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NORMAL SINGULARITIES WITH TORUS ACTIONS

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Abstract. We propose a method to compute a desingularization of a normal affine variety X endowed with a torus action in terms of a combinatorial description of such a variety due to Altmann and Hausen. This desingularization allows us to study the structure of the singularities of X. In particular, we give criteria for X to have only rational, ($\boldsymbol{Q}$-)factorial, or ($\boldsymbol{Q}$-)Gorenstein singularities. We also give partial criteria for X to be Cohen-Macaulay or log-terminal. Finally, we provide a method to construct factorial affine varieties with a torus action. This leads to a full classification of such varieties in the case where the action is of complexity one.

CONTENTS

Introduction. The theory of singularities on toric varieties is well established. All toric singularities are log-terminal and thus rational and Cohen-Macaulay. Furthermore, there are explicit combinatorial criteria to decide if a given toric variety is ($\boldsymbol{Q}$-)factorial or ($\boldsymbol{Q}$-)Gorenstein (see [Dai02]). In this paper we elaborate the analog criteria for more general varieties admitting torus actions.

Let X be a normal variety endowed with an effective torus action. The complexity of this action is the codimension of the maximal orbits. By a classic theorem of Sumihiro [Sum74], every point $x \in X$ posses an affine open neighborhood invariant under the torus action. Hence, local problems can be reduced to the affine case.

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There are well-known combinatorial descriptions of normal T-varieties. We refer the reader to [Dem70] and [Oda88] for the case of toric varieties, to [KKMS73, Chaps. 2 and 4] and [Tim08] for the complexity one case, and to [AH06, AHS08] for the general case.

Let us fix some notation. We let k be an algebraically closed field of characteristic 0, M a lattice of rank n, and T the algebraic torus $T = \operatorname{Spec} k[M] \simeq (k^*)^n$. A T-variety X is a variety endowed with an effective algebraic action of T. For an affine variety $X = \operatorname{Spec} A$, introducing a T-action on X is the same as endowing A with an M-grading.

We let $N_{\boldsymbol{Q}} = N \otimes \boldsymbol{Q}$, where $N = \operatorname{Hom}(M, \boldsymbol{Z})$ is the dual lattice of M. Any affine toric variety can be described via a polyhedral cone $\sigma \subseteq N_{\boldsymbol{Q}}$. Similarly, the combinatorial description of normal affine T-varieties due to Altmann and Hausen [AH06] involves the data $(Y, \sigma, \mathcal{D})$ where Y is a normal semiprojective variety, $\sigma \subseteq N_{\boldsymbol{Q}}$ is a polyhedral cone, and $\mathcal{D}$ is a polyhedral divisor on Y, i.e., a divisor whose coefficients are polyhedra in $N_{\boldsymbol{Q}}$ with tail cone σ.

The normal affine variety corresponding to the data $(Y, \sigma, \mathcal{D})$ is denoted by $X[\mathcal{D}]$. The construction involves another normal variety $\widetilde{X}[\mathcal{D}]$, which is affine over Y, and a proper birational morphism $r : \widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$ (see Section 1 for more details).

This description is not unique. In Section 2, we show that for every T-variety X there exists a polyhedral divisor $\mathcal{D}$ such that $X = X[\mathcal{D}]$ and $\widetilde{X}[\mathcal{D}]$ is a toroidal variety. Hence, the morphism $r : \widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$ is a partial desingularization of X having only toric singularities.

Let X be a normal variety and let $\psi : Z \to X$ be a desingularization. Usually, the classification of singularities involves the higher direct images of the structure sheaf $R^i\psi_*\mathcal{O}_Z$. In particular, X has rational singularities if $R^i\psi_*\mathcal{O}_Z = 0$ for all $i \geq 1$ (see e.g., [Art66, Elk78]). In Section 3, we compute the higher direct image sheaves $R^i\psi_*\mathcal{O}_Z$ for a T-variety $X[\mathcal{D}]$ in terms of the combinatorial data and we give a criterion for $X[\mathcal{D}]$ to have rational singularities.

A well-known theorem of Kempf [KKMS73, p. 50] states that a variety X has rational singularities if and only if X is Cohen-Macaulay and the induced map $\psi_*\omega_Z \hookrightarrow \omega_X$ is an isomorphism. In Proposition 3.7, we apply Kempf's Theorem to give a partial characterization of T-varieties having Cohen-Macaulay singularities.

Invariant T-divisors were studied in [PS11]. In particular, a description of the class group, and a representative of the canonical class of $X[\mathcal{D}]$ are given. In Section 4, we use this results to state necessary and sufficient conditions for $X[\mathcal{D}]$ to be ($\boldsymbol{Q}$-)factorial or ($\boldsymbol{Q}$-) Gorenstein in terms of the combinatorial data. Furthermore, in Theorem 4.9 we apply the partial desingularization obtained in Section 2 to give a criterion for $X[\mathcal{D}]$ to have log-terminal singularities.

In [Wat81], some of the results in Sections 3 and 4 were proved for a 1-dimensional torus action on X. Our results can be seen as the natural generalization of these results of Watanabe (see also [FZ03, Sec. 4]).

In Section 5, we specialize our results in Sections 3 and 4 for a T-variety $X[\mathcal{D}]$ of complexity one. In this case, the variety Y in the combinatorial data is a smooth curve. This make the criteria more explicit. In particular, if $X[\mathcal{D}]$ has $\boldsymbol{Q}$-Gorenstein or rational singularities, then Y is either affine or the projective line.

Finally, in Section 6 we provide a method to construct factorial T-varieties based on the criterion for factoriality given in Proposition 4.5. In the case of complexity one, this method leads to a full classification of factorial quasihomogeneous affine T-varieties analogous to the ones given in [Mor77] and [Ish77] for dimension two and three, respectively; and in [HHS11] for the general case. A common way to show that an affine variety is factorial is to apply the criterion of Samuel [Sam64] or the generalization by Scheja and Storch [SS84]. However, for the majority of the factorial varieties that we construct with our method, these criteria do not work.

In the entire paper, the term variety means a normal integral scheme of finite type over an algebraically closed field k of characteristic 0.

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1. Preliminaries. First, we fix some notation. In this paper N is always a lattice of rank n, and $M = \mathrm{Hom}(N, \boldsymbol{Z})$ is its dual. The associated rational vector spaces are denoted by $N_{\boldsymbol{Q}} := N \otimes \boldsymbol{Q}$ and $M_{\boldsymbol{Q}} := M \otimes \boldsymbol{Q}$. Moreover, $\sigma \subseteq N_{\boldsymbol{Q}}$ is a pointed convex polyhedral cone, and $\sigma^\vee \subseteq M_{\boldsymbol{Q}}$ is its dual cone. Let $\sigma_M^\vee := \sigma^\vee \cap M$ be the semigroup of lattice points inside $\sigma^\vee$.

We consider convex polyhedra $\Delta \subseteq N_{\boldsymbol{Q}}$ admitting a decomposition as Minkowski sum $\Delta = \Pi + \sigma$ with a compact polyhedron $\Pi \subseteq N_{\boldsymbol{Q}}$; we refer to σ as the *tail cone* of Δ and to Δ as a *σ-polyhedron*. We denote the set of all σ-polyhedra by $\mathrm{Pol}_\sigma(N_{\boldsymbol{Q}})$. With respect to Minkowski addition, $\mathrm{Pol}_\sigma(N_{\boldsymbol{Q}})$ is a semigroup with the neutral element σ.

We are now going to describe affine varieties with an action of the torus $T = \mathrm{Spec}\, k[M]$. Let Y be a normal variety, which is semiprojective, i.e., projective over an affine variety. Fix a pointed convex polyhedral cone $\sigma \subseteq N_{\boldsymbol{Q}}$. A *polyhedral divisor* on Y is a formal sum

$$\mathcal{D} = \sum_Z \Delta_Z \cdot Z\,,$$

where Z runs over the prime divisors of Y and the coefficients Δ_Z are all σ-polyhedra with $\Delta_Z = \sigma$ for all but finitely many of them.

For every $u \in \sigma_M^\vee$ we have the evaluation

$$\mathcal{D}(u) := \sum_Z \min_{v \in \Delta_Z} \langle u, v \rangle \cdot Z\,,$$

which is a $\boldsymbol{Q}$-divisor living on Y. This defines an evaluation map $\mathcal{D}^\vee : \sigma^\vee \to \mathrm{Div}_{\boldsymbol{Q}}(Y)$, which is piecewise linear and the loci of linearity are (not necessarily pointed) subcones of $\sigma^\vee$. Hence, $\mathcal{D}^\vee$ defines a quasifan which subdivides $\sigma^\vee$. We call it the normal quasifan of $\mathcal{D}$.

We call the polyhedral divisor $\mathcal{D}$ on Y *proper* if the following conditions hold:

(i) The divisor $\mathcal{D}(u)$ is $\boldsymbol{Q}$-Cartier and has a base point free multiple for every $u \in \sigma_M^\vee$.
(ii) The divisor $\mathcal{D}(u)$ is big for every $u \in \operatorname{relint} \sigma^\vee \cap M$.

Recall that a divisor D on Y is $\boldsymbol{Q}$-Cartier if there exists $l > 0$ such that lD is Cartier, and big if there exists a divisor D_0 in the linear system $|lD|$, for some $l > 0$, such that $Y \setminus \operatorname{supp} D_0$ is affine.

By construction, every polyhedral divisor $\mathcal{D}$ on a normal variety Y defines a sheaf $\mathcal{A}[\mathcal{D}]$ of M-graded $\mathcal{O}_Y$-algebras and its ring $A[\mathcal{D}]$ of global sections:

$$\mathcal{A}[\mathcal{D}] := \bigoplus_{u \in \sigma_M^\vee} \mathcal{O}(\mathcal{D}(u)) \cdot \chi^u, \quad A[\mathcal{D}] := H^0(Y, \mathcal{A}[\mathcal{D}]).$$

Now suppose that $\mathcal{D}$ is proper. The result of Altmann and Hausen [AH06, Th. 3.1] guarantees that $A[\mathcal{D}]$ is a normal affine algebra. Thus, we obtain an affine varieties $X := X[\mathcal{D}] := \operatorname{Spec} A[\mathcal{D}]$ and $\widetilde{X} := \widetilde{X}[\mathcal{D}] := \mathbf{Spec}_Y\, \mathcal{A}[\mathcal{D}]$. Both varieties X and $\widetilde{X}$ come with an effective action of the torus $T = \operatorname{Spec} k[M]$ and there is a proper birational equivariant morphism $r : \widetilde{X} \to X$. Moreover, by the definition of $\widetilde{X}$ there is an affine morphism $q : \widetilde{X} \to Y$, and the composition

$$\pi := q \circ r^{-1} : X \dashrightarrow Y$$

is a rational map defined outside a closed subset of codimension at least 2.

