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Normal modes of the viscoelastic earth

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Summary. A uniformly valid linear viscoelastic rheology is described which takes the form of a 'generalized' Burgers' body and which appears capable of reconciling the behaviour of the Earth's mantle across the complete spectrum of geodynamic time-scales. This spectrum is bracketed by the short time-scales of body wave and free oscillation seismology on which anelastic effects are dominant, and the long time-scale of mantle convection on which the Earth behaves viscously. The parameters of the model which control the viscous response are fixed by post-glacial rebound data whereas those which govern the anelasticity are to be determined by fitting the model to observations of seismic Q. The paper is concerned primarily with a discussion of the normal mode spectrum of the Earth as a generalized Burgers' body. Focusing upon the homogeneous model, it includes an initial analysis of the accuracy of first-order perturbation theory as a method of calculating the respective Qs of the elastic gravitational free oscillations. Also considered are the quasi-static modes of relaxation which only exact eigenanalysis can reveal. The importance of these modes is assessed within the context of a discussion of the effect of viscoelasticity upon the efficiency of Chandler wobble excitation.

1 Introduction

One of the most controversial questions in geodynamics concerns the precise nature of the rheological law which governs the response of mantle material to an applied stress. On the one hand it has become generally accepted that linear anelastic processes are dominant on the time-scale appropriate to a seismic body wave or to an elastic gravitational free oscillation. Several similar constitutive relations have been suggested to describe the recoverable deformation associated with such dissipative processes, including the modified Lommitz law preferred by Jeffreys (1972, 1973) and the closely related 'absorption band' models adopted by Liu, Anderson & Kanamori (1976) and Minster & Anderson (1980, 1981). On the other hand it is also, although perhaps less generally accepted, that on time-scales in excess of a few hundred years the Earth behaves viscously. This follows from the observed behaviour of

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the planet following melting of the large northern hemisphere ice caps which achieved their maximum extents *c.* 2×10^7 yr BP (e.g. Peltier 1974; Peltier & Andres 1976; Peltier 1981). That the complete rheological law should be one which exhibits such a transition from short-term anelastic to long-term viscous dominance is an hypothesis which is still strongly disputed in some quarters (e.g. Jeffreys 1972, 1973) although such behaviour is a well-known characteristic of crystalline materials at high temperature (Nowick & Berry 1972). If such a transition did not occur then the mantle could not support thermal convection and the hypothesis of continental drift, lacking a rational physical explanation, might be less enthusiastically embraced.

The fact that mantle material does support a steady state viscous mode of deformation has fortunately been very well verified in numerous high temperature creep experiments in the laboratory, mostly on single crystal olivine (e.g. Kohlstedt & Goetze 1974; Durham & Goetze 1977). Such controversy as continues to exist involves either the question of the time-scale over which the transition from anelastic to viscous behaviour occurs, or the issue as to whether the stress-strain relation is linear or non-linear in the visous regime. Both of these issues are of crucial importance in so far as the interpretation of post-glacial rebound is concerned. The model developed in Peltier (1974) for this purpose consisted of a linear viscoeleastic Maxwell solid in which transient anelastic effects were ignored entirely. When this model is fit to the rebound data one obtains a value for the equivalent Newtonian viscosity of the mantle ν which is $0(10^{22}$ poise, cgs units) and accordingly the mantle begins to deform viscously in response to an applied shear stress after a time $T_m = \nu/\mu$ (the Maxwell time) which is on the order of a few hundred years. It has been variously suggested (e.g. Anderson & Minster 1979) that the time-scale of post-glacial rebound is sufficiently short ($\lesssim 10^3$ yr) that it should be governed by transient anelasticity rather than the steady creep assumed in the Maxwell analogue. Peltier, Yuen & Wu (1980) have given several arguments as to why this is unlikely to be the case although the arguments are not completely definitive. One of the most telling concerns the fact that the steady state viscosity inferred by employing the Maxwell model to invert post-glacial rebound data has just the value which is required by mantle convection models of the drift process (Peltier 1980).

The second point of contention which remains in the attempt to achieve a consensus regarding a working model for the rheology of the mantle concerns the linearity of the constitutive relation in the steady state creep regime. Although the laboratory data for single crystal olivine clearly display non-Newtonian behaviour (Kohlstedt & Goetze 1974; Durham & Goetze 1977) at high stress levels, there is mounting evidence that at the low stress levels which are obtained in post-glacial rebound and convection (less than $\approx 10^2$ bar) the creep mechanism may become linear. This might be understood as a consequence of the fact that grain boundary processes become important under these circumstances (the mantle is polycrystalline) and such microphysical processes lead to a linear relation between stress and strain (Twiss 1976; Berckhemer, Auer & Drisler 1979). Recent theoretical studies by Greenwood, Jones & Sritharan (1980) indicate furthermore, that the transition stress which marks the boundary between linear and non-linear behaviour is not as sharply defined as has previously been believed. Laboratory experiments on sintered polycrystalline olivine (Relandeau 1981) have shown that the transition from non-linear to linear creep occurs at around 50 bar for 1600°C. Also germane to the issue of the linearity of the constitutive relation in the viscous regime is the recent review by Breatheau *et al.* (1979) on the flow properties of oxides which shows that a Newtonian creep mechanism cannot be ruled out for oxide assemblages in general.

Given the plausibility that the rheology of the mantle might well be linear (for moderate stress levels) across the entire geodynamic spectrum, which includes the broad range of

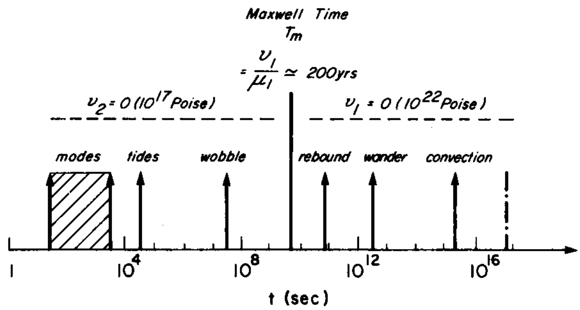

Figure 1. Diagram illustrating the relation of the characteristic time-scale for several geodynamic phenomena to the Maxwell time of the upper mantle. Long time-scale phenomena are goverened by ν_1, while short time-scale events are controlled by ν_2 or a range of ν_2 in the continuous relaxation spectrum (from Peltier *et al.* 1981).

phenomenological time-scales illustrated in Fig. 1, it is not at all unreasonable to enquire as to the form which a complete constitutive relation would then take. Peltier, Wu & Yuen (1981) have suggested that the simplest linear viscoelastic model consistent with the observations consists of a generalized Burgers' Body. This analysis shows that such a model behaves essentially as a Maxwell solid is so far as post-glacial rebound is concerned if the parameters which control the short time-scale response are constrained by fitting the model to the observed Qs of the elastic gravitational free oscillations. The purpose of this paper is to develop further the properties of this general linear viscoelastic model and to apply it to a detailed study of the normal modes of a homogeneous earth model, focusing particularly upon the free oscillations. Section 2 provides a brief discussion of the constitutive relation for the generalized Burgers' body. In Section 3 we give a sketch of the mathematical structure of the free oscillations problem for models with viscoelastic rheology and a discussion of numerical methods. Numerical results for the free oscillations of two different versions of the anelastic component of the Burgers' body rheology are presented in Section 4. First-order perturbation theory for the viscoelastic models is reviewed in Section 5 and applied to infer the frequencies and Qs of the same free oscillations determined by exact eigenanalysis in Section 4. Comparison of these results enables us quantitatively to assess the magnitude of the error incurred through application of first-order perturbation theory. In Section 6 we consider the physical implications of the existence of the quasi-static poles in the anelastic normal mode spectrum. This is pursued within the context of a brief discussion of the effects of anelasticity upon the efficiency of Chandler wobble excitations. Our main conclusions are summarized in Section 7.

2 The mantle as a generalized Burgers' body

Linear viscoelastic constitutive relations may be represented in either differential or integral form with the latter being the more general. Although the conventional models which have simple spring and dashpot analogues may be represented in differential form, the integral representation is required for the more elaborate models necessary to describe the Earth's mantle. Since the simple models will also be employed in our discussion of normal modes we will begin with a brief sketch of their properties.

2.1 DIFFERENTIAL CONSTITUTIVE RELATIONS

Fig. 2 shows a sequence of standard one-dimensional spring and dashpot analogues of the simplest linear viscoelastic rheologies. The solid described by the analogue shown in Fig. 2(c),

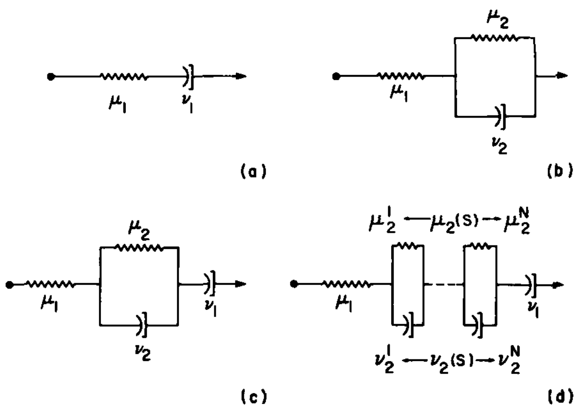

Figure 2. One dimensional spring-and-dashpot mechanical models for the four linear viscoelastic solids discussed in the text: (a) Maxwell, (b) standard linear solid [SLS], (c) Burgers' body with single Debye peak, (d) generalized Burgers' body with a continuous spectrum of relaxation times.

which was first introduced by Burgers' (1935), is the simplest linear model which exhibits the transition from short-term anelastic to long-term viscous behaviour which we expect to be characteristic of the Earth's mantle. It consists essentially of the superposition of a Maxwell element (Fig. 2a) and a standard linear solid (Fig. 2b). Three-dimensional tensor forms for the constitutive relations of these rheological models may be obtained using standard differential methods, which are discussed for example in Eringen (1967). For the Burgers' body the relation between the stress tensor σ_{kl} and the strain tensor e_{kl} is (Peltier *et al.* 1981)

$$\ddot{\sigma}_{kl} + \left[\frac{(\mu_1+\mu_2)}{\nu_2} + \frac{\mu_1}{\nu_2}\right]\left(\dot{\sigma}_{kl} - \frac{1}{3}\dot{\sigma}_{kk}\delta_{kl}\right) + \frac{\mu_1\mu_2}{\nu_1\nu_2}\left(\sigma_{kl} - \frac{1}{3}\sigma_{kk}\delta_{kl}\right)$$
$$= 2\mu_1\ddot{e}_{kl} + e_{kk}\delta_{kl} + \frac{2\mu_1\mu_2}{\nu_2}\left(\dot{e}_{kl} - \frac{1}{3}\dot{e}_{kk}\delta_{kl}\right). \qquad (1)$$

In (1) the dot denotes differentiation with respect to time, μ_1, and λ are the unrelaxed (elastic) Lamé parameters, and μ_2 is the shear modulus associated with the Kelvin-Voigt element (see Fig. 2). For this model the modulus defect $\Delta = \mu_1/\mu_2$. The two viscosities ν_1 and ν_2 are respectively the long and short time-scale parameters and it is clear from Fig. 2(c) that there is only a single relaxation time associated with the Kelvin-Voigt element. The constitutive relations for the simpler Maxwell and standard linear solids may be derived from the general expression (1). Taking the limit $\nu_2 \to \infty$ in (1) yields

$$\dot{\sigma}_{kl} + \frac{\mu_1}{\nu_1}\left(\dot{\sigma}_{kl} - \frac{1}{3}\dot{\sigma}_{kk}\delta_{kl}\right) = 2\mu_1\ddot{e}_{kl} + \lambda\ddot{e}_{kk}\delta_{kl}$$

which may be integrated once in time to give the Maxwell constitutive relation

$$\sigma_{kl} + \frac{\mu_1}{\nu_1}\left(\sigma_{kl} - \frac{1}{3}\tau_{kk}\delta_{kl}\right) = 2\mu_1\dot{e}_{kl} + \lambda\dot{e}_{kk}\delta_{kl} \qquad (2)$$

which was employed by Peltier (1974) in developing the viscoelastic model for glacial isostasy. If in (1) we take the opposite limit $\nu_1 \to \infty$ then we obtain

$$\ddot{\sigma}_{kl} + \frac{(\mu_1+\mu_2)}{2}\left(\dot{\sigma}_{kl} - \frac{1}{3}\dot{\sigma}_{kk}\delta_{kl}\right) = 2\mu_1\ddot{e}_{kl} + \lambda_1\ddot{e}_{kk}\delta_{kl} + \frac{2\mu_1\mu_2}{\nu_2}\left(\dot{e}_{kl} - \frac{1}{3}\dot{e}_{kk}\delta_{kl}\right)$$

which can also be integrated once to yield the constitutive relation for the standard linear solid as

$$\dot{\sigma}_{kl} + \frac{(\mu_1 + \mu_2)}{2}\left(\sigma_{kl} - \frac{1}{3}\sigma_{kk}\delta_{kl}\right) = 2\mu_1\dot{e}_{kl} + \lambda_1\dot{e}_{kk}\delta_{kl} + \frac{2\mu_1\mu_2}{\nu_2}\left(e_{kl} - \frac{1}{3}e_{kk}\delta_{kl}\right). \quad (3)$$

If in (2) we take the limit $\nu_1 \to \infty$ or in (3) the limit $\nu_2 \to \infty$ then we obtain the well-known constitutive relation for the Hookean elastic solid

$$\sigma_{kl} = 2\mu_1 e_{kl} + \lambda_1 e_{kk}\delta_{kl}. \quad (4)$$

The Burgers' body described by (1) is completely specified by the five parameters ($\lambda_1, \mu_1, \nu_1, \mu_2, \nu_2$). Two of these parameters (λ_1, μ_1) are in common with the Hookean elastic solid and for the Earth are reasonably well-known functions of radius determined by the systematic inversion of free oscillations data. In addition the long-term viscosity ν_1, has been determined by the inversion of post-glacial rebound data. The remaining variables μ_2 and ν_2 are to be determined by fitting the model to observations of short time-scale anelastic processes, principally the observed Qs of the elastic gravitational free oscillations.

