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NORMAL SEMIGROUPS OF ENDOMORPHISMS OF PROPER INDEPENDENCE ALGEBRAS ARE IDEMPOTENT GENERATED

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Abstract Let $\mathcal{A}$ be a proper independence algebra of finite rank, let G be the group of automorphisms of $\mathcal{A}$, let a be a singular endomorphism and let a^G be the semigroup generated by all the elements $g^{-1}ag$, where $g \in G$. The aim of this paper is to prove that a^G is a semigroup generated by its own idempotents.

Keywords: independence algebras; normal semigroups; endomorphisms

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1. Introduction

Let X be a finite set. We denote by $T(X)$ the monoid of all (total) transformations on X and by $\mathrm{Sym}(X)$ the symmetric group on X. An element $a \in T(X) \setminus \mathrm{Sym}(X)$ is said to be singular.

Howie proved in 1966 [**5**] that every singular transformation $a \in T(X)$ can be expressed as a product of idempotents of $T(X)$.

This result was generalized by Fountain and Lewin [**3**] for the case of independence algebras as follows. Let $\mathcal{A}$ be an independence algebra of finite rank and let $\mathrm{End}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$ be the monoid of endomorphisms and the group of automorphisms of $\mathcal{A}$, respectively. Given an element $a \in \mathrm{End}(\mathcal{A})$ we define the rank of a, $\mathrm{rank}(a)$, as $\mathrm{rank}(\mathrm{im}(a))$. An endomorphism $a \in \mathrm{End}(\mathcal{A})$ is said to be singular if $\mathrm{rank}(a) < \mathrm{rank}(\mathcal{A})$. Fountain and Lewin proved that every singular endomorphism $a \in \mathrm{End}(\mathcal{A})$ can be expressed as a product of idempotents of $\mathrm{End}(\mathcal{A})$.

In a different direction, Levi and McFadden [**7**] extended Howie's result proving that given a singular transformation $a \in T(X)$, the semigroup generated by all its conjugates $g^{-1}ag$, with $g \in \mathrm{Sym}(X)$, is generated by its own idempotents.

In fact, Levi and McFadden proved something more. As usual, if S is a semigroup, $E(S)$ denotes the set of idempotents of S, and if $\emptyset \neq T \subseteq S$, by $\langle T \rangle$ we denote the semigroup generated by T. Levi and McFadden proved the following theorem.

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Theorem 1.1. *Let* $a \in T_n \setminus \mathrm{Sym}(X)$ *and* $S = \langle \{g^{-1}ag \mid g \in \mathrm{Sym}(X)\}\rangle$. *Then*

$$S = \langle \{a\} \cup \mathrm{Sym}(X)\rangle \setminus \mathrm{Sym}(X) = \langle E(S)\rangle.$$

Our aim is to generalize this result to the case of independence algebras. We would like to have a result reading as follows: let $\mathcal{A}$ be a finite-rank independence algebra and let $a \in \mathrm{End}(\mathcal{A})$ be a singular endomorphism. Then the semigroup generated by the set $\{g^{-1}ag \mid g \in \mathrm{Aut}(\mathcal{A})\}$ is generated by its own idempotents.

However, this result is not true. To see this consider $X = \{0, 1, 2\}$ and the independence algebra $\mathcal{A} = (X, 0)$, an algebra with only one operation which is constant. Let a be the endomorphism of $\mathcal{A}$ defined by $0a = 1a = 0$ and $2a = 1$. Then $a \in \mathrm{End}(\mathcal{A}) \setminus \mathrm{Aut}(\mathcal{A})$ but we have*

$$\langle \mathrm{Aut}(\mathcal{A})\, a\ \mathrm{Aut}(\mathcal{A})\rangle \cong \langle S_2\, (21]\, S_2\rangle.$$

The latter is an inverse semigroup and has elements that are not idempotents, so it is not generated by idempotents and hence $\langle \mathrm{Aut}(\mathcal{A})\, a\ \mathrm{Aut}(\mathcal{A})\rangle$ is not generated by idempotents either.

In the sequel we consider a particular class of independence algebras—proper independence algebras—in which a result corresponding to that of Levi and McFadden holds. This class of algebras is broad enough to contain the most important examples of independence algebras, namely, sets, free G-sets and vector spaces.

In § 2 we introduce proper independence algebras and prove some basic results. In § 3 we prove that, for any singular endomorphism $a \in \mathrm{End}(\mathcal{A})$, the semigroup $\langle \{a\} \cup \mathrm{Aut}(\mathcal{A})\rangle$ is generated by its own idempotents. As a corollary we derive the Fountain and Lewin theorem for proper independence algebras. As sets and vector spaces are proper independence algebras, the result is general enough to contain, as particular cases, both Howie's [**5**] and Erdos's [**2**] classical theorems.