Note that there is a natural inclusion $A[\mathcal{D}] \subset \bigoplus_{u \in M} k(Y) \cdot \chi^u$ which gives rise to a standard representation $f \cdot \chi^u$ with $f \in k(Y)$ and $u \in M$ for every semi-invariant rational function from $k(X) = k(\widetilde{X})$. With this notation, the rational map π is given by the natural inclusion of function field

$$k(Y) \subset k(X) = \operatorname{Quot}\left(\bigoplus_u k(Y) \cdot \chi^u\right).$$

By [AH06, Th. 3.4], every normal affine variety with an effective torus action arises from a proper polyhedral divisor.

EXAMPLE 1.1. Letting $N = \boldsymbol{Z}^2$ and $\sigma = \operatorname{pos}((1,0),(1,6))$, in $N_{\boldsymbol{Q}} = \boldsymbol{Q}^2$ we consider the σ-polyhedra $\Delta_0 = \operatorname{conv}((1,0),(1,1)) + \sigma$, $\Delta_1 = (-1/2, 0) + \sigma$, and $\Delta_\infty = (-1/3, 0) + \sigma$ (see Figure 1).

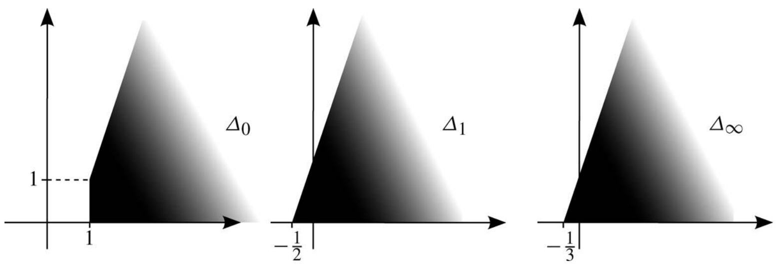

FIGURE 1. The σ-polyhedra Δ_0, Δ_1 and Δ_∞.

Let $Y = \boldsymbol{P}^1$ so that $k(Y) = k(t)$, where t is a local coordinate at zero. We consider the polyhedral divisor $\mathcal{D} = \Delta_0 \cdot [0] + \Delta_1 \cdot [1] + \Delta_\infty \cdot [\infty]$, and we let $A = A[\mathcal{D}]$ and $X = \operatorname{Spec} A$. An easy calculation shows that the elements

$$u_1 = \chi^{(0,1)}, \quad u_2 = \frac{t-1}{t^2}\chi^{(2,0)}, \quad u_3 = \frac{(t-1)^2}{t^3}\chi^{(3,0)}, \quad \text{and} \quad u_4 = \frac{(t-1)^3}{t^5}\chi^{(6,-1)}$$

generate A as an algebra. Furthermore, they satisfy the irreducible relation $u_2^3 - u_3^2 + u_1u_4 = 0$, and so

$$A \simeq k[x_1, x_2, x_3, x_4]/(x_2^3 - x_3^2 + x_1x_4).$$

For a polyhedral divisor $\mathcal{D}$ and a (not necessarily closed) point $y \in Y$, we define the *slice* of $\mathcal{D}$ at y by $\mathcal{D}_y := \sum_{Z \ni y} \Delta_Z$. Note, that $\mathcal{D}_Z$ is equal to the polyhedral coefficient Δ_Z of $\mathcal{D}$.

We want to describe the exceptional divisor of the morphism $\widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$. In general on a T-variety there are two types of prime divisors. Prime divisors of *horizontal* type consist of orbit closures of dimension $\operatorname{rank} N - 1$ and prime divisors of *vertical* type, of orbit closures of dimension $\operatorname{rank} N$. Note, that a generic point on a vertical prime divisor has a finite isotropy group, while on a horizontal prime divisor every point has infinite isotropy.

Let $\rho \in \sigma(1)$ be a ray of the tail cone. We call it a *big ray* of $\mathcal{D}$ if $\mathcal{D}(u)$ is big for $u \in \operatorname{relint}(\sigma^\vee \cap \rho^\perp)$. The set of big rays is denoted by $\operatorname{big}(\mathcal{D})$. For a vertex $v \in \mathcal{D}_Z^{(0)}$, we consider the smallest natural number $\mu(v)$ such that $\mu(v) \cdot v$ is a lattice point. A vertex v is called an *big vertex* if $\mathcal{D}(u)|_Z$ is big for every u in the interior of the normal cone

$$\mathcal{N}(\Delta_Z, v) = \{u\,;\, \langle u, w - v\rangle > 0 \text{ for every } w \in \Delta_Z\}.$$

The set of big vertices in $\mathcal{D}_Z$ is denoted by $\operatorname{big}(\mathcal{D}_Z)$.

THEOREM 1.2 ([PS11, Prop. 3.13]). *For the invariant prime divisors on $\widetilde{X}[\mathcal{D}]$, there are bijections*

(i) *between rays ρ in $\sigma(1)$ and horizontal prime divisors $\widetilde{E}_\rho$ of $\widetilde{X}[\mathcal{D}]$,*
(ii) *between pairs (Z, v), where Z is a prime divisor on Y and v is a vertex in $\mathcal{D}_Z$, and vertical prime divisors $\widetilde{D}_{Z,v}$ of $\widetilde{X}[\mathcal{D}]$.*

Via this correspondences the non-exceptional invariant divisor of $\widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$, and therefore the invariant divisors D_ρ, $D_{Z,v}$ on $X[\mathcal{D}]$ correspond to the elements of $\rho \in \operatorname{big}(\mathcal{D})$ or $v \in \operatorname{big}(\mathcal{D}_Z)$, respectively.

For a semi-invariant function $f \cdot \chi^u$, the corresponding invariant principal divisor on $X[\mathcal{D}]$ is

$$\sum_{Z,v} \mu(v)(\langle u, v\rangle + \operatorname{ord}_Z f) \cdot D_{Z,v} + \sum_\rho \langle u, n_\rho\rangle \cdot E_\rho. \tag{1}$$

Hence, for the pullbacks of a prime divisor Z on Y to $\widetilde{X}[\mathcal{D}]$ and $X[\mathcal{D}]$, we obtain

$$q^*Z = \sum_{v \in \mathcal{D}_Z^{(0)}} \mu(v) \cdot \widetilde{D}_{Z,v} \quad \text{and} \quad \pi^*Z = \sum_{v \in \operatorname{big}(\mathcal{D}_Z)} \mu(v) \cdot D_{Z,v},$$

respectively.

2. Toroidal desingularization. The combinatorial description of affine T-varieties in Section 1 is not unique. The following Lemma is a specialization of [AH06, Cor. 8.12]. For the convenience of the reader, we provide a short argument.

LEMMA 2.1. *Let $\mathcal{D}$ be a proper polyhedral divisor on a normal variety Y. Then for any projective birational morphism $\psi : \widetilde{Y} \to Y$, the variety $X[\mathcal{D}]$ is equivariantly isomorphic to $X[\psi^*\mathcal{D}]$.*

PROOF. We only need to show that

$$H^0(Y, \mathcal{O}_Y(\mathcal{D}(u))) \simeq H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\psi^*\mathcal{D}(u))) \text{ for all } u \in \sigma_M^\vee .$$

We let r be such that $r\mathcal{D}(u)$ is Cartier for all $u \in \sigma_M^\vee$. By Zariski's main theorem $\psi_*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ and by the projection formula, for all $u \in \sigma_M^\vee$ we have

$$H^0(Y, \mathcal{O}_Y(\mathcal{D}(u))) \simeq \left\{ f \in k(\widetilde{Y});\ f^r \in H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\psi^* r\mathcal{D}(u))) \right\} = H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\psi^*\mathcal{D}(u))) .$$

□

In the previous lemma, $\widetilde{X} = \widetilde{X}[\mathcal{D}]$ is not equivariantly isomorphic to $\widetilde{X}[\psi^*\mathcal{D}]$, unless ψ is an isomorphism.

DEFINITION 2.2. We define the *support* of a proper polyhedral divisor as

$$\operatorname{supp}\mathcal{D} = \{Z \text{ prime divisor}\,;\, \mathcal{D}_Z \neq \sigma\} \cup \{Z \text{ prime divisor}\,;\, \mathcal{D}_Z = \sigma \text{ and } \operatorname{big}(\mathcal{D}_Z) = \emptyset\} .$$

We say that $\mathcal{D}$ is an

(i) *SNC polyhedral divisor* if $\mathcal{D}$ is proper, Y is smooth, and $\operatorname{supp}\mathcal{D}$ is a simple normal crossing (SNC) divisor,
(ii) *strictly ample* if $\mathcal{D}(u)$ is ample for *every* $u \in \operatorname{relint}\sigma^\vee$.

REMARK 2.3. The above notion of strictly ampleness has the following geometric interpretation. A proper polyhedral divisor $\mathcal{D}$ is strictly ample if and only if Y is the *unique* maximal element in the inverse system of GIT-quotients of $X[\mathcal{D}]$ (see [AH06, p. 597] for the details of this construction). Hence, the existence of a strictly ample polyhedral divisor is a quite restrictive condition for a T-variety.

In the case of complexity one, i.e., when Y is a curve, any proper polyhedral divisor is SNC and strictly ample.

COROLLARY 2.4. *For any affine T-variety X there exists an SNC polyhedral divisor on a smooth variety Y such that $X = X[\mathcal{D}]$.*

PROOF. Let $\mathcal{D}'$ be proper polyhedral divisor on a semi-projective normal variety Y' such that $X = \operatorname{Spec} X[\mathcal{D}']$. Let $\psi : Y \to Y'$ be a projective resolution of singularities of Y' such that $\operatorname{supp}\psi^*\mathcal{D}'$ is SNC. By Lemma 2.1, $\mathcal{D} = \psi^*\mathcal{D}'$ is an SNC polyhedral divisor such that $X = X[\mathcal{D}]$. □

Now we elaborate a method to effectively compute an equivariant partial desingularization of an affine T-variety in terms of the combinatorial data $(Y, \mathcal{D})$. A key ingredient for our results is the following example (Cf. [Lie10, Ex. 3.20]).