As we shall show explicitly in Section 4, however, it is probably not possible to fit the single Debye peak model, embodied within the standard linear solid part of the general constitutive relation, to the observed free oscillations. The observed dependence of Q upon frequency is far too weak to be explicable by the simplest model which must consequently be generalized. In this respect the Earth's mantle bears a strong resemblance, rheologically, to amorphous polymers which require for the correct description of their anelastic behaviour the superposition of a number of elementary molecular processes acting essentially independently of one another (Ferry 1980). This circumstance is illustrated schematically in Fig. 2(d) in which a chain of Kelvin-Voigt elements represents this superposition of distinct processes. Such models may be described most economically by introducing the notion of a continuous relaxation spectrum (Gross 1947; Zener 1948; MacDonald 1961; Liu *et al.* 1976).

2.2 INTEGRAL CONSTITUTIVE RELATIONS

As the number of Kelvin-Voigt elements in the generalized Burgers' body of Fig. 2(d) approaches infinity, the relaxation spectrum becomes continuous. This in turn necessitates the use of the integral representation of the constitutive relation which follows from the Boltzmann superposition principle. Although the differential constitutive relations have obvious mechanical interpretations in terms of springs and dashpots they are not completely general. Although every differential law may be written in terms of Boltzmann hereditary integrals, the converse is not true (Fluegge 1975). In linear theory the most general functional form of an anelastic relation between stress σ_{ij} and strain e_{ij} may be written

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\tau)\,\dot{e}_{kl}(\tau)d\tau \quad (5)$$

in which C_{ijkl} is a fourth-order tensor function for stress relaxation (Christensen 1971) and the convolution integral in τ is to be regarded as a Stieltjes integral.

For an isotropic material, (5) reduces to

$$\sigma_{ij}(t) = \delta_{ij}\int_{-\infty}^{t} \lambda(t-\tau)\,\dot{e}_{kk}(\tau)d\tau + 2\int_{-\infty}^{t} \mu(t-\tau)\,\dot{e}_{ij}(\tau)d\tau \quad (6)$$

where λ and μ are the two stress relaxation functions required to describe an isotropic linear viscoelastic solid. If the anelasticity of the mantle were felt only in shear and not in compression, then (6) could be rewritten

$$\sigma_{ij}(\tau) = \delta_{ij} \int_{-\infty}^{t} \left(K - \frac{2}{3}\mu(t-\xi)\right) \frac{de_{kk}}{d\xi}\, d\xi + 2\int_{-\infty}^{t} \mu(t-\xi) \frac{de^{1}_{ij}(\xi)}{d\xi}\, d\xi \tag{7}$$

where

$$K = \lambda + \frac{2}{3}\mu$$

is the bulk modulus. Although dislocation theory provides no simple rationale for the existence of bulk dissipation in crystalline solids there is some evidence from both free oscillation and body wave data (Anderson 1980; Cormier 1981) that it could be important in the liquid core.

Restricting ourselves first to a description of the anelastic component of the viscoelasticity it is useful to introduce the notion of a normalized relaxation function by expressing the time dependent modulus $\mu(t)$ as

$$\mu(t) = \mu_{\mathrm{R}}\, \Delta\phi(t) \tag{8}$$

where μ_{R} is the relaxed shear modulus of the generalized Burgers' body, $\Delta = (\mu_1 - \mu_{\mathrm{R}})/\mu_{\mathrm{R}}$ is the modulus defect or relaxation strength, and $\phi(t)$ is the normalized relaxation function. This relaxation function, following standard works on linear viscoelasticity (Gross 1953; Christensen 1971), may be expressed in terms of the equivalent relaxation spectrum R through the following integral transform,

$$\phi(t) = \int_{-\infty}^{+\infty} R(\tau_\epsilon) \exp(-t/\tau_\epsilon)\, d(\ln \tau_\epsilon). \tag{9}$$

General thermodynamic considerations (Christensen 1971) require $\phi(t)$ to decrease from unity to zero as time tends to infinity.

In their studies of transient wave propagation in an absorption band solid, Minster (1978a, b) and Chin (1980) employed a relaxation spectrum which was hyberbolic in shape This model was

$$R(\tau_\epsilon) = \frac{B}{\tau_\epsilon} H(\tau_\epsilon - T_1) H(T_2 - \tau_\epsilon) \tag{10}$$

with τ_ϵ the relaxation time at constant strain and B a normalization constant. For this relaxation spectrum we may obtain a simple analytic form for $\mu(s)$ by substitution in (9) and (8) and direct Laplace transformation. This process gives (Minster 1978a, b)

$$\mu(s) = \mu_1 \left[1 + \frac{2}{\pi Q_{\mathrm{m}}} \ln\left(\frac{s + 1/T_2}{s + 1/T_1}\right)\right] \tag{11}$$

where Q_{m} is a parameter which determines the relatively constant quality factor Q within the absorption band (Kanamori & Anderson 1977). The modulus defect Δ for this model is related to Q_{m} by

$$\Delta = \frac{(2/\pi Q_{\mathrm{m}}) \ln (T_2/T_1)}{1 - (2/\pi Q_{\mathrm{m}}) \ln (T_2/T_1)} \tag{12}$$

$$\cong \frac{2}{\pi Q_{\mathrm{m}}} (T_2/T_1)$$

with the approximation obtaining with reasonable accuracy for geophysically plausible parameters such as those employed by Minster (1978b). Using Minster's values ($Q_m = 250$, $T_2 = 10^4$ s, $T_1 = 10^{-2}$ s) we obtain $\Delta = 0.03$ [i.e. $0(10^{-2})$]. Models such as (10) are commonly employed in analyses of the viscoelastic behaviour of polymers (Ferry 1980) and we shall restrict our attention to it since $\mu(s)$ is known analytically from (11) and since it does seem to suffice as a good first approximation to mantle anelasticity.

This model is not, of course, capable of describing the phenemenon of post-glacial rebound or of mantle convection since it is purely anelastic in character. In order to describe such phenomena we require a rheology whose eventual behaviour is viscous. Some assistance in obtaining an appropriate transformed shear modulus for such material is available by inspection of the forms of the transformed moduli for the simple differential constitutive relations discussed in Section 2.1. These were derived by Peltier *et al.* (1981) and for the Burgers' body shown in Fig. 2(c) are

$$\mu(s) = \frac{\mu_1 s}{(s + \mu_1/\nu_1)} \left[\frac{(s + \mu_1/\nu_2)(s + \mu_1/\nu_1)}{(s + \mu_2/\nu_2)(s + \mu_1/\nu_1) + \mu_1 s/\nu_2}\right] \tag{13a}$$

$$\lambda(s) = \frac{\lambda s^2 + \left\{\left[\frac{(\mu_1 + \mu_2)}{\nu_2} + \frac{\mu_1}{\nu_1}\right]\left[\lambda_1 + \frac{2}{3}\mu_1\right] - \frac{2}{3}\frac{\mu_1\mu_2}{\nu_2}\right\} s + \frac{\mu_1\mu_2}{\nu_1\nu_2}\left(\lambda_1 + \frac{2}{3}\mu_1\right)}{s^2 + \left[\frac{(\mu_1 + \mu_2)}{\nu_2} + \frac{\mu_1}{\nu_1}\right] s + \frac{\mu_1\mu_2}{\nu_1\nu_2}}. \tag{13b}$$

The transformed moduli for the standard linear solid with the one dimensional analogue shown in Fig. 2(b) can be obtained from (13a, b) in the limit $\nu_1 \to \infty$ as

$$\mu(s) = \frac{\mu_1(s + \mu_2\nu_2)}{s + [(\mu_1 + \mu_2)/\nu_2]} \tag{13c}$$

$$\lambda(s) = \frac{\lambda s + [(\mu_1 + \mu_2/\nu_2)(\lambda + 2/3\,\mu_1) - 2/3\,(\mu_1\mu_2/\nu_2)]}{[s + (\mu_1 + \mu_2/\nu_2)]}. \tag{13d}$$

Whereas the moduli for the Maxwell solid employed by Peltier (1974) in the model of glacial isotasy can be obtained from (13a, b) in the limit $\nu_2 \to \infty$ as

$$\mu(s) = \frac{\mu_1 s}{s + \mu_1/\nu_1} \tag{13e}$$

$$\lambda(s) = \frac{\lambda s + \mu_1 K/\nu_1}{s + \mu_1/\nu_1}. \tag{13f}$$

It is now clear that if we replace the term in square brackets in (13a) by the term in square brackets in (11) we will have an expression for the transformed shear modulus of a material which behaves on a short time-scale like an anelastic solid if the Maxwell time $\nu_1/\mu_1 = T_m$ is much longer than time-scales in the anelastic regime [i.e. $T_m \gg (T_1, T_2)$]. Since the shear modulus (11) or a simple variant of it provides quite a good fit to the available seismic data, so will the *generalized Burgers' body* constructed in this fashion. Furthermore this new rheological law (stated in Peltier *et al.* 1981) also reconciles the data of post-glacial rebound since the material it describes behaves like a Newtonian viscous fluid for times in excess of the Maxwell time. The transformed shear modulus for our simple version of the generalized

Burgers' body is then

$$\mu(s) = \frac{\mu_1 s}{(s + \mu_1/\nu_1)} \left[1 + \frac{2}{\pi Q_m} \ln\left(\frac{s + 1/T_2}{s + 1/T_1}\right)\right]. \tag{14}$$

The shape of the relaxation spectrum for such a model may be obtained following the methods outlined in Gross (1953) or Ferry (1980, chapter 4). Theoretically it may be feasible in the future to invert for the relaxation spectrum from laboratory experiments, as is commonly done in polymer dynamics (Ferry 1980). When this inverse procedure is followed one finds that the relaxation spectrum consists of the superposition of a continuous part and a discrete peak associated with the eventual plastic response. Gross & Pelzer (1951) demonstrated that the incorporation of plasticity always produces this effect on the spectrum which therefore has the schematic form

$$\hat{R}(\tau_\epsilon) = \Delta'\delta\,(\tau_\epsilon - T_m) + R(\tau_\epsilon).$$

In analysing the mechanical behaviour of an earth model composed of material with rheology described by (14) we intend to employ the correspondence principle in the same way it was employed in Peltier (1974) to develop the viscoelastic theory of glacial isostasy. This requires the Laplace transform domain representation of the constitutive relation which we may write as

$$\tilde{\sigma}_{ij} = \lambda(s)\,\tilde{e}_{kk}\,\delta_{ij} + 2\mu(s)\,\tilde{e}_{ij} \tag{15}$$

in which $\mu(s)$ is given by (14) and

$$\lambda(s) = K \frac{2}{3} \mu(s)$$

with K the elastic bulk modulus in order to ensure that the model has no bulk dissipation.