Finally, § 4 is devoted to the study of semigroups generated by the set $\{g^{-1}ag : g \in \mathrm{Aut}(\mathcal{A})\}$, where $a \in \mathrm{End}(\mathcal{A}) \setminus \mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$ is a periodic group. The main result of this section generalizes Theorem 1.1.

2. Preliminaries

We assume that the reader has a basic knowledge of both the theory of independence algebras and the theory of semigroups. For independence algebras we recommend [**3**] and [**4**] as references, and for general semigroup theory we recommend [**6**].

The first step in the definition of independence algebras is the introduction of a notion of independence valid for universal algebras. A subset X of an algebra is said to be *independent* if $X = \emptyset$ or if, for every element $x \in X$, we have $x \notin \langle X \setminus \{x\}\rangle$; a set is *dependent* if it is not independent.

Lemma 2.1. *For an algebra* $\mathcal{A}$*, the following conditions are equivalent.*

* In a notation that is now standard, the partial one-to-one mapping on the set $\{1, 2\}$ which sends 2 to 1, is represented by (21].

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(1) *For every subset X of A and all elements u, v of A, if the element $u \in \langle X \cup \{v\} \rangle$ and $u \notin \langle X \rangle$, then $v \in \langle X \bigcup \{u\} \rangle$.*

(2) *For every subset X of A and every element $u \in A$, if X is independent and $u \notin \langle X \rangle$, then $X \cup \{u\}$ is independent.*

(3) *For every subset X of A, if Y is a maximal independent subset of X, then $\langle X \rangle = \langle Y \rangle$.*

(4) *For subsets X, Y of A with $Y \subseteq X$, if Y is independent, then there is an independent set Z with $Y \subseteq Z \subseteq X$ and $\langle Z \rangle = \langle X \rangle$.*

Proof. See [**9**, Exercise 6, p. 50]. □

An algebra $\mathcal{A}$ is said to have the *exchange property* or to satisfy [EP] if it satisfies the equivalent conditions of Lemma 2.1. A basis for $\mathcal{A}$ is a subset of A which generates A and is independent. It is clear from Lemma 2.1 that any algebra with [EP] has a basis. Furthermore, for such an algebra, bases may be characterized as minimal generating sets or maximal independent sets, and all bases for $\mathcal{A}$ have the same cardinality [**4**, Proposition 3.30]. This cardinal is called the rank of $\mathcal{A}$ and is written $\operatorname{rank}(\mathcal{A})$. If $\mathcal{A}$ is an algebra satisfying [EP] and $\alpha \in \operatorname{End}(\mathcal{A})$, then $\operatorname{rank}(\alpha)$ is the rank of the image of α, that is, $\operatorname{rank}(\alpha) = \operatorname{rank}(\operatorname{im}(\alpha))$.

We observe that part (4) of Lemma 2.1 tells us that any independent subset of A can be extended to a basis for $\mathcal{A}$. We also remark that if $\mathcal{A}$ satisfies [EP], then so does any subalgebra of $\mathcal{A}$.

Throughout this paper $\mathcal{A}$ will always denote an independence algebra of finite rank with at most one constant. Thus, in the algebras under consideration, $Con = \emptyset$ or $Con = \{0\}$, where Con denotes the set of constants of $\mathcal{A}$.

Let $\mathcal{A}$ be an independence algebra and let X, Y be two disjoint and independent subsets of A. Then $\mathcal{A}$ is said to be *strong* if $\langle X \rangle \cap \langle Y \rangle = Con$ implies that $X \cup Y$ is an independent set. From now on we restrict our study to the case of strong independence algebras.

Lemma 2.2. *Let $\mathcal{B}$ and $\mathcal{C}$ be subalgebras of $\mathcal{A}$. If B is a basis for $\mathcal{B} \cap \mathcal{C}$, $B \cup C$ is a basis for $\mathcal{B}$ and $B \cup D$ is a basis for $\mathcal{C}$, then $B \cup C \cup D$ is a basis for the algebra generated by $\mathcal{B}$ and $\mathcal{C}$.*

Proof. See [**3**, Lemma 1.6]. □

Definition 2.3. Let I be a set and for a symbol $0 \notin I$, let $I_0 = I \cup \{0\}$. Moreover, let $\mathcal{A}$ be an independence algebra and let $(A_i)_{i \in I}$ be a partition of a basis of $\mathcal{A}$. Consider the endomorphism $\alpha \in \operatorname{End}(\mathcal{A})$ defined by $A_i\alpha = \{a_i\}$, for $i \in I$, where the set $\{a_i : i \in I\}$ is an independent set (and hence a basis for $\operatorname{im}(\alpha)$), and let $A_0\alpha = \{0\}$. An endomorphism $\alpha \in \operatorname{End}(\mathcal{A})$ under these conditions is represented by the following matrix

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i \in I}.$$

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This matrix is said to be a *fundamental representation* of α. The set A_0 in the fundamental representation is said to be the *constant component*.