EXAMPLE 2.5. Let H_i, $i = 1, \ldots, n$ be the coordinate hyperplanes in $Y = \boldsymbol{A}^n = \operatorname{Spec} k[t_1, \ldots, t_n]$, and let $\mathcal{D}$ be the SNC divisor on Y given by

$$\mathcal{D} = \sum_{i=0}^{n} \Delta_i \cdot H_i, \quad \text{where } \Delta_i \in \operatorname{Pol}_\sigma(N_{\boldsymbol{Q}}).$$

We let $h_i = \min_{v \in \Delta_i} \langle u, v \rangle$ be the support function of Δ_i. Since $k(Y) = k(t_1, \ldots, t_n)$ we obtain

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(\mathcal{D}(u))) &= \left\{ f \in k(Y);\, \operatorname{div}(f) + \mathcal{D}(u) \geq 0 \right\} \\ &= \left\{ f \in k(Y);\, \operatorname{div}(f) + \sum_{i=1}^{n} \min_{v \in \Delta_i} \langle u, v \rangle \cdot H_i \geq 0 \right\} \\ &= \bigoplus_{\substack{(r_1, \ldots, r_n) \\ r_i \geq -h_i(u)}} k \cdot t_1^{r_1} \cdots t_n^{r_n}. \end{aligned}$$

Let $N' = N \times \boldsymbol{Z}^n$, $M' = M \times \boldsymbol{Z}^n$ and σ' be the Cayley cone in $N'_{\boldsymbol{Q}}$, i.e., the cone spanned by $(\sigma, \overline{0})$ and (Δ_i, e_i), for $i \in \{1, \ldots, n\}$, where e_i is the i-th vector in the standard base of $\boldsymbol{Q}^n$. A vector $(u, r) \in M'$ belongs to the dual cone $(\sigma')^\vee$ if and only if $u \in \sigma^\vee$ and $r_i \geq -h_i(u)$.

With these definitions we have

$$A[\mathcal{D}] = \bigoplus_{u \in \sigma_M^\vee} H^0(Y, \mathcal{O}_Y(\mathcal{D}(u))) = \bigoplus_{(u,r) \in (\sigma')^\vee \cap M'} k \cdot t_1^{r_1} \cdots t_n^{r_n} \simeq k[(\sigma')^\vee \cap M'].$$

Hence $X[\mathcal{D}]$ is isomorphic as an abstract variety to the toric variety with the cone $\sigma' \subseteq N'_{\boldsymbol{Q}}$. Since Y is affine, $\widetilde{X} \simeq X$ is also a toric variety.

We say that a variety X is *toroidal* if for every $x \in X$ there is a formal neighborhood isomorphic to a formal neighborhood of a point in an affine toric variety.

PROPOSITION 2.6. *Let $\mathcal{D} = \sum_Z \Delta_Z \cdot Z$ be a proper polyhedral divisor on a semiprojective normal variety Y. If $\mathcal{D}$ is SNC then $\widetilde{X} = \widetilde{X}[\mathcal{D}]$ is a toroidal variety.*

PROOF. For $y \in Y$ we consider the fiber X_y over y for the morphism $\varphi : \widetilde{X} \to Y$. We let also $\mathfrak{U}_y$ be a formal neighborhood of X_y.

We let $n = \dim Y$ and

$$S_y = \{Z \text{ prime divisor}\,;\, y \in Z \text{ and } \Delta_Z \neq \sigma\}.$$

Since $\operatorname{supp} \mathcal{D}$ is SNC, we have that $\operatorname{card}(S_y) \leq n$. Letting $j : S_y \to \{1, \ldots, n\}$ be any injective function, we consider the smooth σ-polyhedral divisor

$$\mathcal{D}'_y = \sum_{Z \in S_y} \Delta_Z \cdot H_{j(Z)}, \quad \text{on} \quad \boldsymbol{A}^n.$$

Since Y is smooth, $\mathfrak{U}_y$ is isomorphic to a formal neighborhood of the fiber over zero for the canonical morphism $\pi' : \widetilde{X}[\mathcal{D}'_y] = \mathbf{Spec}_{A^n} \widetilde{A}[\mathcal{D}'_y] \to A^n$. Finally, Example 2.5 shows that $\widetilde{X}[\mathcal{D}'_y]$ is toric for all y and so X is toroidal. This completes the proof. □

REMARK 2.7. Proposition 2.6 holds in the less restrictive case where only

$$\{Z \text{ prime divisor};\ \Delta_Z \neq \sigma\}$$

is SNC. The definition of supp $\mathcal{D}$ given in Definition 2.2 will be useful in Section 4.

REMARK 2.8. The proof of Proposition 2.6 shows the following stronger statement: if $\mathcal{D}$ is SNC polyhedral divisor, then $(\widetilde{X}[\mathcal{D}], U = \widetilde{X}[\mathcal{D}] \setminus (q^{-1}(\operatorname{supp}\mathcal{D}) \cup \bigcup_\rho \widetilde{E}_\rho))$ is a toroidal embedding without self-intersection in the sense of [KKMS73, p. 57]. Indeed, the only thing remaining to be proved is that the irreducible components of $\widetilde{X}[\mathcal{D}] \setminus U$ are normal, but this follows from the fact that orbit closures on a toric variety are normal [Oda88, Prop. 1.6].

Since the morphism $\varphi : \widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$ is proper and birational, to obtain a desingularization of $X[\mathcal{D}]$ it is enough to have a desingularization of $\widetilde{X}[\mathcal{D}]$. Since $\widetilde{X}[\mathcal{D}]$ is a toroidal embedding without self-intersection, there exists desingularization with toric methods [KKMS73, Chap. II, Th. 11]. We won't use this fact in the sequel.

3. Higher direct images sheaves. In this section we apply the partial desingularization $\varphi : \widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$ to compute the higher direct images of the structure sheaf of any desingularization W of $X[\mathcal{D}]$. This allows us to provide information about the singularities of X in terms of the combinatorial data $(Y, \mathcal{D})$. We recall the following notion.

DEFINITION 3.1. A variety X has rational singularities if there exists a desingularization $\psi : W \to X$, such that

$$\psi_*\mathcal{O}_W = \mathcal{O}_X\,, \quad \text{and} \quad R^i\psi_*\mathcal{O}_W = 0 \quad \text{for all } i > 0\,.$$

The sheaves $R^i\psi_*\mathcal{O}_W$ are independent of the particular choice of a desingularization of X. The first condition $\psi_*\mathcal{O}_W = \mathcal{O}_X$ is equivalent to X being normal.

The following well-known lemma is obtained by applying the Leray spectral sequence.

LEMMA 3.2. *Let $\varphi : \widetilde{X} \to X$ be a proper surjective, birational morphism, and let $\psi : W \to X$ be a desingularization of X. If $\widetilde{X}$ has only rational singularities, then*

$$R^i\psi_*\mathcal{O}_W = R^i\varphi_*\mathcal{O}_{\widetilde{X}} \quad \textit{for all } i \geq 0\,.$$

In the following theorem, for a T-variety $X = X[\mathcal{D}]$ and a desingularization $\psi : W \to X$, we provide an expression for $R^i\psi_*\mathcal{O}_Z$ in terms of the combinatorial data $(Y, \mathcal{D})$. As usual for an A-module M, $M^\sim$ denotes the associated sheaf on $X = \operatorname{Spec} A$.

THEOREM 3.3. *Let $X = X[\mathcal{D}]$, where $\mathcal{D}$ is an SNC polyhedral divisor on Y. If $\psi : W \to X$ is a desingularization, then for every $i \geq 0$, the higher direct image $R^i\psi_*\mathcal{O}_W$ is the sheaf associated to*

$$\bigoplus_{u\in\sigma^\vee_M} H^i(Y, \mathcal{O}(\mathcal{D}(u)))\,.$$

PROOF. Let $\psi : W \to X$ be a desingularization of X. Consider the proper birational morphism $\varphi : \widetilde{X} := \widetilde{X}[\mathcal{D}] \to X$. By Lemma 2.6 $\widetilde{X}$ is toroidal, thus it has only toric singularities which are rational [KKMS73, p. 52]. By Lemma 3.2 we have

$$R^i\psi_*\mathcal{O}_W = R^i\varphi_*\mathcal{O}_{\widetilde{X}}\,, \quad i \geq 0\,.$$

Since X is affine, we have

$$R^i\varphi_*\mathcal{O}_{\widetilde{X}} = H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^{\sim}\,, \quad i \geq 0$$

(see [Har77, Ch. III, Prop. 8.5]). For $\widetilde{A} = \widetilde{A}[\mathcal{D}] = \bigoplus_{u\in\sigma_M^\vee} \mathcal{O}_Y(\mathcal{D}(u))$, we let π be the affine morphism $\pi : \widetilde{X} = \mathbf{Spec}_Y\, \widetilde{A} \to Y$. Since the morphism π is affine, we have

$$H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(Y, \widetilde{A}) = \bigoplus_{u\in\sigma_M^\vee} H^i(Y, \mathcal{O}_Y(\mathcal{D}(u)))\,, \quad i \geq 0\,,$$

by [Har77, Chap. III, Ex. 4.1], proving the theorem. □

As an immediate consequence of Theorem 3.3, in the following theorem, we characterize T-varieties having rational singularities.

THEOREM 3.4. *Let $X = X[\mathcal{D}]$, where $\mathcal{D}$ is an SNC polyhedral divisor on Y. Then X has rational singularities if and only if*

$$H^i(Y, \mathcal{O}_Y(\mathcal{D}(u))) = 0\,, \quad i = 1, \ldots, \dim Y\,,$$

for every $u \in \sigma_M^\vee$.

PROOF. Since X is normal, by Theorem 3.3 we only have to prove that

$$\bigoplus_{u\in\sigma_M^\vee} H^i(Y, \mathcal{O}_Y(\mathcal{D}(u))) = 0 \quad \text{for all } \ i > 0\,.$$

This direct sum is trivial if and only if each summand is. Hence X has rational singularities if and only if $H^i(Y, \mathcal{O}_Y(\mathcal{D}(u))) = 0$ for all $i > 0$ and all $u \in \sigma_M^\vee$.

Finally, $H^i(Y, \mathscr{F}) = 0$ for all $i > \dim Y$ and for any sheaf $\mathscr{F}$ (see [Har77, Chap. III, Th. 2.7]). Now the theorem follows. □

In particular, we have the following corollary.

COROLLARY 3.5. *Let $X = X[\mathcal{D}]$ for some SNC polyhedral divisor $\mathcal{D}$ on Y. If X has only rational singularities, then the structure sheaf $\mathcal{O}_Y$ is acyclic, i.e., $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.*

PROOF. This is the "only if" part of Theorem 3.4 for $u = 0$. □

Recall that a local ring is Cohen-Macaulay if its Krull dimension is equal to its depth. A variety is Cohen-Macaulay if all its local rings are. The following lemma is well known (see for instance [KKMS73, p. 50]).

LEMMA 3.6. *Let $\psi : W \to X$ be a desingularization of X. Then X has rational singularities if and only if X is Cohen-Macaulay and the morphism $\psi_*\omega_W \to \omega_X$ is isomorphic.*

As in Lemma 3.2, applying the Leray spectral sequence shows that the previous Lemma is still valid if we allow W to have rational singularities. In the next proposition, we give a partial criterion as to when a T-variety is Cohen-Macaulay.