3 Viscoelastic normal modes

Our preliminary analysis will be restricted to consideration of a spherical, non-rotating, viscoelastic and isotropic continuum which is perturbed from its hydrostatic equilibrium configuration by oscillations of infinitesimal amplitude. Such self-gravitating motions satisfy the following linearized momentum and Poisson equations (see Gilbert 1981, for a recent discussion).

$$\nabla\cdot\boldsymbol{\sigma} - \nabla(\rho g\mathbf{u}\cdot\hat{e}_r) - \rho\nabla\phi + g\nabla\cdot(\rho\mathbf{u})\,\hat{e}_r = -\rho s^2\mathbf{u} \tag{16}$$

$$\nabla^2\phi = -4\pi G\nabla\cdot(\rho\mathbf{u}) \tag{17}$$

which have been written in the domain of the Laplace transform variable s. The scalar fields $\rho(r)$, $g(r)$ represent the variation of density and gravitational acceleration with radius in the unperturbed sphere (we will take $\rho(r) = \rho_0$ since our discussion will be restricted to homogeneous models), $\boldsymbol{\sigma}$ is the stress tensor, ϕ the associated perturbation of the gravitational potential, and G is the gravitational constant. $\hat{e}_r$ is a radial unit vector and $s = s_r + is_i$ is the (complex) Laplace transform variable. Since s is complex so is the shear modulus (14) as is the second Lame parameter $\lambda(s)$ which is required to represent the stress tensor in terms of the strain tensor. In attacking the viscoelastic free oscillations problem in this fashion, by solving an equivalent elastic problem with complex moduli, we are employing the so-called Correspondence Principle and in so doing following the same approach as in Peltier (1974) for the post-glacial rebound problem. A similar approach to the analysis of the vibration spectrum of polymers has been exploited by Hunter (1965), Christensen (1971), Rabotnov (1980).

We lose no important generality by seeking solutions to (16) and (17) in the form

$$\mathbf{u} = \sum_{l=0}^{\infty} \left[U_l(r, s) P_l(\cos\theta)\, \hat{e}_r + V_l(r, s) \frac{\partial}{\partial\theta} P_l(\cos\theta)\, \hat{e}_\theta + W_l(r, s) \frac{\partial}{\partial\theta} P_l(\cos\theta)\, \hat{e}_\sigma \right] \tag{18a}$$

$$\phi = \sum_{l=0}^{\infty} \phi_l(r, s) P_l(\cos\theta) \tag{18b}$$

where $\hat{e}_\theta$ and $\hat{e}_\sigma$ are unit vectors in the directions of increasing latitude and longitude respectively and P_l is the Legendre polynomial of degree l. Substitution of (18) into (16) and (17) leads to two decoupled sets of first-order ordinary differential equations of the form

$$\frac{d\mathbf{X}}{dr} = B\mathbf{X} \tag{19a}$$

$$\frac{d\mathbf{Y}}{dr} = A\mathbf{Y} \tag{19b}$$

where $\mathbf{X} = (W_l,\, T_{\phi l})^{\mathrm{T}}$, $\mathbf{Y} = (U_l,\, V_l,\, T_{rl},\, T_{\theta l},\, \phi_l,\, Q_l)^{\mathrm{T}}$, and A and B are the 2×2 and 6×6 matrices given explicitly in Gilbert (1980). The parameters T_{rl}, $T_{\theta l}$, and $T_{\phi l}$ are the coefficients in the spherical harmonic expansions of the σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{r\phi}$ components of the stress tensor and the Q_n are the coefficients in the expansion of the auxilliary variable $q = \partial\phi/\partial r + (l+1)\phi/r + 4\pi G\rho u_r$. The systems (19a) and (19b) respectively govern the toroidal and spheroidal free oscillations. The spheroidal system with $l = 0$ describes motion which is purely radial so that $V_0 = T_{\theta_0} = 0$ and (ϕ_0, Q_0) decouple from U_0, T_{r_0} leading to a simplification of (19b) for radial modes which may be written as

$$\frac{d\mathbf{Z}}{dr} = C\mathbf{Z} \tag{19c}$$

where

$$\mathbf{Z} = (U_n, T_{rn})^{\mathrm{T}},$$

n is the index used to label the radial eigenstates, and C is the 2×2 matrix given in Gilbert (1980).

The present study will be restricted to an analysis of the effect of viscoelasticity upon the vibration spectrum of the homogeneous earth model with elastic properties listed in Table 1 which are appropriate to the average earth. Solutions for the elastic free oscillations of a homogeneous sphere are well known and have been discussed by Love (1911), Pekeris &

Table 1. Model parameters of a homogeneous earth model.

Parameter	Symbol	Value
Volumetrically averaged density	ρ_0	$5517\ \mathrm{kg\ m^{-3}}$
Volumetrically averaged	λ	$3.5288 \times 10^{11}\ \mathrm{N\ m^{-2}}$
Lamé constants	μ_1	$1.4519 \times 10^{11}\ \mathrm{N\ m^{-2}}$
Gravitational acceleration	g	$9.82\ \mathrm{m\ s^{-2}}$
Earth's radius	a	$6.371 \times 10^6\ \mathrm{m}$
Long-term viscosity	ν_1	$10^{22}\ \mathrm{P}$

Jarosch (1958), and Gilbert & Backus (1968), among others. These solutions are also quoted in the recent texts by Garland (1979) and Aki & Richards (1980). For convenience of numerical comparison we have adopted here the notation of Takeuchi & Saito (1972).

3.1 SECULAR FUNCTIONS AND NUMERICAL METHODS FOR COMPLEX EIGENVALUES

Complex eigenvalues of the homogeneous systems (19) are determined from simultaneous zero crossings of the real and imaginary parts of the secular functions $D_j(s, l)$ associated with each set of equations. The secular functions are determined by the boundary conditions at the Earth's surface. For the toroidal system (19a) the explicit form of the secular function is (Takeuchi & Saito 1972)

$$D_1(s, l) = (l-1)j_l(k_1 a) - k_1 a j_{l+1}(k_1 a) \tag{20a}$$

where the complex wavenumber

$$k_1(s) = [\rho_0/\mu(s)]^{1/2}$$

and ρ_0 is the constant density of the 'average' earth model. For the spheroidal system (19b) the characteristic equation is of the form

$$D_2(s, l) = \det\begin{pmatrix} T_{\theta l}^1 & T_{\theta l}^2 & T_{\theta l}^3 \\ T_{rl}^1 & T_{rl}^2 & T_{rl}^3 \\ Q_l^1 & Q_l^2 & Q_l^3 \end{pmatrix} \tag{20b}$$

where the superscripts 1, 2, 3 denote the three linearly independent solutions which are regular at the origin, each of which consists of a combination of two spherical Bessel functions $j_l(z)$ with different complex arguments and a polynomial in r of degree l. Explicit forms will be found in Takeuchi & Saito (1972). The secular function for the radial system (19c) is simply

$$D_3(s, n) = \frac{\tan k_2 a}{\mu(s)} - \frac{k_2 a}{\mu(s)} - \left(\frac{\lambda(s)}{4} + \frac{\mu(s)}{2}\right) k_2^2 a^2 \tag{20c}$$

where $k_2(s) = \{\rho_0 [16\pi G\rho_0/3 - s^2]/[\lambda(s) + 2\mu(s)]\}^{1/2}$.

The values of s for which both real and imaginary parts of a secular determinant vanish constitute the discrete eigenvalues for that class of viscoelastic normal mode. This approach to the forward calculation of complex eigenspectra is exact and simple, yet has not been pursued previously. It should be contrasted with the phenomenological approach employed by Liu *et al.* (1976), Kanamori & Anderson (1977), Minster (1980), and Minster & Anderson (1980, 1981) in which Q, for example, is calculated directly from the Fourier transform of a given creep function. The differences between these two approaches will be discussed in detail in following sections.

All of the eigenspectra to be reported have been determined using double precision complex arithmetic on an IBM 3300 system and are accurate to the number of figures stated in the attached tables. The crucial constraint on the accuracy of the calculations concerns the evaluation of spherical Bessel functions with complex arguments. These calculations have been performed using an algorithm based upon continued fractions due to Lentz (1976) which was found to be at least five fimes faster than the traditional method using downward recursion (e.g. Luke 1977). Because of the efficiency of the numerical computations for the homogeneous model we can afford to do a preliminary reconnaisance of the complex s-plane on a 30×30 (or so) mesh to obtain preliminary zero crossings of the real and imaginary parts of a given secular function. The procedure is commonly employed in hydrodynamic stability problems which have the same mathematical structure as the free oscillations case (e.g. Davis & Peltier 1976, 1977). Further refinement of these approximate eigenvalues is achieved by employing the Muller iteration scheme (Traub 1964) which is interrupted when accuracies of 1 part in 10^8 have been achieved. The Muller scheme requires no derivative evaluation and has the desirable property of nearly quadratic convergence. We found the alternative Newton-Raphson scheme with damping for systems of equations (Dahlquist & Björck 1974) to fail in situations in which the topography of the $D_j(s)$ surface is sufficiently rugged.

3.2 EIGENSPECTRA FOR THE HOMOGENEOUS VISCOELASTIC EARTH

Our first purpose in this section is to contrast the eigenspectra of the homogeneous earth model for the two different anelastic rheologies discussed in Section 2. The first model we shall employ is the standard linear solid (SLS) whose relaxation spectrum consists of a single Debye peak. The second is the absorption band recently popularized by Liu *et al.* (1976) and Kanamori & Anderson (1977). For the SLS the shear modulus $\mu(s)$ is given by (13c), while

$$\lambda(s) = K - \frac{2}{3}\mu(s)$$

as in (13d) ensures zero bulk dissipation. $\mu(s)$ for the absorption band is defined in (11) and in the prototype of this rheology we shall employ $T_1 = 10^{-2} s$, $T_2 = 2 \times 10^4 s$, and $Q_m = 250$, all of which are within the range of values employed by Minster (1978b) in his study of transient wave propagation. Again

$$\lambda(s) = K - \frac{2}{3}\mu(s)$$

to ensure no bulk dissipation.

In Fig. 3 we show the location of the spheroidal modes ${}_nS_2 (0 < n < 5)$ in the complex s-plane for both the SLS and absorption band rheologies with parameters fixed to the values shown. The first effect of anelasticity is to displace the elastic modes from their locations on the imaginary s-axis into the second quadrant where the eigenvalues s have negative real part. The individual modes therefore consist of exponentially decaying sinusoids. The second effect of anelasticity is to introduce a separate feature into the spectrum which has no elastic counterpart. In the case of the SLS this new feature consists of a single pole on the negative real s-axis corresponding to a mode whose temporal behaviour consists of exponential decay. With $\nu_2 = 10^{17}$ poise the relaxation time for this mode which accompanies the ${}_nS_2$ free oscillations is 15.94 hr. Such quasi-static modes, which play a crucial role in the theory of glacial isostasy within the context of the Maxwell model, are viscous-gravitational in nature. For a homogeneous model in which density is constant everywhere there is only one such spheroidal mode for each value of l and this is supported by the density contrast across the free outer surface of the model. For the absorption band rheology no such isolated quasi-static mode exists; rather the negative real s-axis has branch points at $s = -1/T_1$ and $s = -1/T_2$ and must be cut to ensure that functions dependent upon s (such as the secular function) remain single valued. The importance of the discrete quasi-static modes for the SLS rheology or of the equivalent quasi-static continuum for the absorption band model will be illustrated in Section 5 in the context of a brief discussion of Chandler wobble excitation.

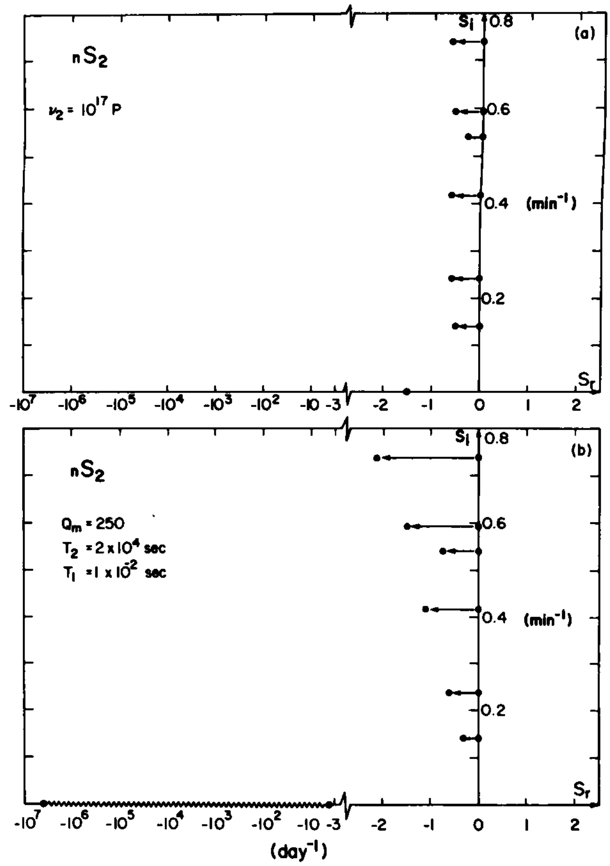

Figure 3. Spectrum of a uniform anelastic earth for spheroidal modes with $l = 2$. (a) and (b) represent SLS and absorption band rheologies respectively. Arrows denote the nature of the analytic continuation of the elastic poles. Eigenvalues occur in pairs $\sigma = s_R + is_I$ and $\sigma^* = s_R - is_I$. Only one is displayed here. Note the single quasi-static pole on the negative real s-axis for (a). In (b) the branch cut betweeen $s = -(1/T_1)$ and $s = -(1/T_2)$ arises from the multi-valued nature of the shear modulus $\mu(s)$ in the absorption band model.