If the algebra has no constants, then the constant component is the empty set and then the endomorphism can be defined by

$$\begin{bmatrix} A_i \\ a_i \end{bmatrix}_{i\in I}.$$

The importance of this concept lies in the following fact.

Theorem 2.4. *Every endomorphism of a strong independence algebra admits a fundamental representation.*

Proof. This follows from Lemma 2.8 and the observations following Corollary 2.10 of [**3**]. □

It is worth observing that the previous theorem does not imply that given $a \in \mathrm{End}(\mathcal{A})$ and a basis B of $\mathcal{A}$, there is a partition of B, say $(A_i)_{i\in I_0}$, such that

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I}$$

is a fundamental representation of a. What the previous theorem implies is that for every $a \in \mathrm{End}(\mathcal{A})$ there exist a basis B of $\mathcal{A}$ and a partition of that basis, say $(A_i)_{I\cup\{0\}}$, such that

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I}$$

is a fundamental representation for a. In fact we can say more. For every $a \in \mathrm{End}(\mathcal{A})$ and every basis C of $\mathrm{im}(a)$, there is a basis of $\mathcal{A}$, say $B = \cup_{I_0} A_i$, such that

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I}$$

is a fundamental representation for a. In short, not every basis of $\mathcal{A}$ induces a fundamental representation of a given $a \in \mathrm{End}(\mathcal{A})$, but, given any $a \in \mathrm{End}(\mathcal{A})$, every basis of $\mathrm{im}(a)$ induces a fundamental representation for a.

We observe that if $e \in E(\mathrm{End}(\mathcal{A}))$, then e has a fundamental representation as follows

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I},$$

where $a_i \in A_i$, for all $i \in I$. Moreover, if C is a basis for $\mathrm{im}(e)$ and $C_0 = C \cup \{0\}$, then there is a basis of $\mathcal{A}$, say $B = \bigcup_{c\in C_0} A_c$, such that $A_c e = c$, for all $c \in C_0$. Thus, e can be represented as (and is defined by)

$$\begin{bmatrix} A_c \\ c \end{bmatrix}_{c\in C_0}.$$

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Lemma 2.5. *Let $\alpha \in \mathrm{End}(\mathcal{A})$, $g, h \in \mathrm{Aut}(\mathcal{A})$ and suppose that*

$$\begin{bmatrix} A_0 & A_1 & \cdots & A_n \\ 0 & a_1 & \cdots & a_n \end{bmatrix}$$

is a fundamental representation for α (where A_0 may be empty). Then

$$\begin{bmatrix} A_0 g & A_1 g & \cdots & A_n g \\ 0 & a_1 h & \cdots & a_n h \end{bmatrix}$$

is a fundamental representation for $g^{-1}\alpha h$.

Proof. Clearly, $\bigcup(A_i g : i \in \{0, \dots, n\})$ is a basis for $\mathcal{A}$ and $\{a_1 h, \dots, a_n h\}$ is an independent set. Moreover, $(A_i g)g^{-1}\alpha h = a_i h$, for all $i \in [n]$, and $(A_0 g)g^{-1}\alpha h = 0$, whenever A_0 is non-empty. The lemma is proved. □

Let $\alpha \in \mathrm{End}(\mathcal{A})$. If the algebra has no constants, then the constant component in every fundamental representation of α is the empty set and then the endomorphism can be defined by

$$\begin{bmatrix} A_1 & \cdots & A_n \\ a_1 & \cdots & a_n \end{bmatrix},$$

where $(A_i)_{i \in [n]}$ is a partition of a basis.

Let α be an endomorphism of $\mathcal{A}$. We say that α is *proper* if it has a fundamental representation with empty constant component. That is, α can be defined by

$$\begin{bmatrix} A_1 & \cdots & A_n \\ a_1 & \cdots & a_n \end{bmatrix},$$

where $\bigcup_{i \in [n]} A_i$ is a basis for $\mathcal{A}$. A proper endomorphism is said to be *reductive* if its rank is less than the rank of $\mathcal{A}$ but greater than zero.

Definition 2.6. A strong independence algebra is said to be *proper* if all endomorphisms of rank at least 1 are proper.

Clearly, an endomorphism of a proper algebra is reductive if and only if it is neither an automorphism nor the null endomorphism.

We observe that strong independence algebras without constants are examples of proper independence algebras. In addition we have the following lemma.

Lemma 2.7. *Let V be a vector space over a field F. Then V is a proper independence algebra.*

Proof. Let B be a basis for V, $B_1 \subset B$ and $b \in B \setminus B_1$. It is an easy exercise to show that $(B_1 + b) \cup B \setminus B_1$, where $B + b = \{a + b \mid a \in B\}$, is a basis for V.

