let us take Σ so that $X \cong V(\Sigma)$. Then $|\Sigma|=9-d$ and the points of Σ are in almost general position by Lemma 3.5 and Theorem 3.3 (f). This concludes the proof of Theorem 3.4.

§ 4. The anti-canonical model of X and the description of singularities of X.

4.1. Let X, $\tilde{X}$, π be as in § 2. We say that X is of rational (resp. of elliptic) type if $\tilde{X}$ is rational (resp. if $\tilde{X}$ is an elliptic ruled surface).

PROPOSITION 4.2. (i) *The anti-canonical system* $|\omega_X^{-1}|$ *of* X *contains a non-singular elliptic curve.*

(ii) *For a non-singular elliptic curve* C *in* $|\omega_X^{-1}|$, *the natural map*

$$H^0(X, \omega_X^m) \longrightarrow H^0(C, \omega_X^m \otimes \mathcal{O}_C)$$

is surjective for all $m \in \boldsymbol{Z}$.

(iii) *If* $\deg X = d$, *then*

$$\dim_k H^0(X, \omega_X^{-m}) = \begin{cases} d \cdot m(m+1)/2+1 & (m \geqq 0) \\ 0 & (m<0) \end{cases}$$

and $H^1(X, \omega_X^m)=0$ *for all* $m \in \boldsymbol{Z}$.

PROOF. If X is of rational type, then by Theorem 3.3 (c), there exists a non-singular elliptic curve C in $|-K_{\tilde{X}}|$. Since the curve C does not meet any rational curve D with $D^2=-2$, the morphism π is an isomorphism on a neighborhood of C. Thus (i) follows from the fact $K_{\tilde{X}}=\pi^*(\omega_X)$. The assertion (ii) follows from Theorem 3.3 (d) and the fact $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})) \cong H^0(X, \omega_X^m)$ for every $m \in \boldsymbol{Z}$. The assertion (iii) from (ii) using induction on m and noticing that $\omega_X \otimes \mathcal{O}_C$ is an invertible sheaf of degree $-d$ on C. If X is of elliptic type, the "infinite section" of the bundle $X=\boldsymbol{P}(\mathcal{O}_C \oplus \mathcal{L})$ is the desired elliptic curve of (i) and if we identify the base curve and the infinite section, we have $\mathcal{L} \cong \omega_X^{-1} \otimes \mathcal{O}_C$. Moreover, if we put T the section of ω_X^{-1} corresponding to the infinite section, we can see easily that the graded ring $\bigoplus_{m \geqq 0} H^0(X, \omega_X^{-m})$ is isomorphic to $R(C, \mathcal{L})[T]$, where $R(C, \mathcal{L})$ is the graded ring $\bigoplus_{m \geqq 0} H^0(C, \mathcal{L}^m)$. Then the assertions (ii) and the former half of (iii) is clear. As for the latter half of (iii), as $R(C, \mathcal{L})[T]$ is a Macaulay ring, the assertion follows from (5.1.6) of [15].

REMARK 4.3. We put $R(X)=\bigoplus_{m \geqq 0} H^0(X, \omega_X^{-m})$ with the grading $(R(X))_m = H^0(X, \omega_X^{-m})$. Then, as ω_X^{-1} is ample, X is naturally isomorphic to $\mathrm{Proj}\,(R(X))$ with $\mathcal{O}_X(1) \cong \omega_X^{-1}$ by (5.1.7) of [15]. Then Proposition 4.2 shows that there is an element $t \neq 0$, $t \in R(X)_1$ such that $R(X)/t \cdot R(X) \cong R(C, \mathcal{L})$, where C is a smooth elliptic curve and $\mathcal{L}$ is a line bundle of degree d on C. As the structure of $R(C, \mathcal{L})$ is well-known (cf. [13]), we have the following

THEOREM 4.4. (i) *If* $d\geqq 4$, $R(X)\cong k[T_0, \cdots, T_d]/I$, $\deg(T_i)=1$ $(0\leqq i\leqq d)$ *and* I *is generated by* $d(d-3)/2$ *quadrics. In particular, if* $d=4$, X *is a complete intersection of two quadrics in* P^4.

(ii) *If* $d=3$, $R(X)\cong k[T_0, T_1, T_2, T_3]/(F)$, $\deg(T_i)=1$ $(0\leqq i\leqq 3)$ *and* $\deg(F)=3$. *That is,* X *is a cubic surface in* P^3.