Our main purpose here is to explore the manner in which anelastic rheology modifies the eigenspectra of the free vibrations. For the SLS the single anelastic parameter is the short time-scale viscosity ν_2. In Fig. 4 we show the variation of the periods of a few fundamental modes (expressed as a fraction of the elastic period τ_E) and their quality factors Q as a function of ν_2. For Q we employ the definition $q = s_R/2s_I$ which has a simple interpretation as the inverse of the width of a resonance line at half-maximum height in the theory of weakly damped linear systems. Inspection of Fig. 4 shows that the normal mode period is increased by anelasticity, as is well known, and that the magnitude of the shift in period maximizes for sufficiently small ν_2. The most rapid change of period τ_A occurs for the same value of ν_2 for which the Q of the mode is minimized. Inspection of the figure shows that the critical value ν_2 for these few modes is in the vicinity of 10^{15} P and this is the value of ν_2 for which the relaxation time is on the order of the period of the oscillation. For a given ν_2 the Q of the radial mode ${}_0S_0$ is higher than the Q of the spheroidal mode ${}_0S_2$ and this in turn is higher than the Q of the toroidal mode ${}_0T_2$. This is of course, a consequence of the relative partitioning of shear and compressional energy among the different modal types.

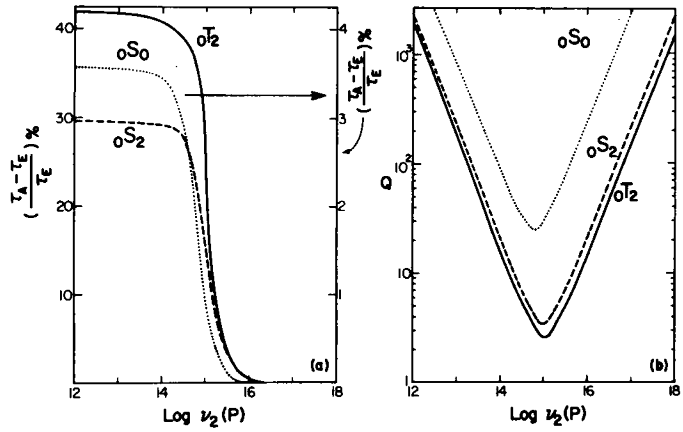

Figure 4. Shift in eigenperiod (a) and Q(b) for SLS model as a function of ν_2.

Since the SLS rheology described through the moduli (13) has no bulk dissipation, it is clear that for fixed ν_2 the Q of a mode will be higher the larger the ratio of compressional to shear energy. The curves in Fig. 4 are reminiscent of the frequency dependence of the complex moduli themselves (Zener 1948).

In order to fit the single Debye peak SLS rheology to the observed Qs of the Earth's free vibrations, which are on the order of a few hundred, and at the same time introduce a negligible shift in normal mode periods away from their elastic values one must clearly be on the high ν_2 side of the Q minimum at $\nu_2 = 10^{15}$ P. The choice $\nu_2 = 0(10^{17}$ P) yields a Q for $_0S_2$ of a few hundred and for the fundamental radial mode $_0S_0$ of a few thousand. These Qs are near the observed values for the real radially stratified Earth. A value of ν_2 on this order for the SLS was previously obtained by Scheidegger (1957) in an analysis of the damping of the Chandler wobble. Smith & Dahlen (1981) have recently reanalysed this datum in terms of an absorption band rheology.

Fig. 5 shows dispersion diagrams for the free vibrations of a homogeneous earth with either of the prototype SLS or absorption band rheologies. Since the period shifts away from the elastic values are small the rheologies cannot be distinguished from one another qualitatively through this effect. We should note here that our exact complex eigenvalue calculations were checked on the non-dissipative limit against the tables previously published for the homogeneous model by Sâto & Usami (1962). In Table 2 we have listed period and Q for a representative set of spheroidal and toroidal modes for the prototype absorption band model.

The Q data for the two rheological models (prototype SLS and absorption band) are shown in Fig. 6 for the same representative set of modes. In terms of these data the two rheologies are clearly distinguished from one another. For the SLS with $\nu_2 = 10^{17}$P, Q increases monotonically with l (and therefore from Fig. 5, with frequency) such that Q becomes proportional to frequency at high frequencies. This is precisely the behaviour to be expected from the single Debye peak model. The monotonic order of magnitude variation in Q for the first 25 fundamental spheroidal modes is not compatible with the observed weak frequency dependence of Q for the real Earth. Although this might be compensated to some extent by radial heterogeneity of ν_2, the required heterogeneity would likely be so extreme as to make the model implausible from a microphysical point of view. This conjectur deserves to be investigated in the context of calculations with a realistic stratified earth model. Also of interest for the SLS rheology is the non-monotonic behaviour of Q with l for

Table 2. Periods and Q of uniform earth model for a representative set of toroidal and spheroidal modes. The parameters of the anelastic model are $Q_m = 250$, $T_1 = 10^{-2}$ s, $T_2 = 2 \times 10^4$ s.

Mode	τ_E(min)	τ_A(min)	Q	Mode	τ_E(min	τ_A(min)	Q
$_0S_2$	44.39896	44.88788	310.341	$_2S_2$	15.05375	15.22837	261.692
$_0S_4$	24.29450	24.56117	291.664	$_2S_4$	11.45600	11.58526	261.308
$_0S_6$	17.60728	17.80037	281.993	$_2S_6$	9.20507	9.30558	263.353
$_0S_8$	13.91895	14.07075	276.414	$_2S_8$	7.66033	7.74258	262.505
$_0S_{10}$	11.53770	11.66260	272.787	$_2S_{10}$	6.56755	6.63801	258.198
$_0S_{12}$	9.86332	9.96925	270.239	$_2S_{12}$	5.77465	5.83681	253.598
$_0S_{14}$	8.61851	8.71031	268.353	$_2S_{14}$	5.17762	5.23339	250.263
$_0S_{16}$	7.65543	7.73632	266.901	$_2S_{16}$	4.70968	4.76027	248.212
$_0S_{18}$	6.88754	6.95975	265.749	$_2S_{18}$	4.33000	4.37627	247.018
$_0S_{20}$	6.26063	6.32578	264.815	$_2S_{20}$	4.01356	4.05618	246.325
$_1S_2$	26.19460	26.49103	248.318	$_3S_2$	11.59225	11.65618	527.073
$_1S_4$	15.55849	15.73126	274.464	$_3S_4$	8.90743	9.00174	269.869
$_1S_6$	11.32725	11.45458	262.120	$_3S_6$	7.50368	7.58458	260.647
$_1S_8$	9.12235	9.22584	253.660	$_3S_8$	6.49478	6.56392	259.781
$_1S_{10}$	7.74594	7.83367	249.489	$_3S_{10}$	5.72074	5.78077	259.684
$_1S_{12}$	6.77891	6.85511	247.598	$_3S_{12}$	5.10796	5.16109	258.532
$_1S_{14}$	6.04863	6.11597	246.700	$_3S_{14}$	4.61585	4.66374	256.142
$_1S_{16}$	5.47158	5.53190	246.239	$_3S_{16}$	4.21683	4.26060	253.326
$_1S_{18}$	5.00133	5.05594	245.988	$_3S_{18}$	3.88914	3.92951	250.851
$_1S_{20}$	4.60938	4.65925	245.844	$_3S_{20}$	3.61554	3.65303	249.003
$_4S_2$	10.55482	10.66386	286.338	$_0T_2$	52.03158	52.76394	246.750
$_4S_4$	7.79216	7.83494	514.628	$_0T_4$	25.54420	25.87962	245.178
$_4S_6$	6.37281	6.43226	292.513	$_0T_6$	17.57766	17.79976	244.815
$_4S_8$	5.59749	5.65501	264.008	$_0T_8$	13.52644	13.69267	244.679
$_4S_{10}$	5.01684	5.06870	259.442	$_0T_{10}$	11.03607	11.16872	244.623
$_4S_{12}$	4.54916	4.59588	258.203	$_0T_{12}$	9.33907	9.44927	244.604
$_4S_{14}$	4.16119	4.20359	257.486	$_0T_{14}$	8.10411	8.19822	244.605
$_4S_{16}$	3.83402	3.87287	256.409	$_0T_{16}$	7.16315	7.24517	244.615
$_4S_{18}$	3.55563	3.59156	254.797	$_0T_{18}$	6.42136	6.49396	244.631
$_4S_{20}$	3.31734	3.35083	252.901	$_0T_{20}$	5.82100	5.88605	244.651
$_5S_2$	8.46778	8.56511	248.269	$_1T_2$	18.23679	18.46811	244.841
$_5S_4$	7.10043	7.17467	269.046	$_1T_4$	13.39900	13.56350	244.675
$_5S_6$	5.90683	5.94308	453.896	$_1T_6$	10.69650	10.82463	244.618
$_5S_8$	5.00471	5.04202	352.195	$_1T_8$	8.94514	9.05018	244.603
$_5S_{10}$	4.47407	4.51709	273.688	$_1T_{10}$	7.70861	7.79762	244.608
$_5S_{12}$	4.08914	4.13001	261.564	$_1T_{12}$	6.78485	6.86205	244.622
$_5S_{14}$	3.77322	3.81108	258.252	$_1T_{14}$	6.06644	6.13457	244.642
$_5S_{16}$	3.50461	3.53965	256.915	$_1T_{16}$	5.49058	5.55152	244.665
$_5S_{18}$	3.27224	3.30481	256.026	$_1T_{18}$	5.01798	5.07308	244.690
$_5S_{20}$	3.06913	3.09955	255.050	$_1T_{20}$	4.62273	4.67299	244.715

overtone number $n > 2$. The local Q maxima along each overtone branch occur for modes in which the ratio of compression to shear energy is largest. Such Q determinations as have been made for the spheroidal overtones (Anderson & Hart 1978) very clearly show the same non-monotonic behaviour as predicted by even the simple homogeneous model considered here (e.g. the measurements for $_4S_l$ and $_5S_l$). Recent inversions of the free oscillations data to obtain intrinsic Q as a function of depth which employ only the fundamental mode data (Stein, Mills & Geller 1981) show that these data offer very little in terms of depth resolution. In view of the poor quality of the toroidal Q measurements simultaneous inversion of the spheroidal and toroidal fundamental mode data (e.g. Deschamps 1977) cannot be expected to resolve the problem. The overtone data must clearly play an important role in future analyses.