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Consider the following fundamental representation of α:

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I}.$$

As $\operatorname{rank}(\alpha) > 0$ it follows that for some $i \in I$ we have $(A_i)\alpha = \{a_i\} \neq \{0\}$. Suppose $a_1 \neq 0$. Then α has a fundamental representation as follows:

$$\begin{bmatrix} A_0 & A_1 & A_i \\ 0 & a_1 & a_i \end{bmatrix}_{i\in I\setminus\{1\}}.$$

Let $a \in A_1$. Then $B = (A_o + a) \cup (\cup_{i\in I} A_i)$ is a basis for V.

Now α acts on the basis B in the following way:

$$\begin{bmatrix} A_0 + a & A_1 & A_i \\ a_1 & a_1 & a_i \end{bmatrix}_{i\in I\setminus\{1\}}.$$

As $C = \{a_i \mid i \in I\}$ generates the image of α, it follows that

$$\begin{bmatrix} (A_0 + a) \cup A_1 & A_i \\ a_1 & a_i \end{bmatrix}_{i\in I\setminus\{1\}}$$

is a fundamental representation of α with empty constant component. □

In what follows we restrict our study to the case of *proper independence algebras of finite rank*. Thus $\mathcal{A}$ will always denote an algebra of this kind (we keep assuming that $\mathcal{A}$ has at most one constant).

We now prove a technical lemma which will be very useful in what follows.

Lemma 2.8. *Let $\{b_1, \dots, b_n\}$ be an independent set in $\mathcal{A}$ where $n < \operatorname{rank}(\mathcal{A})$. Suppose that a belongs to the subalgebra $\langle b_1, \dots, b_n\rangle \setminus Con$. Then there exist two bases as follows*

$$B = \{b_1, \dots, b_n, y, d_1, \dots, d_k\},$$
$$C = \{b_1, \dots, b_{i-1}, a, b_{i+1}, b_n, y, e_1, \dots, e_k\}.$$

Proof. As $n < \operatorname{rank}(\mathcal{A})$ there is an element $y \in \mathcal{A}$ such that the set $\{b_1, \dots, b_n, y\}$ is independent. Hence there is a basis $B = \{b_1, \dots, b_n, y, d_1, \dots, d_k\}$. Take the minimum $i \in [n]$ such that $a \in \langle b_1, \dots, b_i\rangle$. Then, as $a \notin \langle b_1, \dots, b_{i-1}\rangle$, it follows that $\{b_1, \dots, b_{i-1}, a\}$ is an independent set and, by [EP], we can say that $b_i \in \langle b_1, \dots, b_{i-1}, a\rangle$.

We claim that $\{b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n\}$ is independent. In fact, if $a \in \langle b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n\rangle$, then we have

$$b_i \in \langle b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n\rangle = \langle b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n\rangle,$$

a contradiction as $\{b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_n\}$ is an independent set. It now follows that $\{b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n, y\}$ is independent. In fact,

$$\begin{aligned}\langle b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n\rangle \cap \langle y\rangle &\subseteq \langle b_1, \dots, b_i, \dots, b_n, a\rangle \cap \langle y\rangle \\ &= \langle b_1, \dots, b_i, \dots, b_n\rangle \cap \langle y\rangle \\ &= Con.\end{aligned}$$

Thus, $\langle b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n\rangle \cap \langle y\rangle = Con$ and this proves that the set $\{b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n, y\}$ is independent. Therefore it can be extended to a basis of $\mathcal{A}$:

$$C = \{b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_n, y, \dots\}.$$

□

3. The semigroup $\langle \alpha, G\rangle \setminus G$ is idempotent generated

The aim of this section is the proof of the following theorem.

Theorem 3.1. *If $\alpha \in \mathrm{End}(\mathcal{A})$ is a reductive endomorphism, then $\langle \alpha, G\rangle \setminus G$ is generated by its own idempotents.*

For an $\alpha \in \mathrm{End}(\mathcal{A})$ the semigroup $\langle \alpha, G\rangle \setminus G$ will be denoted by α^G. As an element in the semigroup α^G has the form $g_1\alpha g_2 \dots \alpha g_k$, for some $g_1, \dots, g_n \in G$, we see that α^G is generated by its idempotents if and only if $g\alpha h$ is a product of idempotents in α^G, for all $g, h \in G$.

In [**3**, § 3] it is proved that there is an idempotent $e \in \mathrm{End}(\mathcal{A})$ and an automorphism g such that $\alpha = eg$. Thus we have $\alpha \in e^G$ and $e = \alpha g^{-1} \in \alpha^G$. Hence $e^G = \alpha^G$.

Now it remains to prove that the semigroup e^G is generated by its own idempotents. To see this we fix the following fundamental representation of e:

$$\begin{bmatrix} A_1 & \cdots & A_n \\ x_1 & \cdots & x_n \end{bmatrix},$$

where $x_i \in A_i$, for all $i = 1, \dots, n$. This is possible because α is proper and thus, by Lemma 2.5, e is proper as well.