PROPOSITION 3.7. *Let $X = X[\mathcal{D}]$, where $\mathcal{D}$ is a proper polyhedral divisor on Y. If* $\operatorname{big}(\mathcal{D}) = \sigma(1)$, *and* $\operatorname{big}(\mathcal{D}_Z) = \mathcal{D}_Z^{(0)}$ *for all prime divisor $Z \in Y$, then X is Cohen-Macaulay if and only if X has rational singularities.*

PROOF. By Theorem 1.2, the contraction $\varphi : \widetilde{X} \to X$ is an isomorphism of open subsets $\varphi^{-1}(U) \subset \widetilde{X}$ and $U \subset X$, where $\widetilde{X} \setminus \varphi^{-1}(U)$ and $X \setminus U$ are of codimension at least two. Furthermore, by the normality of X and $\widetilde{X}$, the dualizing sheaves $\omega_{\widetilde{X}}$ and ω_X are reflexive [Dai02, Prop. 1.2] and $\varphi_*\omega_{\widetilde{X}}$ and ω_X agree on U. In particular, for every open $V \subseteq X$ we have

$$\begin{aligned} H^0(V, \omega_X) &= H^0(V \cap U, \omega_X) = H^0(\varphi^{-1}(V \cap U), \omega_{\widetilde{X}}) \\ &= H^0(\varphi^{-1}(V) \cap \varphi^{-1}(U), \omega_{\widetilde{X}}) = H^0(\varphi^{-1}(V), \omega_{\widetilde{X}}) . \end{aligned}$$

Thus $\varphi_*\omega_{\widetilde{X}} \simeq \omega_X$. The result now follows from Lemma 3.6. □

For isolated singularities, we can give a full classification whenever $\operatorname{rank} N \geq 2$.

COROLLARY 3.8. *Let $X = X[\mathcal{D}]$, where $\mathcal{D}$ is an SNC polyhedral divisor on Y. If* $\operatorname{rank} N \geq 2$ *and X has only isolated singularities, then X is Cohen-Macaulay if and only if X has rational singularities.*

PROOF. We only have to prove the "only if" part. Assume that X is Cohen-Macaulay and let $\psi : W \to X$ be a resolution of singularities. Since X has only isolated singularities, we have that $R^i\psi_*\mathcal{O}_W$ vanishes except possibly for $i = \dim X - 1$ (see [Kov99, Lemma 3.3]). Now Theorem 3.3 shows that $R^i\psi_*\mathcal{O}_W$ vanishes also for $i = \dim X - 1$ since $\dim Y = \dim X - \operatorname{rank} N$ and $\operatorname{rank} N \geq 2$. □

REMARK 3.9. In [Wat81] a criterion of X to be Cohen-Macaulay is given in the case where $\operatorname{rank} N = 1$. In this particular case, a partial criterion for X to have rational singularities is given.

4. Canonical divisors and discrepancies. In the following, we will restrict to the case that Y is projective and σ has the maximal dimension. This corresponds to the fact that there is a unique fixed point lying in the closure of all other orbits. In particular, there is an embedding $k^* \hookrightarrow T$ inducing a good k^*-action on X. Hence, the singularity at the vertex is quasihomogeneous.

LEMMA 4.1 ([PS11, Prop. 3.1]). *If σ is full-dimensional and Y is projective, then every T-invariant Cartier divisor on $X[\mathcal{D}]$ is principal.*

THEOREM 4.2 ([PS11, Cor. 3.15]). *The divisor class group of $X[\mathcal{D}]$ is isomorphic to*

$$\operatorname{Cl} Y \oplus \bigoplus_{\rho} \boldsymbol{Z} D_\rho \oplus \bigoplus_{Z,v} \boldsymbol{Z} D_{Z,v}$$

modulo the relations

$$[Z] = \sum_{v \in \mathrm{big}(\mathcal{D}_Z)} \mu(v) D_{Z,v} ,$$
$$0 = \sum_{\rho} \langle u, \rho \rangle E_\rho + \sum_{Z,v} \mu(v) \langle u, v \rangle D_{Z,v} ,$$

where u runs over all elements of M (or equivalently over a spanning subset).

Fix a canonical divisor $K_Y = \sum_Z b_Z \cdot Z$ on Y. Then by [PS11], T-invariant canonical divisors on $\widetilde{X}[\mathcal{D}]$ and $X[\mathcal{D}]$ are given by

$$K_{\widetilde{X}} = q^* K_Y + \sum_v (\mu(v) - 1) \widetilde{D}_v - \sum_\rho \widetilde{E}_\rho , \tag{2}$$
$$K_X = \pi^* K_Y + \sum_v (\mu(v) - 1) D_v - \sum_\rho E_\rho ,$$

respectively. Here, the sums in the first formula run over all rays and vertices and in the second only over the big rays and big vertices.

Since $X[\mathcal{D}]$ has an attractive point, its Picard group is trivial. Hence, $X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein of index l (i.e., $l \cdot K_X$ is Cartier and l is the minimal positive integer with this property) if and only if there is a character $u \in M$ and a principal divisor $\mathrm{div}(f) = \sum_Z a_Z \cdot Z$ on Y such that $\mathrm{div}(f \cdot \chi^u) = l \cdot K_X$ and l is the minimal positive integer with this property. Due to [PS11], the Weil divisor $\mathrm{div}(f \cdot \chi^u)$ can be calculated as

$$\mathrm{div}(f \cdot \chi^u) = \sum_{Z,v} \mu(v)(\langle u, v \rangle + \mathrm{ord}_Z f) \cdot D_{Z,v} + \sum_\rho \langle u, n_\rho \rangle \cdot E_\rho .$$

Our first aim is to express this formula as a matrix multiplication. We assume that $\mathrm{supp}\, \mathcal{D} \cup \mathrm{supp}\, f \subset \{Z_1, \dots, Z_s\}$. We set $\mathrm{big}(\mathcal{D}_{Z_i}) = \{v_i^1, \dots, v_i^{r_i}\}$, $\mathrm{big}(\mathcal{D}) = \{\rho_1, \dots, \rho_r\}$ and $n_{\rho_1}, \dots, n_{\rho_r}$ are the primitive lattice generators. We denote $\mu(v_j^i)$ by μ_j^i. Then we can read of the coefficients of $\mathrm{div}(f \cdot \chi^u)$ as components of the product vector

$$\left(\begin{array}{ccccc}
\mu_1^1 & 0 & \dots & 0 & \mu_1^1 v_1^1 \\
\vdots & \vdots & & \vdots & \vdots \\
\mu_1^{r_1} & 0 & \dots & 0 & \mu_1^{r_1} v_1^{r_1} \\
 & & \ddots & & \\
0 & 0 & \dots & \mu_s^1 & \mu_s^1 v_s^1 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & \mu_s^{r_s} & \mu_s^{r_s} v_s^{r_s} \\
\hline
0 & 0 & \dots & 0 & n_{\rho_1} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & n_{\rho_r}
\end{array}\right)
\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ u \end{pmatrix} \tag{3}$$

Here, we fix isomorphisms $N = \boldsymbol{Z}^n$ and write elements of N as row vectors and elements of $M = N^*$ as column vectors.

Since we want to have $\operatorname{div}(f \cdot \chi^u) = l \cdot K_X$, the equation (2) leeds to a system of linear equations. Before writing down this system, we want to incorporate the condition that $\sum_i a_i Z_i$ is principal. This implies that $0 = \sum a_i \overline{Z}_i$ in $\mathrm{NS}(Y) := \operatorname{Div} Y/\overset{\text{num}}{\sim} \cong \boldsymbol{Z}^r$. We end up with the following system of equations

$$\tag{4} \underbrace{\left(\begin{array}{cccc|c} \overline{Z}_1 & \overline{Z}_2 & \dots & \overline{Z}_s & 0 \\ \hline \mu_1^1 & 0 & \dots & 0 & \mu_1^1 v_1^1 \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_1^{r_1} & 0 & \dots & 0 & \mu_1^{r_1} v_1^{r_1} \\ & & \ddots & & \\ 0 & 0 & \dots & \mu_s^1 & \mu_s^1 v_s^1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \mu_s^{r_s} & \mu_s^{r_s} v_s^{r_s} \\ \hline 0 & 0 & \dots & 0 & n_{\rho_1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & n_{\rho_r} \end{array}\right)}_{=:A} \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ u \end{pmatrix} = \left(\begin{array}{c} 0 \\ \hline \mu_1^1 b_1 + \mu_1^1 - 1 \\ \vdots \\ \mu_1^{r_1} b_1 + \mu_1^{r_1} - 1 \\ \\ \vdots \\ \mu_s^1 b_s + \mu_s^1 - 1 \\ \vdots \\ \mu_s^{r_s} b_s + \mu_s^{r_s} - 1 \\ \hline -1 \\ \vdots \\ -1 \end{array}\right)$$

Here, we assume that $\operatorname{supp} \mathcal{D} \cup \operatorname{supp} K_Y \subset \{Z_1, \dots, Z_s\}$ and that these prime divisors span $\mathrm{NS}(Y)$. The classes of Z_i in $\mathrm{NS}(Y) \cong \boldsymbol{Z}^r$ by $\overline{Z}_i$. We fix an isomorphism $\mathrm{NS}(Y) := \operatorname{Div} Y/\overset{\text{num}}{\sim} \cong \boldsymbol{Z}^r$ and write elements of $\mathrm{NS}(Y)$ as column vectors.

PROPOSITION 4.3. *$X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein if and only if the above system has a (unique) solution $u \in (1/l) \cdot M$, $a_1, \dots, a_s \in \boldsymbol{Q}$, such that $l \cdot \sum_{i=1}^s a_i \cdot Z_i$ is principal for some $l > 0$. The Gorenstein index of $X[\mathcal{D}]$ is the minimal l satisfying these two conditions.*

PROOF. This is immediate by the above considerations. □

THEOREM 4.4. *$X = X[\mathcal{D}]$ is $\boldsymbol{Q}$-factorial if and only if*

$$\sum_Z ({}^{\#}\mathrm{big}(\mathcal{D}_Z) - 1) + {}^{\#}\mathrm{big}(\mathcal{D}) = \dim N - \operatorname{rank} \operatorname{Cl} Y\,.$$

In particular, Y has a finitely generated class group if $X[\mathcal{D}]$ is $\boldsymbol{Q}$-factorial.