Inspection of the Q results in Fig. 6 for the absorption band model shows that this model reconciles the gravest difficulty with the SLS rheology which was its prediction of a monotonic

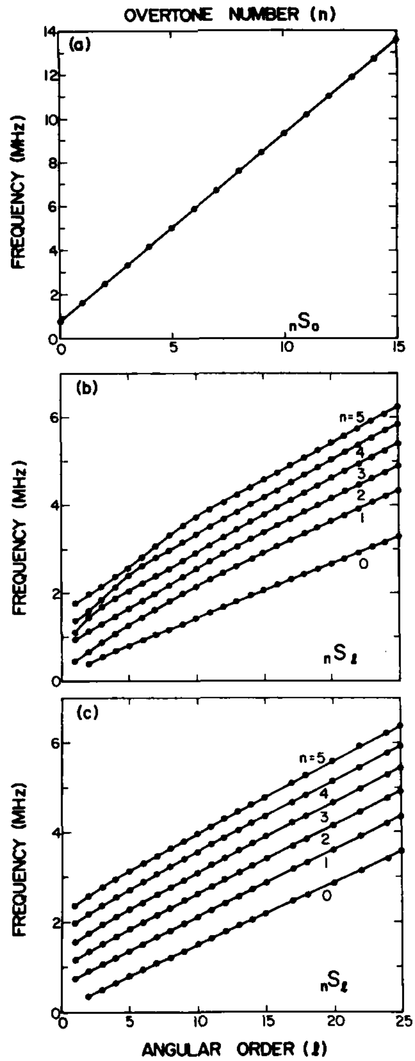

Figure 5. Eigenfrequencies for radial, spheroidal and toroidal modes of a homogeneous anelastic earth. Rheological parameters for the absorption band model are ($Q_m = 250$, $T_1 = 10^{-2}$ s, $T_2 = 2 \times 10^4$ s). Differences in eigenfrequencies between anelastic and elastic normal modes are too small to be discernible in the figure. See Table 2 for a detailed listing of the eigenfrequencies belonging to the elastic and anelastic modes.

increase of Q with frequency (or l) for fixed overtone number. The toroidal mode Qs are all constant (panel f) and equal to the parameter Q_m of the absorption band model. For the spheroidal modes (panel d) there exists a single local Q maximum which is developed across a sequence of three or four modes and which exists for overtone numbers $\bar{n} > 2$. The height of this maximum is, however, a decreasing function of overtone number n. Otherwise the normal mode Qs all cluster about the constant Q_m of the absorption band. Fig. 7 further illustrates the variability of Q with n for fixed $l = 2$, 16. For the absorption band, the trend $Q\,(l)$ for the radial modes of the homogeneous model (panel b) also resembles real data (e.g. Sailor & Dziewonski 1978; Buland, Berger & Gilbert 1979) in that there is a sharp drop in Q

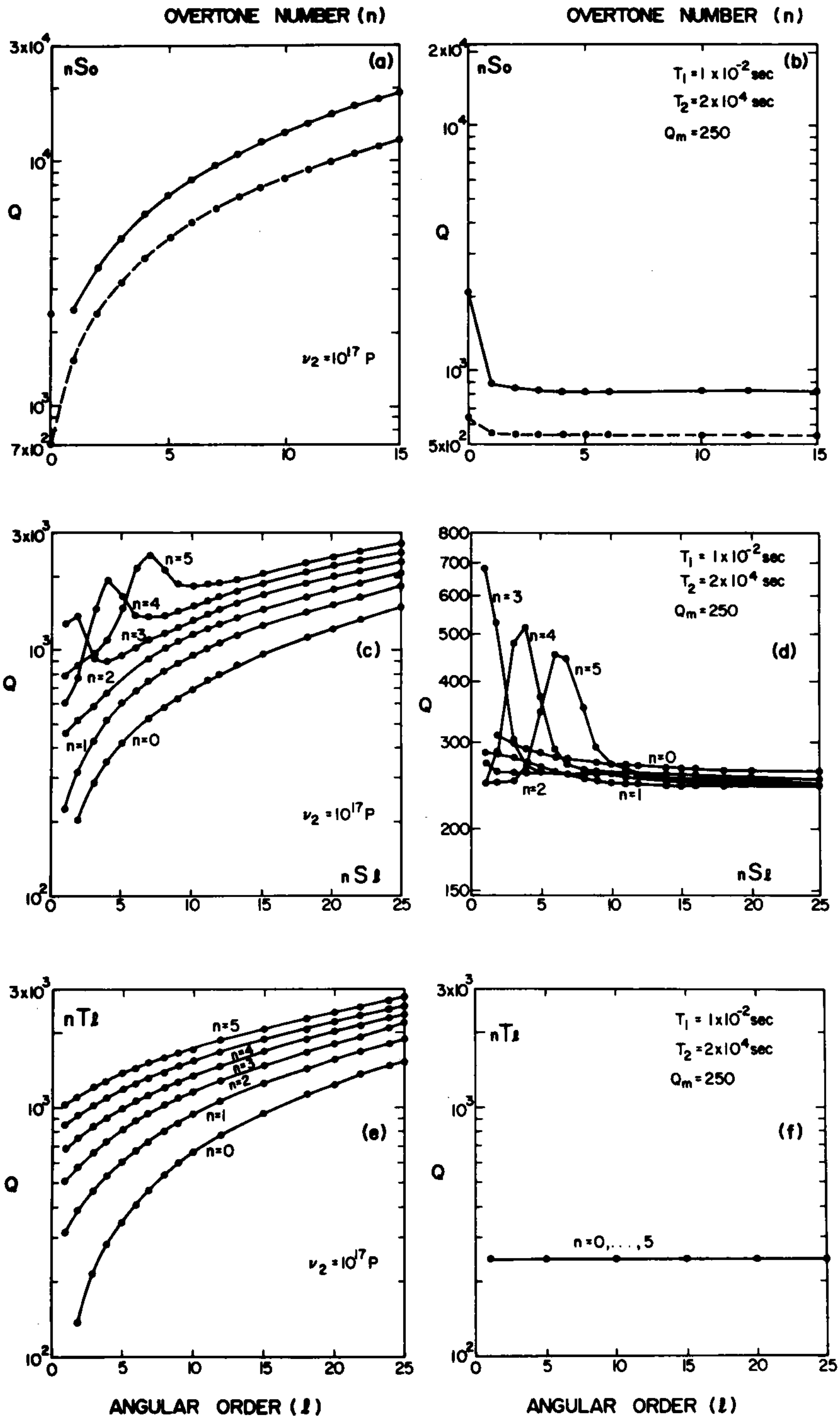

Figure 6. Q corresponding to the eigenfrequencies of Fig. 6. Rheological parameters of the SLS and absorption band model are given in the figure. See Table 2 for a detailed listing of representative Q values associated with the absorption band model. Dashed curves for radial modes are derived from a visco-elastic rheology proposed by Rundle (1978).

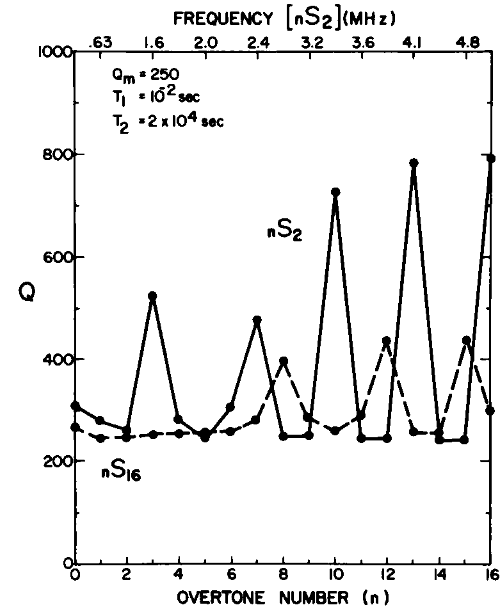

Figure 7. Q as a function of overtone number n for ${}_nS_2$ and ${}_nS_{16}$. The eigenfrequencies of ${}_nS_2$ are given at the top of the figure for convenience.

between ${}_0S_0$ and ${}_1S_0$ with relatively little variation thereafter. The sharpness of this drop is a consequence of the assumption that the viscoelastic model has no bulk dissipation. The viscoelastic model employed by Rundle (1978) in his analysis of post-seismic rebound in contrast, is an example of a viscoelastic rheology which has non-zero bulk dissipation. We have done calculations of $Q(l)$ using his rheology for the radial modes and these are shown as the dashed lines in panel (a) and (b) for a single Debye peak and absorption band versions of this physical model. For the absorption band version the ratio of $Q({}_0S_0)$ to $Q({}_1S_0)$ is very much smaller than in the model with no bulk dissipation (and in the Earth) so that the rheology is untenable.

Similar calculations to those presented here were carried out previously for the torsional mode sequence by Akopyan, Zharkov & Lyubimov (1978a, b). These authors also employed the correspondence principle and considered torsional free vibration spectra for several viscoelastic rheologies. However, they restricted their analysis to the real part of the complex secular function and used this to determine the complex eigenfrequencies. Our exact analysis shows that this procedure is highly inaccurate, even for the toroidal modes to which their attention was confined, and leads to errors in Q as large as 20 per cent. Their approximation cannot be employed at all for the spheroidal modes.

3.3 BOUNDS UPON THE ABSORPTION BAND PARAMETERS FROM THE HOMOGENEOUS MODEL

Although we cannot expect to constrain the absorption band parameters precisely by fitting the homogeneous model to a small subset of the observed complex vibration frequencies of the real Earth we can nevertheless illustrate the manner in which radical changes in the model parameters might serve to make it untenable. This is the purpose of the present subsection. Fig. 8 illustrates the effect upon the complex eigenfrequency of ${}_0S_2$ of separately

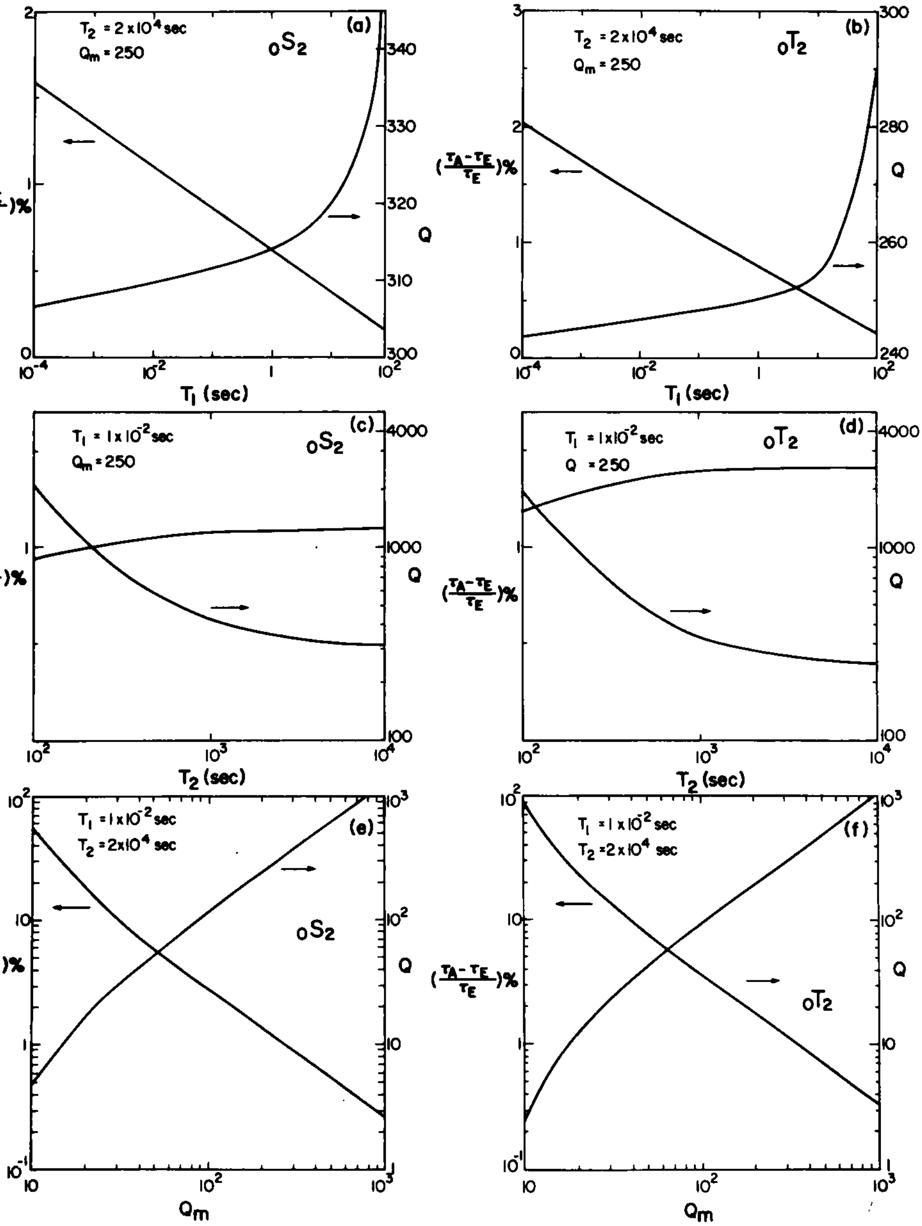

Figure 8. Shifts in eigenfrequencies and Q from variations in T_1, T_2 and Q_m of the absorption band model. Parameters which are held constant are displayed in the figure.

varying each of the three parameters (T_1, T_2, Q_m) of our absorption band. Inspection of (Fig. 9a, b) shows that T_1 cannot exceed 10^2 s without dramatically increasing the Q of $_0S_2$. Excessively small values of $T_1 \lesssim 10^{-4}$ s are likewise unacceptable since such would result in a noticeable shift (> 1 per cent) of the eigenfrequency. This would of course further complicate the issue of the baseline correction (Jeffreys 1965; Carpenter & Davies 1966; Davies 1967). From Fig. (8c–d) we see that T_2 is similarly constrained; $T_2 \lesssim 500$ s is implausible since this would effect a drastic rise in the Q of $_0S_2$. No upper bound on T_2 can be obtained from the seismic data alone. If we were to take $T_1 = 0.1$ s as suggested by Lundquist & Cormier

(1980) on the basis of body wave spectra and $T_2 = 500$ s (the minimum allowed) we obtain an absorption bandwidth of almost four decades. Minster & Anderson (1981) have suggested a somewhat narrower bandwidth of 2–3 decade. The effect of varying Q_m is illustrated in Fig. (8e–f). Reducing $Q_m (= 2/\pi\Delta)$ causes a dramatic shift in period and a precipitous decrease in Q. These data certainly require $Q_m > 10^2$ which implies $\Delta \gtrsim 9 \times 10^{-2}$. This conservative bound is nearly equal to the relaxation strength $\Delta = 8 \times 10^{-2}$ suggested by Minster & Anderson (1981) from their microphysical model based upon dislocation glide.