If, for all $g, h \in G$, we have $geh \in \langle E(e^G)\rangle$, then $e^G = \langle E(e^G)\rangle$. Thus one only has to prove that, for all $g, h \in G$, the element geh belongs to $\langle E(e^G)\rangle$. Let $a = geh$. Then it follows from Lemma 2.5 that the following matrix

$$\begin{bmatrix} A_1g^{-1} & \cdots & A_ng^{-1} \\ x_1h & \cdots & x_nh \end{bmatrix}$$

is a fundamental representation for a. Let $A_ig^{-1} = Z_i$, $x_ih = a_i$. Thus a fundamental representation of a is given by the matrix

$$\begin{bmatrix} Z_1 & \cdots & Z_n \\ a_1 & \cdots & a_n \end{bmatrix}.$$

We observe that geg^{-1} is idempotent and has a fundamental representation as follows

$$\begin{bmatrix} A_1g^{-1} & \cdots & A_ng^{-1} \\ x_1g^{-1} & \cdots & x_ng^{-1} \end{bmatrix}.$$

As $A_ig^{-1} = Z_i$, for $i = 1, \dots, n$, it follows that $E(e^G)$ contains an element (namely, geg^{-1}) with the following fundamental representation:

$$\begin{bmatrix} Z_1 & \cdots & Z_n \\ x_1g^{-1} & \cdots & x_ng^{-1} \end{bmatrix}.$$

We introduce now what will be the main tool in our proof. Let $u \in \langle E(e^G)\rangle$ with fundamental representation

$$\begin{bmatrix} Z_{\sigma 1} & \cdots & Z_{\sigma k} & Z_{\sigma(k+1)} & \cdots & Z_{\sigma n} \\ a_1 & \cdots & a_k & b_{k+1} & \cdots & b_n \end{bmatrix},$$

where σ is a permutation of $[n]$. Moreover, suppose that if there is $v \in \langle E(e^G)\rangle$ and $\varsigma \in \mathrm{Sym}([n])$ such that $Z_{\varsigma i}v = a_i$, for $i \in [j]$, then $j \leqslant k$. Roughly speaking, u is a maximal element with respect to the property 'coinciding with a *twisted* a in the first j image elements'.

We claim that $k = n$.

We start by introducing some notation. Let $\beta \in \mathrm{End}(\mathcal{A})$ and let the following matrix be a fundamental representation for β:

$$\begin{bmatrix} Y_1 & \cdots & Y_n \\ w_1 & \cdots & w_n \end{bmatrix}.$$

Moreover, let $Y = \bigcup_{i\in[n]} Y_i$ and for all $i \in [n]$ let y_i be an element in Y_i. Then we represent Y_i as $[y_i]_Y$, and hence the given fundamental representation for β becomes

$$\begin{bmatrix} [y_1]_Y & \cdots & [y_n]_Y \\ w_1 & \cdots & w_n \end{bmatrix}.$$

If one of the sets $(Y_i)_{i\in[n]}$, say Y_1, has more than one element, say w_1, v_1, we can represent Y_1 as $[w_1]_Y$ or as $[v_1]_Y$ or as $[w_1, v_1]_Y$. The same applies for a set having more than two elements. The aim of this notation is just to isolate one or more elements of a set in order to make it easier for the reader to follow composition of fundamental representations.

With this notation, the fixed fundamental representation of e with respect to the basis $Z = \bigcup A_i$ is

$$\begin{bmatrix} [x_1]_Z & \cdots & [x_n]_Z \\ x_1 & \cdots & x_n \end{bmatrix}.$$

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Lemma 3.2. *Let $B = \{b_1, \dots, b_n\}$ be contained in a basis C. Then, there exists in e^G an idempotent ε defined by*

$$\begin{bmatrix} [b_1]_C & \cdots & [b_n]_C \\ b_1 & \cdots & b_n \end{bmatrix}.$$

Moreover, we can choose ε in such a way that $|[b_i]_C| = |[x_i]_Z|$, for all $i \in [n]$.

Proof. Consider a partition $(B_i)_{i\in[n]}$, of C, such that

(1) $b_i \in B_i$, for all $i \in [n]$; and

(2) $|[x_i]_Z| = |[b_i]_C|$, for all $i \in [n]$.

Now, consider any bijection $g : (\bigcup [x_i]_Z : i \in [n]) \longrightarrow C$ satisfying two properties:

(1) $x_i g = b_i$, for all $i \in [n]$; and

(2) $([x_i]_Z)g = [b_i]_C$, for all $i \in [n]$.