PROOF. We consider any set of prime divisors $Z_1, \dots, Z_s$ which contains the support of $\mathcal{D}$. Then by Theorem 4.2 the vector space $\mathrm{Cl}(X) \otimes \boldsymbol{Q}$ is generated by a basis of $\mathrm{Cl}(Y) \otimes \boldsymbol{Q}$ and the divisors E_ρ, $D_{Z_i,v}$ with $1 \leq i \leq s$ and $v \in \mathrm{big}\, \mathcal{D}_{Z_i}$. These are

$$\tag{5} \operatorname{rank} \mathrm{Cl}(Y) + {}^{\#}\mathrm{big}(\mathcal{D}) + \Big(s + \sum_Z ({}^{\#}\mathrm{big}(\mathcal{D}_Z) - 1)\Big)$$

generators. The relations are

$$[Z_i] = \sum_{v \in \mathrm{big}(\mathcal{D}_{Z_i})} \mu(v) D_{Z,v}\,, \quad 1 \leq i \leq s\,,$$

$$0 = \sum_{\rho} \langle u, \rho \rangle E_\rho + \sum_{Z,v} \mu(v) \langle u, v \rangle D_{Z,v}\,, \quad u \in \{u_1, \ldots, u_r\} \text{ a basis of } M\,.$$

Hence, $\operatorname{rank} \mathrm{Cl}(X) < \infty$ if and only if $\mathrm{Cl}(Y) < \infty$. Moreover, $\mathrm{Cl}(X) \otimes \boldsymbol{Q}$ is isomorphic to the cokernel of the matrix A in (4). Lemma 4.6, given below, shows that the rank of the matrix is $s + r$. Now, $\mathrm{Cl}(X) \otimes \boldsymbol{Q} = 0$ holds if and only if the number of rows of that matrix and hence (5), the number of our generators, equals $r + s$. □

Note that the condition for $\boldsymbol{Q}$-factoriality in Theorem 4.4 is equivalent to the fact that $\mathrm{Cl}\, Y$ has finite rank and the matrix is square. Moreover, for factoriality we get the following stronger condition.

PROPOSITION 4.5. *$X[\mathcal{D}]$ is factorial if and only if $\mathrm{Cl}(Y) \cong \boldsymbol{Z}^l$ and the above matrix is square and has determinant ± 1.*

PROOF. If we consider an arbitrary T-ivariant Weil divisor instead of the canonical one, we end up with the system of equations from (4) but with an almost arbitrary right-hand side

$$A \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \hline \vdots \\ \hline \vdots \end{pmatrix}.$$

Here, A denotes the matrix from (4). Now, being factorial means that for every choice on the right-hand side, we find an integral solution such that $\sum a_i Z_i$ is principal.

Lemma 4.6 shows that the columns of the matrix M are linearly independent. By Theorem 4.4 we know that $\mathrm{Cl}(Y)$ has finite rank. Moreover, in the case of a free class group, we have $\mathrm{Cl}(Y) \cong \mathrm{NS}(Y)$ and every solution $\binom{a}{u}$ automatically corresponds to an principal divisor $\operatorname{div} f = \sum a_i Z_i$ on Y. Now, we will find an integral solution for every right-hand side if and only if M has determinant ± 1 (since this implies that M is invertible).

Let's now assume that $D = \sum_i b_i Z_i$ gives a torsion element in $\mathrm{Cl}(Y)$. Then $\pi^* D$ would give a torsion element in $\mathrm{Cl}(X)$. Let us assume, that $\operatorname{div}(f \cdot \chi^u) = \pi^* D$. Then $A \cdot (\operatorname{ord}_{Z_1} f - b_1, \ldots, \operatorname{ord}_{Z_s} f - b_s, u)^t = 0$. By Lemma 4.6 this implies that $u = 0$ and $D = \operatorname{div} f$. Hence, every torsion element in $\mathrm{Cl}(Y)$ has to be trivial. □

LEMMA 4.6. *The columns of the matrix in* (4) *are linearly independent.*

PROOF. We choose a non-big ray $\rho \in \operatorname{tail} \mathcal{D}$ and a maximal cone δ from the normal quasifan of $\mathcal{D}$ such that $\rho^\perp \cap \delta$ is a facet, and we denote this facet by τ.

We have a linear map $F : u \mapsto \overline{\mathcal{D}(u)} \in \boldsymbol{Q}^r \cong \mathrm{NS}_{\boldsymbol{Q}}(Y)$. Now we choose any interior element $w \in \operatorname{relint} \delta$, hence $\mathcal{D}(u)$ is big by the properness of $\mathcal{D}$. We consider the subspaces

$$V := V_1 + V_2\,, \quad V_1 := \operatorname{span}(\overline{Z}\ ;\ \mathcal{D}(w)|_Z \text{ is not big})\,, \quad V_2 := \operatorname{span}(F(\tau))\,.$$

We claim that $F(w) \notin V$. The semi-ample and big divisor $\mathcal{D}(w)$ defines a birational morphism

$$\varphi : Y \to \operatorname{Proj} \bigoplus_{i \geq 0} H^0(Y, i \cdot \mathcal{D}(w)).$$

By definition $\varphi_* \mathcal{D}(u)$ is ample, hence big and φ contracts every prime divisor Z such that $\mathcal{D}(u)|_Z$ is not big. Let us assume that $\overline{\mathcal{D}(w)} \in V$. It follows that

$$\varphi_* \overline{D(w)} \in \varphi_*(V) = \varphi_*(V_1) + \varphi_*(V_2) = 0 + \varphi_* V_2.$$

But since V_2 does not contain any big class, the same is true for $\varphi_* V_2$. This contradicts the ampleness of $\varphi_* \mathcal{D}(w)$.

Now we choose a basis B of V and complement $\{\overline{\mathcal{D}(w)}\} \cup B$ to get a basis of $\mathrm{NS}_{\boldsymbol{Q}}(Y)$. This leads to a coordinate map $x_1 : \mathrm{NS}_{\boldsymbol{Q}} \to \boldsymbol{Q}$ corresponding to the basis element $\overline{\mathcal{D}(w)}$. For every Z_i there is a vertex $v_i \in \mathcal{D}_{Z_i}$ such that $\langle w, \cdot \rangle$ is minimized at this vertex. Now we sum up the corresponding rows in the matrix with multiplicity $x_1(Z_i)/\mu(v_i)$ (by choice of the matrix, all non-big vertices v_i have $x_1(Z_i) = 0$) and get $\left(x_1(\overline{Z_1}), \ldots, x_1(\overline{Z_r}), v_\rho\right)$, where $v_\rho := \sum_i x_1(\overline{Z_i}) \cdot v_i$. By construction we have $x_1(F(u)) = \langle u, v_\rho \rangle$ for $u \in \delta$. Since $x_1(F(u)) = 0$ and $x_1(F(u + \alpha w)) = \alpha$ for $u \in \tau$ and $\alpha > 0$, it follows that v_ρ is a non-zero element in ρ.

Now assume that $\sum_i \lambda_i c_i = 0$, where the c_i's are the columns of the matrix. Then for every big ray ρ we get $\sum_{i=1}^n \lambda_{r+i} \cdot (n_\rho)_i = 0$, where $(n_\rho)_i$ denotes the i-th coordinate of the primitive generator of ρ. Since $\sum_{i=1}^r \lambda_i \cdot \overline{Z}_i = 0$ holds because of the first rows of the matrix, we get $\sum_{i=1}^n \lambda_{r+i} \cdot (v_\rho)_i = 0$ for every non-big ray of $\mathcal{D}$. The fact that the tail cone $\operatorname{tail} \mathcal{D}$ has maximal dimension implies that $\lambda_{r+1}, \ldots, \lambda_{r+n}$ are zero.

Let us assume that the first r' columns correspond to prime divisors with $\operatorname{big}(\mathcal{D}_Z) \neq \emptyset$. By construction of the matrix, these columns have staircase structure. Hence, the coefficients $\lambda_1, \ldots, \lambda_{r'}$ vanish. The remaining columns are of the form $\binom{\overline{Z}_i}{0}$, i.e., all but the first r entries vanish. Since the sets of big vertices $\operatorname{big}(\mathcal{D}_{Z_i})$ are empty, $\mathcal{D}(u)|_{Z_i}$ is not big for every $u \in \operatorname{relint} \sigma^\vee$. Hence, the Z_i are exceptional prime divisor of the birational projective map

$$\vartheta_u : Y \to Y_u \operatorname{Proj} \left(\bigoplus_{j \geq 0} H^0 \left(Y, \mathcal{O}(j \cdot \mathcal{D}(u))\right) \right). \tag{6}$$

In particular, their images in $\mathrm{NS}(Y)$ are linearly independent, which completes the proof. □

Let us assume that X is $\boldsymbol{Q}$-Gorenstein. Remember that, for a birational proper morphism $r : \widetilde{X} \to X$, we have a canonical divisor $K_{\widetilde{X}}$ on $\widetilde{X}$ such that that the discrepancy divisor $\operatorname{Discr}(r) = K_{\widetilde{X}} - r^* K_X$ is supported only at the exceptional divisor $\sum_i E_i$ of r. Hence, it has the form $\sum_i \alpha_i E_i$. The coefficient α_i of $\operatorname{Discr}(r)$ is called discrepancies of r at E_i. Similarly, the discrepancies of a pair (X, B), consisting of a normal variety and a $\boldsymbol{Q}$-Cartier divisor, are the coefficients β_i of $\operatorname{Discr}(r, B) := K_{\widetilde{X}} - r^*(K_X + B) = \sum \beta_i E_i$. With this notation we have

$$\mathrm{Discr}(r' \circ r) = \mathrm{Discr}(r', -\mathrm{Discr}(r))\,. \tag{7}$$

Consider an SNC polyhedral divisor $\mathcal{D}$. Fix $y \in Y$ and consider the prime divisors $Z_1, \ldots, Z_m$ from the support of $\mathcal{D}$ containing y. We may choose additional prime divisors $Z_{m+1}, \ldots, Z_{\dim(Y)}$, such that $Z_1, \ldots, Z_{\dim(Y)}$ intersect tranversally at y.

From Section 2, we know that the formal neighborhood of every fiber $\widetilde{X}_y$ of $\widetilde{X}[\mathcal{D}] \to Y$ is isomorphic to that of of a closed subset of a toric variety corresponding to some cone $\sigma'_y \in N_{\boldsymbol{Q}} \oplus \boldsymbol{Q}^{\dim Y}$. Moreover, the isomorphism identifies $\widetilde{D}_{Z_i,v}$ and $V(\boldsymbol{Q}_{\geq 0}(v, e_i))$ as well as E_ρ and $V(\rho \times \boldsymbol{0})$.

Now we may calculate a representation $K_X = \pi^* H + \mathrm{div}(\chi^u)$ of the canonical divisor on X by solving a system of linear equations as in Proposition 4.3. Here, $H = \sum_Z a_Z \cdot Z$ is a principal divisor on Y. Having such a representation, we get the discrepancies of $\widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$ at $\widetilde{D}_{Z,v}$ or $\widetilde{E}_\rho$, respectively as

$$\mathrm{discr}_{Z,v} = \mu(v)(b_Z - a_Z - \langle u, v\rangle + 1) - 1\,, \quad \mathrm{discr}_\rho = -1 - \langle u, n_\rho\rangle\,. \tag{8}$$

We may also consider a toroidal desingularization $\varphi : \overline{X} \to \widetilde{X}[\mathcal{D}]$, obtained by toric desingularisations of the $X_{\sigma'_y}$. Since the discrepancies $\mathrm{discr}_{Z,v}$ vanish for $Z \notin \operatorname{supp}\mathcal{D}$, the discrepancy divisor on $\widetilde{X}[\mathcal{D}]$ corresponds to a toric divisor $B \subset X_{\sigma'_y}$ and we are able to calculate the discrepancy divisor $\mathrm{Discr}(\varphi, B)$ by toric methods.