(iii) *If* $d=2$, $R(X)\cong k[x, y, z, w]/(F)$, $\deg(x)=\deg(y)=\deg(z)=1$, $\deg(w)=2$ *and* $\deg(F)=4$. *That is,* X *is a hypersurface of degree* 4 *in the weighted projective space* $P(1, 1, 1, 2)$.

(iv) *If* $d=1$, $R(X)\cong k[x, y, z, w]/(F)$, $\deg(x)=\deg(y)=1$, $\deg(z)=2$, $\deg(w)=3$ *and* $\deg(F)=6$. *That is,* X *is a hypersurface of degree* 6 *in* $P(1, 1, 2, 3)$.

PROOF. This follows from Remark 4.3 (cf. [5], § 3).

COROLLARY 4.5. (i) *If* $d\geqq 3$, *then* ω_X^{-1} *is very ample and its global sections yield an embedding of* X *in* P^d *as a subvariety of degree* d.

(ii) *If* $d=2$, *then* ω_X^{-2} *is very ample and its sections yield an embedding of* X *in* P^6 *as a subvariety of degree* 8.

(iii) *If* $d=1$, *then* ω_X^{-3} *is very ample and its sections yield an embedding of* X *in* P^6 *as a subvariety of degree* 9.

Moreover, the above embeddings define projectively normal varieties and they are defined by quadratic equations except for the case d=3.

PROOF. (i) (resp. (ii)) is clear since $R(X)$ (resp. $R(X)^{(2)}$) is generated by its elements of degree 1. As for (iii), to show that $R(X)^{(3)}$ is generated by its element of degree 1, it suffices to check that the coefficient of z^3 of the equation F in Theorem 4.4, (iv) is not 0, which will be shown in Proposition 4.6. (For a graded ring $R=\bigoplus_{n\geq 0} R_n$, we write $R^{(d)}=\bigoplus_{n\geq 0} R_{nd}$ with the grading $(R^{(d)})_n=R_{nd}$.)

PROPOSITION 4.6. *Assume that* $\mathrm{ch}(k)\neq 2$.

(i) *If* $d=2$, *then* X *is a double covering of* P^2 *and the ramification divisor is a quartic curve without multiple component. In this case,* X *is of elliptic type if and only if the ramification divisor is four lines meeting in a point.*

(ii) *If* $d=1$, *then* X *is a double covering of a quadratic cone in* P^3 *and the ramification divisor is an intersection of the cone with a cubic surface without multiple components which does not pass the vertex of the cone. In this case,* X *is of elliptic type if and only if the ramification divisor is the intersection of the cone with three hyperplanes* *meeting in a point.*

PROOF. (i) It is easy to see that the point $p=(0, 0, 0, 1)$ of $P(1, 1, 1, 2)$ is a singular point which is not Gorenstein (cf. [16], Theorem 1.7, Proposition 2.3). As X is Gorenstein, X does not pass the point p. Noting this fact and that ch$(k)\neq 2$, we may assume $F=w^2-Q(x, y, z)$ in Theorem 4.4, (iii), where Q is a quartic form. Consider the projection $P(1, 1, 1, 2)\to P^2$ defined by $(x, y, z, w)\to(x, y, z)$. As the center p does not lie on X, X is a double covering of P^2 and the ramification divisor is defined by $Q=0$. Since X is normal, Q has no multiple components. If X is of elliptic type, we may assume that Q does not include the variable x as in the proof of Proposition 4.2 and so Q is the equation of four lines meeting at the point $(1, 0, 0)$. Note that this double-covering is defined by $|\omega_X^{-1}|$ since $H^0(X, \omega_X^{-1})$ is spanned by x, y, z.