In Fig. 9 we address more closely the issue concerning the lower bound for T_2 by comparing $Q(l)$ for representative spheroidal overtones and $Q(n)$ for the radial modes for $T_2 = 10^2$ s and $T_2 = 2 \times 10^4$ s. For ${}_0S_0$ Q increases from 2086 to 8452 as T_2 is decreased from the upper to the lower value. Although the highest value for $Q({}_0S_0)$ is not outrageous (Buland

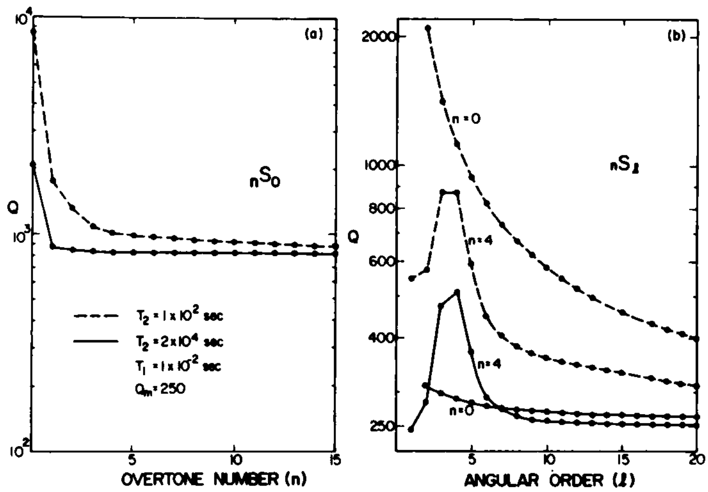

Figure 9. Changes in Q from variations of T_2 for spheroidal and radial modes. The number adjacent to each curve in (b) represents the overtone number. Dashed and solid curves denote $T_2 = 10^2$ and 2×10^4 s respectively. T_1 is held fixed at 10^{-2} s and Q_m is set to 250.

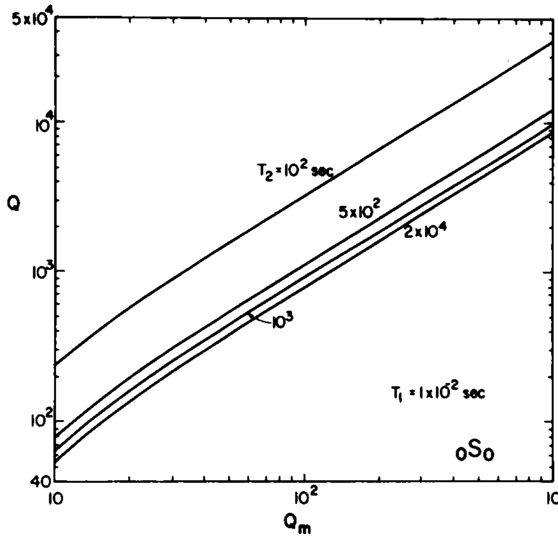

Figure 10. Q as a function of Q_m for a range of values in T_2. T_1 is held constant at 10^{-2} s.

Table 3. Comparison of period and Q of the mode ${}_0S_2$ for the generalized Burgers' body rheology (τ_B, Q_B) with those for the absorption band rheology (τ_0, Q_0). The parameters of the absorption band are $Q_m = 250$, $T_1 = 10^{-2}$ s, and $T_2 = 2\times10^4$ s.

ν_1 (poise)	$(Q_B - Q_0)/Q_0$ (per cent)	$(\tau_B - \tau_0)/\tau_0$ (per cent)
10^{22}	-6×10^{-4}	1×10^{-5}
10^{21}	-1×10^{-3}	3×10^{-5}
10^{20}	-2×10^{-3}	7×10^{-5}
10^{19}	-8×10^{-3}	4×10^{-4}
10^{18}	-3×10^{-1}	6×10^{-3}
10^{17}	-4×10	2×10^{-1}

et al. 1979; Knopoff *et al.* 1979) the decrease of T_2 simultaneously effects an increase of $Q({}_0S_2)$ from 310 to 2104. This effect on the modes ${}_4S_l$ is not so extreme. That a value of T_2 as low as 10^2 s is unacceptable is reinforced in Fig. 8(d) where we see that such a change increases the Q of ${}_0T_2$ from 247 to 1953. Fig. 10 shows $Q({}_0S_0)$ as a function of Q_m for a range of values of T_2 from 10^2 to 2×10^4 s and confirms our stated preference of $T_2 = 500$ s as a lower bound on the parameter.

One final point which we wish to establish concerning the eigenvibration spectra of the anelastic absorption band model discussed in this section is that these complex spectra do not depend upon the existence of the transition in the complete model from anelastic to viscous behaviour. The transformed shear modulus for the generalized Burgers' body advocated here is given in (14) in which the Maxwell time $T_m = \nu_1/\mu_R$. So long as T_m exceeds a few days or equivalently ν_1 exceeds about 10^{17}P, the presence of the transition to viscous behaviour produces no effect upon the complex eigenspectra. This is demonstrated in Table 3 where we show the variation of Q and period for ${}_0S_2$ between the complete generalized Burgers' body model and that for the prototype absorption band. The value of ν_1 required by the post-glacial rebound data is of the order of 10^{22}P (Peltier 1981) and inspection of the table shows that the long time-scale viscous behaviour of the complete model then contributes negligibly to the free vibration frequencies and Qs.

4 The accuracy of first-order perturbation theory

The complex eigenfrequencies for the anelastic normal modes discussed in the last section were computed exactly using the correspondence principle approach. Although this is relatively straightforward it is not the approach normally taken in conventional normal mode theory. In such analysis one normally exploits the fact that the dissipation is weak and employs the form of first-order perturbation theory embodied in Rayleigh's principle in order to correct the elastic vibration frequencies in such a way as to approximate their exact anelastic counterparts. Here we wish to assess the magnitude of the error which one might make in applying this approximation to the Earth.

Rayleigh's principle (Rayleigh 1877) has been employed in several forms for application to spherically symmetric elastic models (Pekeris & Jarosch 1958; Backus & Gilbert 1967) and to their anelastic counterparts (Gilbert 1980; Woodhouse 1980). Dahlen (1981) has recently extended the formalism to include the effects of lateral variations of anelasticity. The principle is based upon the fact that in the following relation (Backus & Gilbert 1967)

$$-s^2 \int_V dv \rho \mathbf{u}\cdot\mathbf{u} = \int_V dv\{K\hat{C} + \mu\hat{M} + \rho_0 u_i u_j \partial_i\partial_j\phi_0 + \rho_0\partial_j\phi_0(u_i\partial_i u_j - u_j\partial_i u_i)\}$$

$$+\int_E dv\left\{\frac{1}{4\pi G}|\nabla\phi_1|^2 + 2\rho_0 u_i\partial_i\phi_1\right\}, \qquad (21)$$

the quotient for s^2 is stationary under arbitrary allowable variations in the solution vector $(\mathbf{u}, \phi_1)$ if and only if $(\mathbf{u}, \phi_1)$ is an eigenfunction with squared eigenfrequency $-s^2$. In (21) ϕ_0 is the ambient gravitational potential, V the volume of the sphere, and E is the whole space. The other quantities in (21) are as defined in Backus & Gilbert (1967)

$$\hat{C} = |\nabla \cdot \mathbf{u}|^2$$
$$\Delta_{ij}\ \tfrac{1}{2}(\partial_i u_j - \partial_j u_i) - \tfrac{1}{3}\,\partial_k u_k \delta_{ij} \tag{22}$$
$$\hat{M} = 2\Delta_{ij}\Delta_{ij}.$$

We follow Peltier (1976) in using (21) to determine the sensitivity of the eigenvalue s against small changes of the shear modulus μ, effected here by the introduction of anelasticity. Since the models in which we are interested have no bulk dissipation, the first variation of (21) yields

$$\delta s = \frac{-\int_V \delta v\, \delta\mu(s)\Delta_{ij}\Delta_{ij}}{s_0 \int_V dv \rho_0 \mathbf{u}\cdot\mathbf{u}} \tag{23}$$

in which s_0 is the unperturbed (purely imaginary) elastic eigenvalue and the displacement vector $\mathbf{u}$ is constructed of the corresponding eigenfunction. Since s in (23) is complex, so is δs. Separating (23) into real and imaginary parts we obtain the following expressions for the shift of frequency (δ^1) and the Q of the mode as:

$$\delta^1 = \frac{I_m(\delta s)}{s_0} = \frac{R_e(D)}{2s_0^2 T} \tag{24a}$$

$$Q^1 = \frac{s_0}{2\,R_e(\delta s)} = \frac{-I_m(D)}{s_0^2 T} \tag{24b}$$

in which

$$D = \int_V \delta\mu(s)\,\Delta_{ij}\,\Delta_{ij}\,dv \tag{25a}$$

and

$$T = \int_V \rho\,\mathbf{u}\cdot\mathbf{u}\,dv. \tag{25b}$$

In terms of the scalar function W in (18) the integrals D and T for toroidal modes are

$$D_t = \int_0^a \delta\mu(s, r)\left[\left(\partial_r W - \frac{W}{r}\right)^2 + (l+2)(l-1)\frac{W^2}{r^2}r^2\right] dr \tag{26a}$$

$$T_t = \int_0^a \rho W^2 r^2\, dr. \tag{26b}$$

Equations (24a, b) were previously derived by Liu & Archambeau (1975) for toroidal modes using a Rayleigh-Schrödinger perturbation expansion. For spheroidal modes the integrals D

and T in (24) may be expressed in terms of the scalars U and V which appear in (18) as

$$D_s = \int_0^a \delta\mu(s, r) M(r) r^2 \, dr \tag{27a}$$

$$T_s = \int_0^a \rho \left[U^2 + l(l+1) V^2\right] r^2 \, dr \tag{27b}$$

where

$$M = \frac{1}{3}\left(2\partial_r U - \frac{2U}{r} + (l+1)\frac{lV}{r}\right)^2 + l(l+1)\left[\left(\partial_r V + \frac{U}{r} - \frac{V}{r}\right)^2 + \frac{(l-1)(l+2)}{r^2} V^2\right]. \tag{27c}$$

To determine D_t or D_s for use in (24) we need $\delta\mu(s, r)$ and this is clearly a function of the rheology through the shear modulus $\mu(s, r)$.

For the SLS the transformed shear modulus is given in (13a). This may be written in the form $\mu(s) = \mu_1 + \delta\mu$ where μ_1 is the elastic modulus so that

$$\delta\mu = \frac{\mu_1}{(s_0^2 + B^2)} \left[(s_0^2 + AB) + i(B - A) s_0\right] \tag{28a}$$

in which we have taken $s = is_0$ in evaluating $\delta\mu$ since the dissipation is weak. The constants A and B in the above expression are

$$A = (\mu_1 + \mu_2)/\nu_2 \tag{28b}$$

$$B = \mu_2/\nu_2. \tag{28c}$$

For the absorption band rheology, on the other hand, the shear modulus is given by (14) which may similarly be written in the form $\mu(s) = \mu_1 + \delta\mu$ where now $\delta\mu$ is given by

$$\delta\mu = \frac{2\mu_1}{\pi Q_m} \ln\left[\frac{is_0 + 1/T_2}{is_0 + 1/T_1}\right]. \tag{29}$$

For the homogeneous earth models such as concern us here, which are subject to depth-independent imperfections of elasticity, $\delta\mu$ may be taken outside the integral in (25a) and expressions (24) are considerably simplified. This is particularly true for the toroidal modes since for them the total energy is equipartitioned between shear and kinetic. Equations (24) then become

$$\delta_T^1 = -\frac{R_e(\delta\mu)}{2s_0\mu_1} \tag{30a}$$

$$Q_T^1 = \frac{\mu_1}{I_m(\delta\mu)}. \tag{30b}$$

In general the integrals (25) which appear in (24) must be evaluated numerically and for this purpose we have employed the simple adaptive quadrature algorithm in Lyness (1969). Our purpose here is to compare the predictions of perturbation theory embodied in (24) to the exact results discussed in the last section. As a check on the accuracy of the quadrature required to evaluate the predictions (24) for radial and spheroidal modes we have evaluated an energy budget for each mode as

$$F(\mathbf{u}, \mathbf{u}) = S(\mathbf{u}, \mathbf{u}) + C(\mathbf{u}, \mathbf{u}) + G(\mathbf{u}, \mathbf{u}) \tag{31}$$