DEFINITION 4.7. We say that a pair (X, B) is log-terminal if, for a log-resolution of (X, B), the discrepancies are greater than -1. We say that X is log-terminal if the pair $(X, 0)$ is log-terminal.

For the toric case we have the following lemma

LEMMA 4.8. *A toric pair (X_σ, B) is log-terminal as long as $-B + \sum_\rho V(\rho)$ is effective and $\boldsymbol{Q}$-Cartier.*

PROOF. We may argue as in the proof of [Fuj03, Lemma 5.1]. Since we have the equality $K_{X_\sigma} = -\sum_\rho V(\rho)$, the $\boldsymbol{Q}$-divisor $B + K_{X_\sigma}$ corresponds to an element $u \in M_{\boldsymbol{Q}}$ such that $\langle u, n_\rho\rangle < 0$ for every $\rho \in \sigma(1)$. But then the primitive generator $n_{\rho'}$ of a ray ρ' in a subdivision Σ of σ is a positive combination of primitive generators n_ρ of rays ρ of σ. Hence, $\langle u, n_{\rho'}\rangle < 0$ holds. But now we have $\mathrm{discr}_{V(\rho')} = -1 - \langle u, n_{\rho'}\rangle > -1$. □

A $\boldsymbol{Q}$-divisor $B = \sum_Z b_Z \cdot Z$ is called a boundary divisor if $0 < b_Z \leq 1$. For a strictly ample polyhedral divisor on Y, we define the boundary divisor $B := \sum_Z ((\mu_Z - 1)/\mu_Z) \cdot Z$ on Y, where μ_Z is defined as $\max\{\mu(v);\, v \in \mathcal{D}_Z\}$.

THEOREM 4.9. *Assume that $\mathcal{D}$ is strictly ample and $X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein, then $X[\mathcal{D}]$ is log-terminal if and only if (Y, B) is log-terminal and $-B - K_Y$ is ample.*

PROOF. Let $K_X = \pi^* H + \mathrm{div}(\chi^w)$ be a representation as above. By (2) we have

$$K_Y + B = H + \sum_Z \langle w, v_Z\rangle \cdot Z\,, \tag{9}$$

here $v_Z \in \mathrm{big}(\mathcal{D}_Z)$ denotes the vertex where μ obtains its maximum.

For any ray $\rho \in \sigma(1)$, the value $\langle w, n_\rho \rangle$ has to be negative because of the condition $\mathrm{discr}_\rho = -1 - \langle w, n_\rho \rangle > -1$ for non-big rays or $\langle w, n_\rho \rangle = -1$ for big rays, respectively. It follows that $-w \in \mathrm{relint}(\sigma^\vee)$.

Let v'_Z denote vertex in $\mathcal{D}_Z$, where $-w$ is minimized. On the one hand, we get $\mathcal{D}(-w) \leq \sum_Z \langle w, v_z \rangle \cdot Z$. On the other hand we have

$$\langle w, v_Z \rangle - \frac{\mu(v_Z) - 1}{\mu(v_Z)} = \langle w, v'_Z \rangle - \frac{\mu(v'_Z) - 1}{\mu(v'_Z)},$$

and by the maximality of $\mu(v_Z)$ we infer that $\langle w, v_Z \rangle \leq \langle w, v'_Z \rangle$. We conclude that

$$K_Y + B = H - \mathcal{D}(-w) = H + \sum_Z \langle w, v'_Z \rangle \cdot Z \,. \tag{10}$$

Since $\mathcal{D}(-w)$ is ample this implies the Fano property for the pair (Y, B).

Now consider a birational proper morphism $\varphi : \widetilde{Y} \to Y$. Also denote $\widetilde{X}[\varphi^*\mathcal{D}]$ by $\widetilde{X}$. Consider a prime divisor $E \subset \widetilde{Y}$ and denote by $(\varphi^* Z)_E$ the coefficient of $\varphi^* Z$ at E. Note that $v'_E := \sum_Z (\varphi^* Z)_E \cdot v'_Z$ is a vertex in $(\varphi^*\mathcal{D})_E$. If v'_E is not a big vertex, by (8) we get the discrepancy

$$\begin{aligned} \mathrm{discr}_{v'_E} &= \mu(v'_E)\big((K_{\widetilde{Y}})_E - (\varphi^* H)_E - \langle w, v'_E \rangle + 1\big) - 1 \\ &= \mu(v'_E)\big((K_{\widetilde{Y}})_E - \varphi^*(K_Y + B)_E + 1\big) - 1 \,. \end{aligned} \tag{11}$$

For the case that E is an exceptional divisor of φ, this proves the log-terminal property for (Y, B).

For the other direction, we first show that the Fano property for (Y, B) implies that $-w \in \sigma^\vee$. For big rays $\rho \in \mathrm{big}(\mathcal{D})$ we have $\langle w, n_\rho \rangle = -1$ by (2). For a non-big ray ρ we consider a maximal chamber of linearity $\delta \subset \sigma^\vee$ such that $\tau = \rho^\perp \cap \delta$ is a facet. This corresponds to a family of vertices v_Z^u such that $\mathcal{D}(u) = \sum_Z \langle u, v_Z^u \rangle \cdot Z$ for $u \in \mathrm{relint}\,\delta$. Now there exists a decomposition $-w = \alpha \cdot u + u_\tau$ such that $u_\tau \in \tau$ and $u \in \mathrm{relint}\,\delta$. Hence, we have

$$-K_Y - B \;\leq\; -H + \sum_Z \langle -w, v_Z^u \rangle \cdot Z \;\sim\; \mathcal{D}(u_\tau) + \alpha \mathcal{D}(u) \,.$$

By our precondition $-K_Y - B$ is big. This implies that the right-hand side is big, too. Then we must have $\alpha > 0$ since $\mathcal{D}(u)$ is big but $\mathcal{D}(u_\tau)$ is not. By $\langle -w, n_\rho \rangle = \alpha \cdot \langle u, n_\rho \rangle$, we conclude that $\langle -w, n_\rho \rangle > 0$ and hence $\mathrm{discr}_\rho = -1 - \langle w, n_\rho \rangle > -1$ and $-w \in \sigma^\vee$. Let $\varphi : \widetilde{Y} \to Y$ be a desingularization such that $\varphi^*\mathcal{D}$ is SNC. By the equation (11), we infer that $\mathrm{disc}_{v_E} > -1$ for every exceptional divisor E and every vertex $v_E \in (\varphi^*\mathcal{D})_E$. By Lemma 4.12 this completes the proof. □

REMARK 4.10. As a special case of the theorem, we recover the fact that the log-terminal property of a section ring characterizes log-terminal Fano varieties [SS10, Prop. 5.4].

REMARK 4.11. A variety is called *of Fano type* if there exists a boundary divisor such that the pair (X, B) is Fano and log-terminal. In recent papers [Bro11, GOST12, KO12], varieties Y of Fano type are characterized by the log-terminality of the Cox ring $\mathbf{Cox}(Z) := \bigoplus_{D \in \mathrm{Cl}\, Z} \mathcal{O}\big(Y, \mathcal{O}(D)\big)$. This observation is very much related to Theorem 4.9. Indeed, there exists a projective morphism $Y \to Z$ and a polyhedral divisor $\mathcal{D}$ on Y describing the multigraded ring $\mathbf{Cox}(Z)$ (see [AW11]). If we omit the condition that $\mathcal{D}$ is strictly ample, we may at least conclude that $-(K_Y + B) = \mathcal{D}(-w)$ is semi-ample and big in the proof of Theorem 4.9. Hence, as in (6) we get a birational contraction morphism $Y \to Y_{-w}$ corresponding to this divisor. By replacing Y with Y_{-w} in the proof of Theorem 4.9, we conclude that (Y_{-w}, B) is Fano and log-terminal, hence Y is of Fano type. By construction Y_{-w} is a small birational modification of Z (i.e., they are isomoriphic outside closed subsets of codimension 1). Hence, by [GOST12, Lemma 3.1] Z is also of Fano type.

LEMMA 4.12. *Let $\mathcal{D}$ be an SNC polyhedral divisor. Then $X[\mathcal{D}]$ is log-terminal if and only if the discrepancies of $\psi : \widetilde{X}[\mathcal{D}] \to X[\mathcal{D}]$ are all greater than -1.*

PROOF. Proposition 2.6 shows that $\widetilde{X}[\mathcal{D}]$ is toroidal. Moreover, the exceptional locus of ψ is a toroidal subset. Hence, by Lemma 4.8 and (7) we get our claim. □

COROLLARY 4.13. *Every $\boldsymbol{Q}$-Gorenstein T-variety X of complexity c with singular locus of codimension greater than $c+1$ is log-terminal.*

PROOF. We may assume that X is affine. Given an SNC polyhedral divisor for X, we consider exceptional divisors $\widetilde{D}_{Z,v}$, $\widetilde{E}_\rho$ of $\widetilde{X}[\mathcal{D}] \to X$ with discrepancies at most -1. By the orbit decomposition of $X[\mathcal{D}]$ given in [AH06], we know that $\widetilde{E}_\rho$ is contracted via r to a closed subvariety of codimension at most $c+1$ in $X[\mathcal{D}]$ and $\widetilde{D}_{Z,v}$ to a subvariety of codimension at most c. But $r(\widetilde{E}_\rho)$ and $r(\widetilde{D}_{Z,v})$ are necessarily parts of the singular locus. □

5. Complexity one. As an application, in this section we restate our previous results in this particular setting. This allows us to rediscover some well-known results with our methods.

Let $\mathcal{D}$ be a proper polyhedral divisor on Y. If the corresponding T-action on $X = X[\mathcal{D}]$ has complexity one then Y is a curve. Since any normal curve is smooth and any proper birational morphism between smooth curves is an isomorphism, the base curve Y is uniquely determined by the T-action on X.

Furthermore, any curve Y is either affine or projective, and any proper polyhedral divisor $\mathcal{D}$ on Y is SNC and strictly ample. Let $\mathcal{D}$ and $\mathcal{D}'$ be two proper polyhedral divisors on Y with the same tail cone σ. Then $X[\mathcal{D}] \simeq X[\mathcal{D}']$ equivariantly if and only if the application

$$\varDelta : \sigma^\vee \to \mathrm{Div}_{\boldsymbol{Q}}(Y)\,, \quad u \mapsto \mathcal{D}(u) - \mathcal{D}'(u)\,,$$

is the restriction of a linear map and $\varDelta(u)$ is a principal Cartier divisor for all $u \in \sigma_M^\vee$.