Then g can be extended to an automorphism of $\mathcal{A}$ and, by Lemma 2.5, the matrix

$$\begin{bmatrix} [b_1]_C & \cdots & [b_n]_C \\ b_1 & \cdots & b_n \end{bmatrix}$$

is a fundamental representation of $g^{-1}eg$ and $g^{-1}eg \in E(e^G)$. The lemma follows. □

As e is reductive, for some $i \in [n]$ the set $A_i = [x_i]_Z$ has at least two elements. We can suppose that $[x_{k+1}]_Z$ contains more than one element. In fact, if $[x_i]_Z$ contains more than one element, instead of using e, we would work with the idempotent $\varepsilon = (x_i x_{k+1})_Z e (x_i x_{k+1})_Z$, and now it is the $\mathrm{Ker}(\varepsilon)$-class of x_{k+1} which contains more than one element. The point is that we have to change only if $[x_{k+1}]_Z$ has only one element. If this is the case, then we consider the element $\varepsilon = (x_i x_{k+1})_Z e (x_i x_{k+1})_Z$, and with this element we have $\langle E(\varepsilon^G)\rangle = \langle E(e^G)\rangle$. Thus if $u \in \langle E(\varepsilon^G)\rangle$, then $u \in \langle E(e^G)\rangle$, which is what we want to prove.

Thus we can assume that the fundamental representation of e chosen above has the following shape

$$\begin{bmatrix} [x_1]_Z & \cdots & [x_k]_Z & [x_{k+1}, w_{k+1}]_Z & \cdots & [x_n]_Z \\ x_1 & \cdots & x_k & x_{k+1} & \cdots & x_n \end{bmatrix}.$$

To prove that $k = n$ we are going to prove that if $k < n$, then there is $\varepsilon \in E(e^G)$ such that $u\varepsilon$ coincides with a twisted a in the first $k+1$ elements, contradicting the maximality of u.

Suppose first that

$$a_{k+1} \notin \langle a_1, \dots, a_k, b_{k+1}, \dots, b_n \rangle.$$

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Then the set $\{a_1, \dots, a_k, b_{k+1}, \dots, b_n, a_{k+1}\}$ is independent and hence can be extended to B, a basis of $\mathcal{A}$. Now it follows from Lemma 3.2 that there is an idempotent ε in e^G with the following fundamental representation

$$\begin{bmatrix} [a_1]_B & \cdots & [a_k]_B & [a_{k+1}, b_{k+1}]_B & [b_{k+2}]_B & \cdots & [b_n]_B \\ a_1 & \cdots & a_k & a_{k+1} & b_{k+2} & \cdots & b_n \end{bmatrix}.$$

But now, $u\varepsilon$ is defined by

$$\begin{bmatrix} Z_{\sigma 1} & \cdots & Z_{\sigma k} & Z_{\sigma(k+1)} & Z_{\sigma(k+2)} & \cdots & Z_{\sigma n} \\ a_1 & \cdots & a_k & a_{k+1} & b_{k+2} & \cdots & b_n \end{bmatrix},$$

which coincides with a twisted a in the first $k+1$ image elements, contradicting the maximality of u.

Thus we can now suppose that $a_{k+1} \in \langle a_1, \dots, a_k, b_{k+1}, \dots, b_n \rangle$. It follows from Lemma 2.8 that the following two bases exist:

(1) $B = \{a_1, \dots, a_k, b_{k+1}, \dots, b_{k+j-1}, b_{k+j}, b_{k+j+1}, \dots, b_n, y, \dots\}$, and

(2) $C = \{a_1, \dots, a_k, b_{k+1}, \dots, b_{k+j-1}, a_{k+1}, b_{k+j+1}, \dots, b_n, y, \dots\}$,

where $j \geqslant 1$.

Using Lemma 3.2 once again, it follows that there are two idempotents in e^G with the following associated fundamental representations:*

$$\eta \longleftrightarrow \begin{bmatrix} [a_1]_B & \cdots & [a_k]_B & [b_{k+1}]_B & \cdots & [b_{k+j}, y]_B & \cdots & [b_n]_B \\ a_1 & \cdots & a_k & b_{k+1} & \cdots & y & \cdots & b_n \end{bmatrix}$$

and

$$\zeta \longleftrightarrow \begin{bmatrix} [a_1]_C & \cdots & [a_k]_C & [b_{k+1}]_C & \cdots & [a_{k+1}, y]_C & \cdots & [b_n]_C \\ a_1 & \cdots & a_k & b_{k+1} & \cdots & a_{k+1} & \cdots & b_n \end{bmatrix}.$$

Thus, the element $u\eta\zeta$ is defined by

$$\begin{bmatrix} Z_{\sigma 1} & \cdots & Z_{\sigma k} & Z_{\sigma(k+1)} & \cdots & Z_{\sigma(k+j)} & \cdots & Z_{\sigma n} \\ a_1 & \cdots & a_k & b_{k+1} & \cdots & a_{k+1} & \cdots & b_n \end{bmatrix}.$$