(ii) By the same argument as in (i), we may assume that $F=w^2-G(x, y, z)$ in Theorem 4.4, (iv), where G is a weighted homogeneous polynomial of degree 6. Also, the projection $P(1, 1, 2, 3)\to P(1, 1, 2)$ defined by $(x, y, z, w)\to(x, y, z)$ gives a double covering of X onto $P(1, 1, 2)$ and it is easy to see that $P(1, 1, 2)$ is isomorphic to a quadratic cone in P^3. The ramification divisor is defined by $G(x, y, z)=0$ on $P(1, 1, 2)$ which is a cubic hypersurface section of the cone if we identify $P(1, 1, 2)$ with the cone. The point $(0, 0, 1, 0)$ or $P(1, 1, 2, 3)$ is a singular point which is not Gorenstein. So, X does not pass this point since X is Gorenstein. This means that $G(0, 0, 1)\neq 0$. If X is of elliptic type, we may assume that $F=w^2-z(z-y^2)(z-\lambda y^2)$ for some $\lambda\in k$, $\lambda\neq 0, 1$ and conversely. One may also notice that the linear system $|\omega_X^{-2}|$ defines the double covering of X to a quadratic cone in P^3, since $H^0(X, \omega_X^{-2})$ is spanned by x^2, xy, y^2 and z.

REMARK 4.7 (cf. [12], Proposition 2.13). If $\mathcal{O}$ is a normal Gorenstein local ring of dimension 3 with the maximal ideal $\mathfrak{m}$, which is a rational singularity, then M. Reid showed that the projectivized tangent cone (or α-tangent cone for some weighting α) of $\mathcal{O}$ is a Gorenstein surface whose anti-canonical sheaf is ample. So, such a surface is the one we are treating if it is normal.

4.8. Now, let us study the singular points on X. We call a rational double point by the name of the corresponding Dynkin diagram. Also, we say that the configuration of singularities on X is D_4+3A_1, for example, if there are one D_4-singularity and three A_1-singularities on X and no more. In this case, we denote Sing$(X)=D_4+3A_1$. If $d\geqq 3$, all possible configurations of singularities on X are known (cf. [11], [3], [8]).

So we treat the case $d \leqq 2$.

THEOREM 4.9. *Assume that X is of rational type,* ch $(k) \neq 2$ *and* $d \leqq 2$.

(i) *The maximal number of singular points on X is* 6.

(ii) *If $d=2$, then, unless the ramification divisor C of the double covering $X \to P^2$ is four lines,* Sing(X) *is a proper subgraph of the extended Dynkin diagram $\tilde{E}_7$. Mereover, every proper subgraph of $\tilde{E}_7$ actually appears as* Sing(X) *for some X. If C is four lines, then* Sing(X) *is $6A_1$ or D_4+3A_1.*

PROOF. (i) As X is a double covering of P^2 or $P(1,1,2)$, the singular points of X corresponds to singular points of the ramification divisor. (If $d=1$, the point of X which is mapped to the singular point of $P(1,1,2)$ is a smooth point.) If C is the ramification divisor of the double covering, then $\dim_k H^1(\mathcal{O}_C)=3$ (resp. 4) if d=2 (resp. $d=1$). As C has at most 4 (resp. 3) irreducible components if d=2 (resp. $d=1$), there are at most 6 singular points on C.

(ii) Let us list up all the possible singularities which can appear as singularities of C.

TABLE 4.10

The equation of the singularity on C.	The corresponding singularity on X.
$y(y-x^n)$ $(1 \leqq n \leqq 4)$	A_{2n-1}
y^2-x^{2n+1} $(1 \leqq n \leqq 3)$	A_{2n}
$xy(x-y)$	D_4
$xy(x-y)(x-\lambda y)$ $(\lambda \neq 0, 1)$	elliptic
y^3-x^4	E_6
$x(y^2-x^n)$ $(n=3, 4)$	D_{n+2}
$y(y^2-x^3)$	E_7

Thanks to the classification of quartic plane curves (cf. [19], p. 38), we can check that all possible subgraphs actually occur.

EXAMPLE 4.11. We write the defining equation of X as $w^2-Q(x,y,z)$.

(a) If $Q=x(x^2z-y^3)$, Sing$(X)=E_7$.

(b) If $Q=xy(xz-y^2)$, Sing$(X)=D_6+A_1$.

(c) If $Q=(x^2-yz)(x^2-yz-y^2)$, Sing$(X)=A_7$.

(d) If $Q=z(x^3-y^2z)$, Sing$(X)=A_5+A_2$.

(e) If $Q=xy(x^2+y^2-2xz-2yz+z^2)$, Sing$(X)=2A_3+A_1$.

These examples exhaust the proper subgraphs which have 7 vertices.

REMARK. As for the case $d=1$, the situation seems to be much more complicated. In [14], the case where Sing (X) has 8 vertices is treated.

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