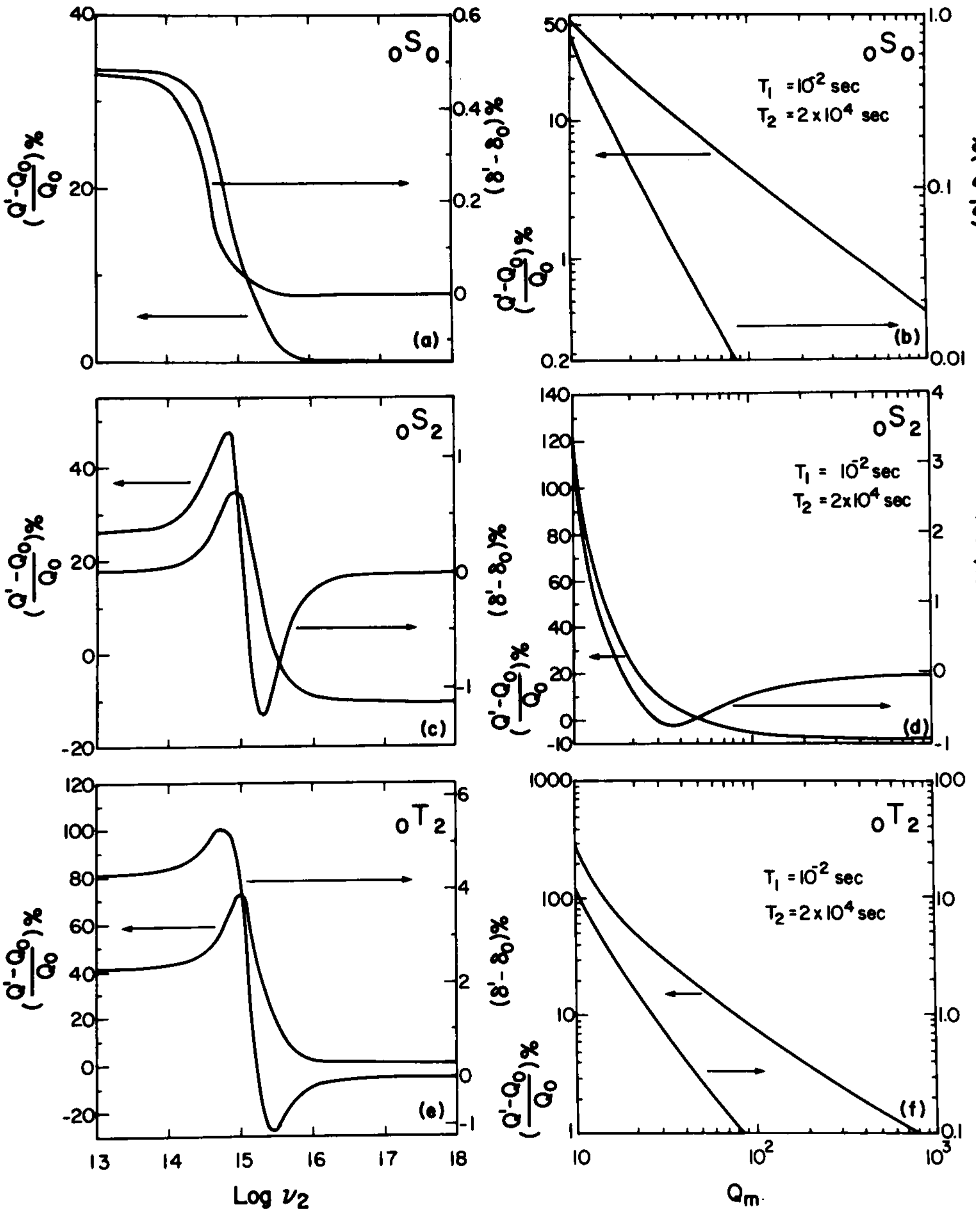

Figure 11. Breakdown of first-order perturbation theory as a result of extreme reduction in ν_2 and Q_m. Rheological parameters which are not varied are shown in the figure. Definitions of $(Q' - Q_0/Q_0)$ per cent and $(\delta' - \delta_0)$ per cent are given in the text.

in which F, S, C and G are respective bilinear forms in the displacement vector **u** which denote the kinetic, shear, compression and gravitational contributions to the energy in the mode. In all cases reported here the energy budget (31) is satisfied to at least one part in 10^8.

Fig. 11 illustrates the gradual breakdown of perturbation theory for both the SLS (panels a, c, e) and for the absorption band (panels b, d, f) as the strength of the anelasticity increases. Results are shown for the fundamental radial mode $_0S_0$, the spheroidal mode $_0S_2$, and the toroidal mode ${}_0T_2$. The quantities compared on the figure are $\delta^1 - \delta_0$ and $(Q^1 - Q_0)/Q_0$ where $\delta_0 = (\tau_A - \tau_E)/\tau_A$ with τ_E the elastic and τ_A the exact anelastic period of the mode and Q_0 the modal Q determined through exact eigenanalysis. Majda, Chin & Followill (1978) discussed the radius of convergence of first-order perturbation theory in connection with an analysis of Love wave propagation in an anelastic half-space. Our analyses appear to be the first to provide a direct investigation of this question for the normal modes. Inspection of Fig. 11 shows that for weak anelasticity the frequency shift is predicted much more accurately by the first-order theory than is modal Q (which is not unexpected). For the SLS, the largest errors for spheroidal modes are found with ν_2 in the neighbourhood of 10^{15} P. In the case of the absorption band, decreasing Q_m (or increasing Δ) such that $Q_m \lesssim 10^2$ the errors can be substantial indeed. It may be implied by this result that the inference of 'intrinsic' Q in high attenuation zones within the mantle (such as exist beneath young oceanic lithosphere) using first-order perturbation theory could give rise to substantial error. In their explanation of the seismic low-velocity zone, for example, Minster & Anderson (1980) suggest a relaxation strength $\Delta \simeq 0.08$ which translates to a local $Q_m \approx 80$ to account for the approximately 10 per cent decrease of seismic velocity. If Q_m were this low it is not completely clear that it could be estimated accurately using first-order theory. Similarly, most present schemes for the inversion of surface wave dispersion and attenuation data (e.g. Lee & Solomon 1978) are based upon the first-order perturbation theory of Anderson & Archambeau (1964) and such retrievals could be in error if significant low Q zones are present.

A complete set of comparisons for toroidal and spheroidal fundamental modes with $2 < l < 25$ is shown in Fig. 12 for the absorption band rheology. The errors incurred in the

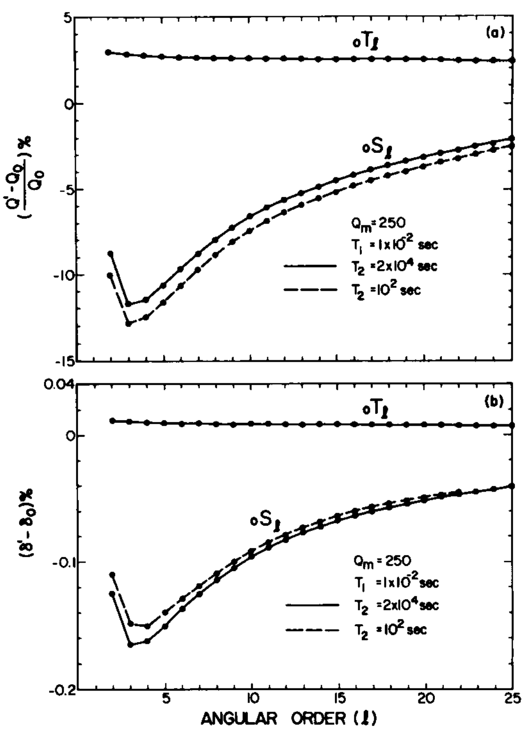

Figure 12. Results for the accuracy of perturbation theory for the fundamental spheroidal and toroidal free oscillations. Solid and dashed curves represent $T_2 = 2 \times 10^4$ s and 10^2 s respectively. $Q_m = 250$ and $T_1 = 10^{-2}$ s are held constant.

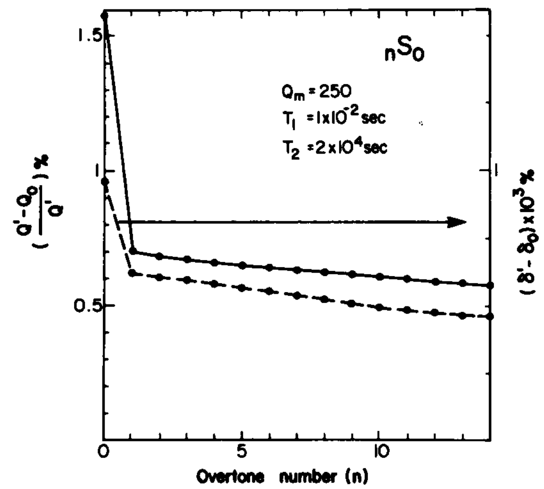

Figure 13. Results showing the accuracy of Rayleigh's principle for the radial modes. Rheological parameters are displayed in the figure.

first-order theory are generally smaller for toroidal than for spheroidal modes and of opposite sign. Maximum errors in $(Q^1 - Q_0)/Q_0$ of ≈ 10 per cent obtain for spheroidal modes of low angular order while the frequency shifts are predicted to an accuracy better than 0.2 per cent. The efficacy of perturbation theory is least for those spheroidal modes in which a significant fraction of the total energy is gravitational potential or compressional. Fig. 13 shows the results of equivalent calculations for radial modes. Except for ${}_0S_0$ for which the error in Q is ≈ 1.5 per cent, the Q errors for the overtones are less than 1 per cent. Frequency shifts are predicted to an accuracy of 10^{-3}. Results for the toroidal and spheroidal mode overtone sequences with $l = 2$ and 16 are shown in Fig. 14. Again errors for the toroidal modes are small with Qs accurate to within a few per cent and frequency shifts accurate to 0.01 per cent. For the spheroidal overtones with $l = 2$ the errors oscillate initially in the

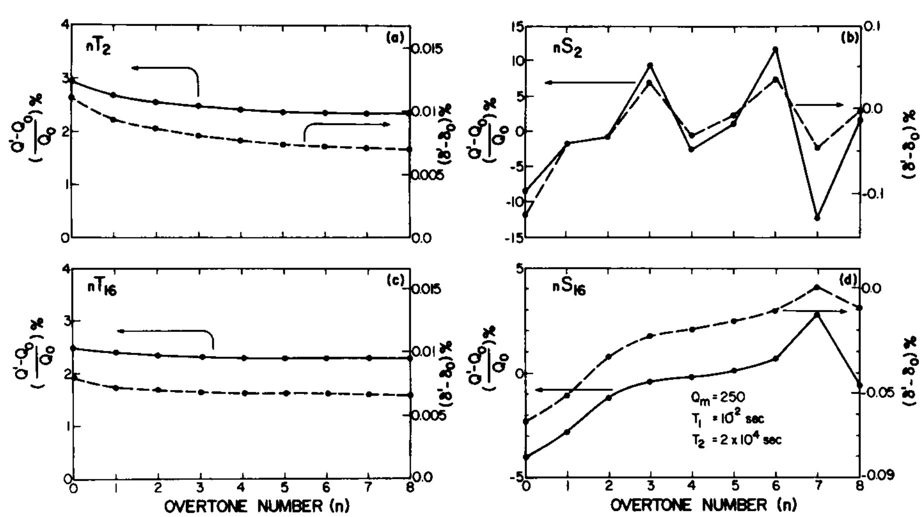

Figure 14. Results of first-order perturbation theory for the toroidal and spheroidal overtone sequence. The rheological constants for all calculations are $Q_m = 250$, $T_1 = 10^{-2}$ s, and $T_2 = 2 \times 10^4$ s.

range ±10 per cent decreasing by $n = 16$ (not shown) to ±5 per cent. The errors for the spheroidal modes with angular order 16 are considerably smaller.

Taken overall the analysis in this section is encouraging in that for the plausible range of parameters in the absorption band, perturbation theory predictions of Q are accurate at the 5 per cent level with the exception of the lowest order spheroidal modes for which the errors may be somewhat in excess of 10 per cent. Even these errors, however, are smaller than the current observational uncertainties (Buland *et al.* 1979; Stein *et al.* 1981). It is important, though, to be cognizant of the fact that these errors could be substantially magnified by sharp radial variations of the anelastic parameters. We have not attempted to assess the possible importance of such error magnification here but intend to do so in the course of future work. As the quality of the observational data set improves it may become necessary to solve the viscoelastic normal mode problem in the self-consistent fashion described in the previous section. In the next section we shall address our attention to the quasi-static modes whose determination requires the exact eigenanalysis since they have no counterpart in the elastic spectrum.