Now we can consider a permutation $\sigma' \in \mathrm{Sym}([n])$ defined as follows:

$$(k+1)\sigma' = (k+j)\sigma, \quad (k+j)\sigma' = (k+1)\sigma \quad \text{and} \quad (i)\sigma' = (i)\sigma,$$

for the remaining elements of $[n]$. The element $u\eta\zeta$ is defined by

$$\begin{bmatrix} Z_{\sigma' 1} & \cdots & Z_{\sigma' k} & Z_{\sigma'(k+1)} & \cdots & Z_{\sigma'(k+j)} & \cdots & Z_{\sigma' n} \\ a_1 & \cdots & a_k & a_{k+1} & \cdots & b_{k+1} & \cdots & b_n \end{bmatrix},$$

* The notation $\eta \leftrightarrow M$, where η is an endomorphism and M is a matrix, means that M is a fundamental representation of η.

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a contradiction with the maximality of u. It is proved that $k = n$ and, hence, the matrix

$$\begin{bmatrix} Z_{\sigma 1} & \cdots & Z_{\sigma k} & Z_{\sigma(k+1)} & \cdots & Z_{\sigma n} \\ a_1 & \cdots & a_k & a_{k+1} & \cdots & a_n \end{bmatrix}$$

is a fundamental representation of u. Thus proving our claim on p. 212.

To finish the proof of Theorem 3.1, it remains to prove, as explained on p. 212, that a belongs to $\langle E(e^G)\rangle$. So we have the following lemma.

Lemma 3.3. *Let $b \in \langle E(e^G)\rangle$ be defined by the matrix*

$$b \leftrightarrow \begin{bmatrix} B_1 & \cdots & B_k & \ldots & B_{k+j} & \cdots & B_n \\ b_1 & \cdots & b_k & \cdots & b_{k+j} & \cdots & b_n \end{bmatrix}.$$

Then the endomorphism defined by

$$c \leftrightarrow \begin{bmatrix} B_1 & \cdots & B_{k-1} & B_k & B_{k+1} & \cdots & B_{k+j-1} & B_{k+j} & B_{k+j+1} & \cdots & B_n \\ b_1 & \cdots & b_{k-1} & b_{k+j} & b_{k+1} & \cdots & b_{k+j-1} & b_k & b_{k+j+1} & \cdots & b_n \end{bmatrix}$$

belongs to $\langle E(e^G)\rangle$ as well.

Proof. Let $B = \{b_1, \ldots, b_n, y, \ldots\}$ be a basis of $\mathcal{A}$. By Lemma 3.2, the following idempotents belong to e^G:

$$\varepsilon \longleftrightarrow \begin{bmatrix} [b_1]_B & \cdots & [b_k]_B & \cdots & [b_{k+j}, y]_B & \cdots & [b_n]_B \\ b_1 & \cdots & b_k & \cdots & y & \cdots & b_n \end{bmatrix},$$

$$\zeta \longleftrightarrow \begin{bmatrix} [b_1]_B & \cdots & [b_{k-1}]_B & [y]_B & [b_{k+1}] & \cdots & [b_k, b_{k+j}]_B & \cdots & [b_n]_B \\ b_1 & \cdots & b_{k-1} & y & b_{k+1} & \cdots & b_{k+j} & \cdots & b_n \end{bmatrix},$$

$$\eta \longleftrightarrow \begin{bmatrix} [b_1]_B & \cdots & [y, b_k]_B & \cdots & [b_{k+j}]_B & \cdots & [b_n]_B \\ b_1 & \cdots & b_k & \cdots & b_{k+j} & \cdots & b_n \end{bmatrix}.$$

Now, $b\varepsilon\zeta\eta$ is defined by

$$\begin{bmatrix} B_1 & \cdots & B_k & \cdots & B_{k+j} & \cdots & B_n \\ b_1 & \ldots & b_{k+j} & \cdots & b_k & \cdots & b_n \end{bmatrix},$$

which is equal to c. The lemma is proved. □

Now, as u, with fundamental representation

$$\begin{bmatrix} Z_{\sigma 1} & \cdots & Z_{\sigma k} & Z_{\sigma(k+1)} & \cdots & Z_{\sigma n} \\ a_1 & \cdots & a_k & a_{k+1} & \cdots & a_n \end{bmatrix},$$

belongs to $\langle E(e^G)\rangle$, it follows by repeated application of the previous lemma that a, defined by

$$\begin{bmatrix} Z_1 & \cdots & Z_n \\ a_1 & \cdots & a_n \end{bmatrix},$$

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belongs to $\langle E(e^G)\rangle$ as well. Thus the semigroup e^G is generated by its idempotents. As $e^G = \alpha^G = \langle \alpha, G\rangle \setminus G$, the theorem is proved.