The simplest case is the one where $N = \boldsymbol{Z}$, i.e., the case of k^*-surfaces. In this particularly simple setting, there are only two non-equivalent pointed polyhedral cones in $N_{\boldsymbol{Q}} \simeq \boldsymbol{Q}$ corresponding to $\sigma = \{0\}$ and $\sigma = \boldsymbol{Q}_{\geq 0}$.

If we assume further that Y is projective, then $\sigma \neq \{0\}$ since $\mathcal{D}(1)$ and $\mathcal{D}(-1)$ can not be big simultaneously and so we have $\sigma = \boldsymbol{Q}_{\geq 0}$. In this case $\mathcal{D}(u) = u\mathcal{D}(1)$. Hence $\mathcal{D}$ is completely determined by $\mathcal{D}_1 := \mathcal{D}(1)$, and $X[\mathcal{D}] \simeq X[\mathcal{D}']$ equivariantly if and only if $\mathcal{D}_1 - \mathcal{D}'_1$ is a principal Cartier divisor. We also let

$$\mathcal{D}_1 = \sum_{i=1}^{r} \frac{e_i}{m_i} \cdot z_i\,, \quad \text{where} \quad \gcd(e_i, m_i) = 1\,, \text{ and } m_i > 0\,. \tag{12}$$

In this case, the algebra $A[\mathcal{D}]$ is also known as the section ring of $\mathcal{D}_1$.

5.1. Rational singularities. The following proposition gives a simple characterization of T-varieties of complexity one having rational singularities.

PROPOSITION 5.1. *Let $X = X[\mathcal{D}]$, where $\mathcal{D}$ is an SNC polyhedral divisor on a smooth curve Y. Then X has only rational singularities if and only if*

(i) *Y is affine, or*
(ii) *$Y = \boldsymbol{P}^1$ and $\deg\lfloor\mathcal{D}(u)\rfloor \geq -1$ for all $u \in \sigma_M^\vee$.*

PROOF. If Y is affine, then the morphism $\varphi : \widetilde{X}[\mathcal{D}] \to X$ is an isomorphism. By Lemma 2.6 X is toroidal, and thus X has only toric singularities and toric singularities are rational.

If Y is projective of genus g, we have $\dim H^1(Y, \mathcal{O}_Y) = g$. Hence, by Corollary 3.5, if X has rational singularities then $C = \boldsymbol{P}^1$. Furthermore, for the projective line we have $H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(D)) \neq 0$ if and only if $\deg D \leq -2$ [Har77, Chap. III, Th. 5.1]. Now the corollary follows from Theorem 3.4. □

In the next proposition, we provide a partial criterion for the Cohen-Macaulay property in the case of complexity one. Recall that if the complexity is one, a ray $\rho \in \sigma(1)$ is a big ray if and only if $\deg \mathcal{D} \cap \rho = \emptyset$.

PROPOSITION 5.2. *Let $X = X[\mathcal{D}]$, where Y is a smooth curve and $\mathcal{D}$ is an SNC polyhedral divisor on Y. Then X is Cohen-Macaulay if either*

(i) *Y is affine, or*
(ii) $\operatorname{rank} M = 1$.

Moreover, if Y is projective and $\operatorname{big}(\mathcal{D}) = \sigma(1)$, *then X is Cohen-Macaulay if and only if X has rational singularities.*

PROOF. If Y is affine then $X = \widetilde{X}[\mathcal{D}]$. Thus X has rational singularities and so X is Cohen-Macaulay. If $\operatorname{rank} M = 1$ then X is a normal surface. By Serre's criterion, any normal surface is Cohen-Macaulay (see [Eis95, Th. 11.5]). Finally, the last assertion is a specialization of Proposition 3.7. □

REMARK 5.3. Corollary 3.8 and Proposition 5.2 give a full classification of isolated Cohen-Macaulay singularities on T-varieties of complexity one.

5.2. Log-terminal and canonical singularities. In the complexity one case, every proper polyhedral divisor is strictly ample since ampleness and bigness coincide. Now, Theorem 4.9 gives rise to the following corollary.

COROLLARY 5.4. *Let $\mathcal{D} = \sum_z \Delta_z \cdot z$ be a proper polyhedral divisor on a curve Y. Assume that $X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein. Then $X[\mathcal{D}]$ is log-terminal if and only if either*

(i) *Y is affine, or*
(ii) *$Y = \boldsymbol{P}^1$ and $\sum_z (\mu_z - 1)/\mu_z < 2$.*

PROOF. By Theorem 4.9 we know that $-K_Y - \sum_z ((\mu_z - 1)/\mu_z) \cdot z$ has to be ample. This is the case exactly under the conditions on the corollary. □

REMARK 5.5. (i) The second condition in the corollary can be made more explicit: there are at most three coefficients $\mathcal{D}_{z_1}, \mathcal{D}_{z_2}, \mathcal{D}_{z_3}$ on $\boldsymbol{P}^1$ having non-integral vertices, and the triple $(\mu_{z_1}, \mu_{z_2}, \mu_{z_3})$ is one of the Platonic triples $(1, p, q)$, $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$. Here $p \geq q \geq 1$, and $r \geq 2$.

(ii) It is well known that log-terminal singularities are rational. Indeed, since $a/b - \lfloor a/b \rfloor \leq (b-1)/b$, the condition $\sum_z (\mu_z - 1)/\mu_z < 2$ ensures that $\deg \lfloor \mathcal{D}(u) \rfloor > \deg \mathcal{D}(u) - 2 \geq -2$. Thus $X[\mathcal{D}]$ has rational singularities by Corollary 5.1.

As a direct consequence, we get the following corollary characterizing quasihomogeneous surfaces having log-terminal singularities. Recall the definition of $\mathcal{D}_1$ in (12).

COROLLARY 5.6. *Every quasihomogeneous log-terminal surface singularity is isomorphic to the section ring of the divisor*

$$\mathcal{D}_1 = \frac{e_1}{m_1} \cdot [0] + \frac{e_2}{m_2} \cdot [1] + \frac{e_3}{m_3} \cdot [\infty]$$

with $\deg \mathcal{D}_1 > 0$ *on* $Y = \boldsymbol{P}^1$. *Here* (m_1, m_2, m_3) *is one of the Platonic triples* $(1, p, q)$, $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$, *and* $(2, 3, 5)$, *where* $p \geq q \geq 1$, *and* $r \geq 2$.

We now characterize quasihomogeneous surfaces having canonical singularities, i.e., double rational points.

THEOREM 5.7. *Every quasihomogeneous canonical surface singularity is isomorphic to the section ring of one of the following $\boldsymbol{Q}$-divisors on $\boldsymbol{P}^1$:*

$$\begin{array}{lll}
\mathrm{A}_i: & \dfrac{i+1}{i} \cdot [\infty]\,, & i \geq 1\,. \\
\mathrm{D}_i: & \dfrac{1}{2} \cdot [0] + \dfrac{1}{2} \cdot [1] - \dfrac{1}{(i-2)} \cdot [\infty]\,, & i \geq 4\,. \\
\mathrm{E}_i: & \dfrac{1}{2} \cdot [0] + \dfrac{1}{3} \cdot [1] - \dfrac{1}{(i-3)} \cdot [\infty]\,, & i = 6, 7, 8\,.
\end{array}$$

PROOF. Canonical singularities are log-terminal. Hence, it suffices to consider a polyhedral divisors $\mathcal{D}$ on $\boldsymbol{P}^1$ as in Corollary 5.6, i.e., those of the form

$$\mathcal{D}_1 = \frac{e_1}{m_1} \cdot [0] + \frac{e_2}{m_2} \cdot [1] + \frac{e_3}{m_3} \cdot [\infty]\,, \quad \text{and} \quad \deg \mathcal{D}_1 > 0\,.$$

Let $1 \leq m_1 \leq m_2 \leq m_3$. Up to linear equivalence, we may assume that $m_1 > e_1 \geq 0$ and $m_2 > e_2 \geq 0$. If $m_1 = 1$ we have $e_1 = 0$ and X is isomorphic to the affine toric variety given by the cone $\operatorname{pos}((e_2, m_2), (e_3, -m_3))$. But every cone is isomorphic to a subcone of $\operatorname{pos}((0, 1), (1, 1))$. Therefore, we may assume that $m_1 = m_2 = 1$, $e_1 = e_2 = 0$ and $e_3 \geq m_3$.

The system of equations from Proposition 4.3 takes the form

$$\begin{array}{cccc|l} 1 & 1 & 1 & 0 & 0 \\ m_1 & 0 & 0 & e_1 & m_1 - 3 \\ 0 & m_2 & 0 & e_2 & m_2 - 1 \\ 0 & 0 & m_3 & e_3 & m_3 - 1\,. \end{array}$$

Any solution (a_1, a_2, a_3, u) must also fulfill

$$u \cdot \deg D = \sum_i \frac{m_i - 1}{m_i} - 2\,. \tag{13}$$

The formula for the discrepancy at E_ρ yields $\operatorname{disc}_\rho = -1 - u$. Hence, we need $u \leq -1$. For the case $(1, 1, q)$, the equation (13) yields $u = -(m_3 + 1)/e_3$. Hence we must have $e_3 = m_3 + 1$. For the case $(2, 2, r)$, the equation (13) takes the form $u(m_3 + e_3)/m_3 = 1/m_3$ and we get $e_3 = 1 - m_3$. For the remaining case $(2, 3, r)$, we get

$$\frac{3 + 2e_2 + 2e_3}{6} = \frac{1}{6}\,, \quad \frac{6 + 4e_2 + 3e_3}{12} = \frac{1}{12}\,, \quad \frac{15 + 10e_2 + 6e_3}{30} = \frac{1}{30}\,.$$

Since $1, 2$ are the only possibilities for e_2, we infer that $e_2 = 1$ and $e_3 = 1 - m_3$. □

5.3. Elliptic singularities. Let (X, x) be a normal singularity, and let $\psi : W \to X$ be a resolution of the singularity (X, x). We says that (X, x) is an *elliptic singularity* if

$$R^i \psi_* \mathcal{O}_W = 0 \quad \text{for all } i \in \{1, \ldots, \dim X - 2\}\,, \quad \text{and} \quad R^{\dim X - 1} \psi_* \mathcal{O}_W \simeq k\,.$$

An elliptic singularity is *minimal* if it is Gorenstein. (see, e.g., [Lau77] and [Dai02]).

In the complexity one case, $R^i \psi_* \mathcal{O}_W = 0$ for all $i \geq 2$. Thus, the only way to have elliptic singularities is to have $M = \boldsymbol{Z}$. That is, the case of k^*-surfaces. In the following, we restrict to this case.

We give now a simple criterion as to when $X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein. This is a specialization of Proposition 4.3. Recall that the boundary divisor is defined in this particular case as $B = \sum_i ((m_i - 1)/m_i) \cdot z_i$. We let $u_0 = \deg(K_Y + B)/\deg(\mathcal{D}_1)$.