5 The quasi-static spectrum and viscoelastic polar motion induced by earthquakes

In Section 3.2 we remarked on the existence in the complete viscoelastic spectrum for the homogeneous SLS model of discrete poles on the negative real axis in the complex s-plane and noted that since these modes had no elastic counterparts they could not be investigated using the methods of perturbation theory discussed in the last section. For the absorption band rheology and equivalent continuum of such modes exists with relaxation times extending from T_1 to T_2; mathematically this continuum manifests itself in the existence of branch-points of the secular function at $s = -1/T_1$ and $s = -1/T_2$. Although the existence of these modes is of no consequence in so far as the free vibrations are concerned, they may play an important role in other geophysical phenomena. Indeed, for the generalized Burgers' body with $\mu(s)$ given by (29) it is precisely the quasi-static modes supported by the long time-scale viscous behaviour of the rheological model which govern the phenomenon of glacial isostatic adjustment (Peltier 1974, 1976, 1980). Even in the absence of the viscous component of the response, however, similar quasi-static modes with much shorter relaxation times are supported by the anelasticity. These modes are the ones which concern us here and they will play an important role in the phenomenon of post-seismic rebound and similar tectonic processes. Half-space models with SLS rheology have been employed in the study of transient tectonic movements following large earthquakes by Nur & Mavko (1974) and Yamashita (1979). These authors found that values of the short time-scale viscosity ν_2 which were required to fit their data had $10^{17}\,\mathrm{P} \gtrsim \nu_2 \lesssim 10^{18}\,\mathrm{P}$. It is important to note that this is the same value for the short time-scale viscosity which is required to explain the observed Qs of the free oscillations (Section 3.2). It seems clear therefore that if we fix ν_2 by fitting the rheological model to normal mode Qs the same linear model will also correctly predict the phenomenon of post-seismic rebound. This seems to be an important cross-check on the validity of the model itself.

In Table 4 we list the relaxation times $\tau = 1/s_r$ for the homogeneous SLS sphere with $\nu_2 = 10^{17}\,\mathrm{P}$. For each spherical harmonic degree l there is one toroidal mode and one spheroidal mode. The radial equation also supports a mode of relaxation. In the table we show the relaxation times for the quasi-static modes ${}_{\mathrm{R}}S_0$, ${}_{\mathrm{R}}S_2$, ${}_{\mathrm{R}}T_2$ where the letter R denotes quasi-static. The 15.94 hr relaxation time for ${}_{\mathrm{R}}S_2$ of the compressible models may be compared with the 15.91 hr relaxation time computed from the analytic expression for the incompressible homogeneous sphere given in Wu & Peltier (1982). As an example of the

Table 4. *E*-folding times of quasi-static modes for standard linear solid rheology with $\nu_2 = 10^{17}$P

Mode	*E*-folding time (hr)
${}_0S_0$	11.54
${}_0S_2$	15.94
${}_0T_2$	9.59

physical importance of these modes in a problem of global scale we shall proceed to sketch their influence upon the polar motion excited by an earthquake.

5.1 POLAR MOTION FORCED BY A DISLOCATION SOURCE

Two main mechanisms have been seriously considered as possibly being responsible for the continuous re-excitation of the Earth's free Eulerian nutation; the atmospheric circulation (Munk & Macdonald 1960; Munk & Hassan 1961; Wilson & Haubrich 1976), and earthquakes (Mansinha & Smylie 1967; Ben Menahem & Israel 1970; Dahlen 1971, 1973; Smylie & Mansinha 1971; Rice & Chinnery 1972; Mansinha, Smylie & Chapman 1979) or some combination of the two (O'Connell & Dziewonski 1976). It seems well established at this point that the atmospheric excitation is not in itself of sufficient intensity. The efficacy of the earthquake mechanism is difficult to establish since its determination requires knowledge of the seismic moment which must be determined empirically from the earthquake magnitude (e.g. O'Connell & Dziewonski 1976) and the form which the relation between these two parameters takes is rather controversial (Kanamori 1976). Most current analyses, however, agree in concluding that the elastic excitation engendered in an earthquake is also an inadequate means of excitation. One way out of this dilemma is to invoke a substantial contribution to the excitation due to aseismic slip (Kanamori 1976) and there is some evidence to support this possibility. Recently Slade, Melosh & Raefsky (1979) have suggested that viscoelasticity might act in such a way as to enhance the elastic excitation and treat this process in terms of an assumed non-Newtonian rheology. Since such a rheology will behave to first order as a linear rheology with a different relaxation time we may employ the generalized Burgers' body which fits the seismic and post-glacial rebound data to assess the viability of their mechanism.

Assuming that the earthquake induces a small perturbation ΔI_{ij} to the inertia tensor of the undisturbed earth the resulting polar motion may be described in terms of the vector $\mathbf{n} = (n_1, n_2, n_3)^{\mathrm{T}}$. The elements of $\mathbf{n}$ are the direction cosines of the displacement of the axis rotating with angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)^{\mathrm{T}}$ which is the mean angular velocity of the Earth. In the absence of external torques, the Euler equations for the conservation of angular momentum are (Munk & MacDonald 1960)

$$\frac{d\mathbf{n}}{dt} - \mathrm{D} \cdot \mathbf{n} = \mathbf{f} \tag{32}$$

where

$$\mathrm{D} = \begin{pmatrix} 0 & -\sigma & 0 \\ +\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{f} = \begin{pmatrix} \dfrac{\Delta I_{23}\sigma}{(C-A)} \quad \dfrac{\Omega\Delta \dot{I}_{13}\sigma}{(C-A)} \\ \dfrac{-\Delta I_{13}\sigma}{(C-A)} - \dfrac{\Omega\Delta \dot{I}_{23}\sigma}{(C-A)} \\ \dfrac{-\Delta I_{33}\sigma}{C} \end{pmatrix} \tag{33}$$

in which C and A are respectively the axial and equatorial moments of inertia, $\sigma = (C-A/A)\Omega$ is the Chandler frequency for a rigid earth and the dots denote time differentiation. Introducing the complex variables $\mathbf{m} = n_1 + in_2$ and $\mathbf{g} = f_1 + if_2$ the first two elements of the vector equation (32) may be written as

$$\frac{dm}{dt} - i\sigma \mathbf{m} = \mathbf{g}. \tag{34}$$

To deduce the polar motion $\mathbf{m}$ we have only to determine the forcing ΔI_{ij} due to the earthquake. For this purpose we can again invoke the Correspondence Principle. Elastic solutions for the ΔI_{ij} due to a point dislocation in a uniform incompressible self-gravitating sphere were deduced by Rice & Chinnery (1972) from the principal of virtual work. This solution has the form

$$\Delta I_{ij} = \frac{\Delta u \Sigma J_{ij}(\Sigma)}{(1 + 2\rho_0 ga/19\mu_1)} \tag{35}$$

where Δu is the slip on the surface of the fault plane of area Σ and $J_{ij}(\Sigma)$, which has the dimensions of (mass/length) is a tensor function obtained from the geometrical projection of the shear stress on to the fault plane. We may determine the equivalent $\Delta I_{ij}(t)$ for a viscoelastic rheology simply by substituting the appropriate $\mu(s)$ in place of μ_1 and inverting the resulting $\Delta I_{ij}(s)$ as

$$\Delta I_{ij}(t) = \frac{\Delta u \Sigma J_{ij}(\Sigma)}{2\pi i} \int_L \frac{\exp(st)\, ds}{s[1 + 2\rho_0 ga/19\mu(s)]} \tag{36}$$

where we have taken $\Delta u(t) = \Delta u H(t)$ where $H(t)$ is the Heaviside function and where L is the Bromwich path.

For the SLS rheology $\Delta I_{ij}(t)$ is obtained by substituting (15a) into (36). After some simplification the temporal behaviour $\Delta \bar{I}(t) = \Delta I_{ij}(t)/\Delta u \Sigma J_{ij}(\Delta)$ may be written as

$$\Delta \bar{I}(t) = \frac{W}{2\pi i} \int_L ds \, \frac{\exp(st)}{s} \left[1 + \frac{(\mu_1/\nu_2 - b)}{s + b}\right] \tag{37}$$

where

$$W = \frac{1}{1 + 2\rho_0 ga/19\mu_1}; \qquad b = \frac{19\mu_1\mu_2 + 2\rho_0 ga(\mu_1 + \mu_2)}{\nu_2(19\mu_1 + 2\rho_0 ga)}. \tag{38}$$

The Laplace inversion in (37) may be expressed analytically and the result is

$$\Delta \bar{I}_{\mathrm{SLS}}(t) = W\left[1 + \left(\frac{\mu_2}{\nu_2 b} - 1\right)(1 - \exp(-bt))\right]. \tag{39}$$

The inverse relaxation time $b = s_r$ is just the relaxation time of the quasistatic pole in the viscoelastic spectrum ${}_{\mathrm{R}}S_2$. The result of anelasticity is clearly to produce a very slight *decrease* with time of the wobble excitation. It will also of course endow the wobble with a finite Q.

For a Maxwell rheology this decrease of excitation would tend towards a zero limit rather than to the finite limit which obtains for the SLS. This can be seen by substituting $\mu(s) = \mu_1 s/(s + \mu_1/\nu_1)$ for the Maxwell model (Peltier 1974) into (36). Inverting the transform then gives

$$\Delta \bar{I}_{\mathrm{M}}(t) = W \exp(-ct) \tag{40}$$

where the inverse of the relaxation time is

$$c = \frac{2\rho_0 a\mu g}{\nu_1(19\mu_1 + 2\rho_0 g a)}. \tag{41}$$

Since the excitation eventually vanishes this means that there is no net displacement of the spin axis as a consequence of the earthquake. Such a net offset would be produced by the SLS result (39). Since the Earth seems to be a generalized Burgers' body in which the slight anelastic relaxation of the SLS is followed by complete viscous relaxation of the Maxwell element there can never be any net polar wander effected by earthquakes in the internal body of the planet. Furthermore, because the strain field produced by the dislocation is of internal origin it cannot be enhanced by viscoelastic effects. Such effects can only decrease the efficiency of Chandler wobble excitation. This result will not be affected by the introduction of an absorption band description of the anelasticity, but might conceivably be altered for models with radial variations of the rheological parameters such as are produced by the presence of the lithosphere.

6 Conclusions

This paper has been concerned with a first assessment of the generalized Burgers' body representation of the viscoelasticity of the Earth's mantle which was introduced in Peltier *et al.* (1981). Even though the frequency-dependent shear modulus for such material may be well approximated for most purposes by the simple analytic expression (14), it nevertheless seems possible with this expression to simultaneously reconcile all of the geodynamic phenomena noted on Fig. 1. These phenomena include the elastic gravitational free oscillations which are sensitive only to the anelastic component of the model and mantle convection, which is an entirely viscous phenomenon. The importance of post-glacial rebound is that it is simultaneously sensitive to both the elastic and viscous components of the response. When seismically determined elastic moduli are employed in the theory of post-glacial rebound (Peltier 1981a) then a viscosity of order 10^{22} P (cgs units) is required to reconcile the observational data. This is the same viscosity which is required by convection models of the seafloor spreading process (Peltier 1981a). We are therefore in a good position to suggest the generalized Burgers' body as a uniformly valid representation of the rheology of the Earth's mantle.

By direct numerical calculation of the complex vibration spectra for homogeneous earth models, we have established the generalized Burgers' body as an appropriate vehicle for the description of seismic Q. With characteristic time-scales less than the Maxwell time, geodynamic phenomena are essentially oblivious of the eventual viscous behaviour of the rheology. For the Earth, the Maxwell time appears to be several hundred years and this ensures that even the Chandler wobble will be governed entirely by anelastic processes.

Our analysis of the viscoelastic free vibrations has established the first direct estimate of the errors incurred in the use of first-order perturbation theory as a means of calculating free oscillation Qs from a given model of intrinsic dissipation. Although such errors are not yet observationally significant they may exceed 10 per cent and are largest for the lowest order modes in the homogeneous models to which we have confined our attention. An equally important point which has emerged from the calculations discussed here concerns the existence of the quasi-static modes which are supported by the anelastic component of the complete rheology. These modes are the short time-scale counterparts of the viscous gravitational free decay modes in terms of which the post-glacial rebound phenomenon is described. They too require exact eigenanalysis for their determination since they have no counterparts in the elastic spectrum. These modes play a crucial role in short time-scale geodynamic phenomena such as a post-seismic rebound but have yet to be exploited in this context.

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