Corollary 3.4. *Let $\mathcal{A}$ be a proper independence algebra. Then every ideal of the semigroup* $\mathrm{End}(\mathcal{A}) \setminus \mathrm{Aut}(\mathcal{A})$ *is generated by its idempotents.*

Proof. It is proved in [**3**, remark after Proposition 1.3] that the ideals of $\mathrm{End}(\mathcal{A}) \setminus \mathrm{Aut}(\mathcal{A})$ are precisely the sets

$$I_r = \{\alpha \in \mathrm{End}(\mathcal{A}) : \mathrm{rank}(\alpha) \leqslant r\}, \quad \text{for } r < \mathrm{rank}(\mathcal{A}).$$

Now, if $r = 0$, then I_r has only one element, which is idempotent, and hence the result holds. If $0 < r < \mathrm{rank}(\mathcal{A})$, then every $\alpha \in I_r \setminus I_0$ is reductive and hence

$$\alpha \in \alpha^G = \langle E(\alpha^G)\rangle \subseteq I_r.$$

Thus α is a product of idempotents of I_r. □

Corollary 3.5 (Howie). *Let X be a finite set. Then $T(X) \setminus \mathrm{Sym}(X)$ is idempotent generated.*

Corollary 3.6 (Erdos). *Let V be a finite-dimensional vector space. Then the semigroup* $\mathrm{End}(V) \setminus \mathrm{Aut}(V)$ *is idempotent generated.*

4. Normal semigroups of endomorphisms

Let $\mathcal{A}$ be an independence algebra, let $\alpha \in \mathrm{End}(\mathcal{A})$ and $g \in G$. We will denote the element $g\alpha g^{-1}$ by α^g. A subsemigroup S of $\mathrm{End}(\mathcal{A})$ is said to be a *normal semigroup* if $s^g \in S$, for all $s \in S$ and all $g \in G$. The smallest normal semigroup containing $\alpha \in \mathrm{End}(\mathcal{A})$ is the semigroup generated by the set $\{\alpha^g \mid g \in G\}$ and will be denoted by $\langle \alpha : G\rangle$.

The proof of the next theorem turns out to be very easy when we use some techniques developed by McAlister [**8**].

Theorem 4.1. *Let $\mathcal{A}$ be a proper independence algebra such that G, the automorphism group of $\mathcal{A}$, is a periodic group. Moreover, let α be a reductive endomorphism of $\mathcal{A}$. Then the semigroup $\langle \alpha : G\rangle$ is equal to $\langle \alpha, G\rangle \setminus G$ and hence is generated by its idempotents.*

Proof. It is obvious that $\langle \alpha : G\rangle \subseteq \langle \alpha, G\rangle \setminus G$. So we prove the converse.

Let $u = g_1\alpha g_2\alpha g_3 \dots g_n\alpha g_{n+1}$ be an idempotent of $\langle \alpha, G\rangle \setminus G$. Then we have

$$\begin{aligned} u &= g_1\alpha g_1^{-1}(g_1g_2)\alpha(g_1g_2)^{-1}(g_1g_2g_3)\alpha \dots \alpha(g_1g_2g_3\dots g_n)^{-1}(g_1g_2g_3\dots g_ng_{n+1}) \\ &= \alpha^{g_1}\alpha^{g_1g_2}\alpha^{g_1g_2g_3}\dots\alpha^{g_1g_2g_3\cdots g_n}(g_1\dots g_{n+1}). \end{aligned}$$

Thus $u \in \langle \alpha : G\rangle(g_1 \dots g_{n+1})$, say $u = vg$, where g is equal to $g_1 \dots g_{n+1}$ and v belongs to $\langle \alpha : G\rangle$.

As G is periodic, there is $n \in \mathbb{N}$ such that g^n is the identity. Now, as $u = vg$ is idempotent, we have $vg = (vg)^n$ and hence

$$\begin{aligned} vg &= (vg)^n \\ &= v(gvg^{-1})(g^2vg^{-2})\dots(g^{n-1}vg^{-n+1})g^n \\ &= v(gvg^{-1})(g^2vg^{-2})\dots(g^{n-1}vg^{-n+1}) \in \langle \alpha : G \rangle. \end{aligned}$$

It is proved that $u = vg = v(gvg^{-1})(g^2vg^{-2})\dots(g^{n-1}vg^{-n+1})$ is an idempotent of $\langle \alpha : G \rangle$. Thus all the idempotents of $\langle \alpha, G \rangle \setminus G$ belong to $\langle \alpha : G \rangle$. As $E(\langle \alpha, G \rangle \setminus G)$ generates $\langle \alpha, G \rangle \setminus G$, the theorem is proved. □

We observe that when $\mathcal{A}$ is a vector space, over a field, the above result is true even when the automorphism group is not a periodic group (see [**1**]).

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