LEMMA 5.8. *The surface $X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein if and only if there exists l such that $u_0 \in (1/l) \cdot \boldsymbol{Z}$ and the divisor $l \cdot (u_0 \mathcal{D}_1 - K_Y - B)$ is principal. The Gorenstein index of $X[\mathcal{D}]$ is the minimal positive integer l satisfying these two conditions. Furthermore, if $X[\mathcal{D}]$ is $\boldsymbol{Q}$-Gorenstein of index 1 then $X[\mathcal{D}]$ is Gorenstein.*

PROOF. Let a canonical divisor of the curve Y be given by

$$K_Y = \sum_{i=r+1}^{k} b_i \cdot z_i\,, \quad \text{where} \quad z_i \neq z_j \quad \text{for all } \ i \neq j\,.$$

With the notation of Proposition 4.3, we have that $\mathrm{big}(D_{z_i}) = \{e_i/m_i\}$ for $i \leq r$ and $\mathrm{big}(D_{z_i}) = \{0\}$, otherwise. Furthermore, $\mu_i = m_i$ and $\mu_i v_i = e_i$ for $i \leq r$, and $\mu_i = 1$ and $\mu_i v_i = 0$, otherwise. With this considerations, the system of equations in (4) becomes

$$\begin{aligned} m_i a_i + e_i u &= m_i - 1 \quad \text{for all } \ i \leq r, \\ a_i &= b_i \quad \text{for all } \ i \geq r+1\,, \end{aligned}$$

and so

$$a_i = -u\frac{e_i}{m_i} + \frac{m_i - 1}{m_i} \quad \text{for all } \ i \leq r\,.$$

This yields $D = -u\mathcal{D}_1 + B + K_Y$ and $u = u_0$. This shows the first assertion. The second one follows at once since any normal surface is Cohen-Macaulay. □

REMARK 5.9. In [Wat81], a result similar to Lemma 5.8 is proved for affine k^*-varieties. This result can also be derived from Proposition 4.3 with an argument similar to the proof of Lemma 5.8.

In the following theorem we characterize quasihomogeneous (minimal) elliptic singularities of surfaces.

THEOREM 5.10. *Let $X = X[\mathcal{D}]$ be a normal affine surface with an effective elliptic 1-torus action, and let $\bar{0} \in X$ be the unique fixed point. Then $(X, \bar{0})$ is an elliptic singularity if and only if one of the following two conditions holds.*

(i) $Y = \boldsymbol{P}^1$, $\deg\lfloor u\mathcal{D}_1\rfloor \geq -2$ *for all* $u \in \mathbf{Z}_{>0}$, *and* $\deg\lfloor u\mathcal{D}_1\rfloor = -2$ *for one and only one* $u \in \mathbf{Z}_{>0}$.

(ii) *Y is an elliptic curve, and for every $u \in \mathbf{Z}_{>0}$, the divisor $\lfloor u\mathcal{D}_1\rfloor$ is not principal and* $\deg\lfloor u\mathcal{D}_1\rfloor \geq 0$.

Moreover, $(X, \bar{0})$ is a minimal elliptic singularity if and only if (i) *or* (ii) *holds, u_0 is integral and $u_0\mathcal{D}_1 - K_Y - B$ is principal.*

PROOF. Assume that Y is a projective curve of genus g, and let $\psi : Z \to X$ be a resolution of singularities. By Theorem 3.3,

$$R^1\psi_*\mathcal{O}_Z = \bigoplus_{u\geq 0} H^1(Y, \mathcal{O}_Y(u\mathcal{D}_1))\,.$$

Since $\dim R^1\psi_*\mathcal{O}_Z \geq g = \dim H^1(Y, \mathcal{O}_Y)$, if X has an elliptic singularity then $g \leq 1$.

If $Y = \boldsymbol{P}^1$ then $(X, \bar{0})$ is an elliptic singularity if and only if $H^1(Y, \mathcal{O}_Y(u\mathcal{D}_1)) = k$ for one and only one value of u. This is the case if and only if (i) holds. If Y is an elliptic curve, then $H^1(Y, \mathcal{O}_Y) = k$. So the singularity $(X, \bar{0})$ is elliptic if and only if $H^1(Y, u\mathcal{D}_1) = 0$ for all $u > 0$. This is the case if and only if (ii) holds.

Finally, the last assertion concerning minimal elliptic singularities follows immediately form Proposition 5.8. □

EXAMPLE 5.11. By applying the criterion of Theorem 5.10, the following combinatorial data gives rational k^*-surfaces with an elliptic singularity at the unique fixed point.

(i) $Y = \boldsymbol{P}^1$ and $\mathcal{D}_1 = -\frac{1}{4}[0] - \frac{1}{4}[1] + \frac{3}{4}[\infty]$. In this case $X = \operatorname{Spec} A[Y, m\mathcal{D}_1]$ is isomorphic to the surface in A^3 defined by the equation

$$x_1^4 x_3 + x_2^3 + x_3^2 = 0\,.$$

(ii) $Y = \boldsymbol{P}^1$ and $\mathcal{D}_1 = -\frac{2}{3}[0] - \frac{2}{3}[1] + \frac{17}{12}[\infty]$. In this case $X = \operatorname{Spec} A[Y, m\mathcal{D}_1]$ is isomorphic to the surface

$$V(x_1^4 x_2 x_3 - x_2 x_3^2 + x_4^2,\ x_1^5 x_3 - x_1 x_3^2 + x_2 x_4,\ x_2^2 - x_1 x_4) \subseteq \mathrm{A}^4\,.$$

This last example is not a complete intersection since otherwise $(X, \bar{0})$ would be Gorenstein, i.e., minimal elliptic, which is not the case by virtue of Theorem 5.10. In the first example, the elliptic singularities is minimal since every normal hypersurface is Gorenstein.

6. Factorial T-varieties. Let Y be a normal projective variety having class group $\boldsymbol{Z}$. Hence, we have a canonical degree map $\mathrm{Cl}(Y) \to \boldsymbol{Z}$ by sending the ample generator to 1. We further assume that the complete linear system of the ample generator is of positive dimension. We choose a set $\mathcal{Z} = \{(Z_1, \mu_i), \ldots, (Z_s, \mu_s)\}$ of prime divisors of degree 1 and corresponding tuples $\mu_i = (\mu_{i1}, \ldots, \mu_{ir_i}) \in \boldsymbol{N}^{r_i}$, where $\boldsymbol{N}$ is the set of non-negative integers. We assume that the integers $\gcd(\mu_i)$ are pairwise coprime and define $|\mathcal{Z}| := \sum_i (r_i - 1)$.

We give a construction of a polyhedral divisor on Y with polyhedral coefficients in $N_{\boldsymbol{Q}} = \boldsymbol{Q}^{|\mathcal{Z}|+1}$ by induction on $|\mathcal{Z}|$.

CONSTRUCTION 6.1. If $|\mathcal{Z}| = 0$ then $\mu_{11}, \ldots, \mu_{s1}$ are positive pairwise coprime integers. Also the greatest common divisor of the integers $M_i := \mu_{11} \cdots \mu_{s1}/\mu_{i1}$ for $1 \leq i \leq s$ is 1. Hence, there are integer coefficients $e_1, \ldots, e_s$ such that $1 = \sum e_i M_i$. Now, we define the vertices $v_{i1} := e_i/\mu_{i1} \in N_{\boldsymbol{Q}}$.

If $|\mathcal{Z}| > 0$ there is $j \in \{1, \ldots, s\}$ such that $r_j > 1$. Now, we consider the data $\mathcal{Z}'$ obtained from $\mathcal{Z}$ by replacing μ_j by

$$\mu'_j := (\mu_{j1}, \ldots, \mu_{jr_j-2}, \gcd(\mu_{jr_j-1}, \mu_{jr_j})).$$

By induction, we obtain vertices $v'_{im} \in N_{\boldsymbol{Q}}$ from the data $\mathcal{Z}'$ consisting of the integers μ'_{im} with v'_{jr_j-1} being the vertex corresponding to $\mu'_{jr_j-1} = \gcd(\mu_{jr_j-1}, \mu_{jr_j})$. We find coefficients $\alpha, \beta \in \boldsymbol{Z}$ such that $\mu'_{jr_j-1} = \alpha\mu_{jr_j-1} + \beta\mu_{jr_j}$. Now, we define the vertices

$$v_{jr_j-1} = \left(v'_{j1}, -\frac{\beta}{\mu_{jr_j-1}}\right), \quad v_{jr_j} = \left(v'_{j1}, \frac{\alpha}{\mu_{jr_j}}\right),$$

and $v_{im} = (v'_{im}, 0)$ for $i \neq j$ or $m < r_j - 1$.

For every set of admissible data $\mathcal{Z}$, we can define a polyhedral divisor $\mathcal{D} = \mathcal{D}(\mathcal{Z})$ on Y. The tail cone is spanned by the rays $\boldsymbol{Q}_{\geq 0} \cdot \sum_i v_{im_i}$, where $1 \leq m_i \leq r_i$. Also the vertices of $\mathcal{D}_{Z_i}$ are exactly the $v_{i1}, \dots v_{i,r_i}$. We denote the corresponding algebra $A[\mathcal{D}]$ also by $A[\mathcal{Z}]$.

THEOREM 6.2. *$A[\mathcal{Z}]$ is a normal factorial ring.*

PROOF. For $|\mathcal{Z}| = 0$, the matrix of relations for the class group has the form

$$M_{\mathcal{Z}} = \left(\begin{array}{cccc} 1 & \dots & 1 & 0 \\ \hline \mu_{11} & \dots & 0 & \mu_{11} v_{11} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_{s1} & \mu_{s1} v_{s1} \end{array}\right) = \left(\begin{array}{cccc} 1 & \dots & 1 & 0 \\ \hline \mu_{11} & \dots & 0 & e_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_{s1} & e_s \end{array}\right).$$

We get $\det M_{\mathcal{Z}} = \sum_i e_i M_i = 1$ by the choice of $v_{i1} = e_i / \mu_i$.

By the inductive construction above, we obtain $M_{\mathcal{Z}}$ from $M_{\mathcal{Z}'}$ by adding first a column of zeros on the left and then replacing the row $(0 \cdots 0, \mu'_{jr_j-1}, 0 \cdots 0, \mu'_{jr_j-1} v'_{jr_j-1}, 0)$ by the two rows

$$\begin{array}{lllllll} (\,0 & \cdots & 0 & \mu_{jr_j-1} & 0 \cdots 0 & \mu_{jr_j-1} v'_{jr_j-1} & -\beta\,)\,, \\ (\,0 & \cdots & 0 & \mu_{jr_j} & 0 \cdots 0 & \mu_{jr_j} v'_{jr_j-1} & \alpha\,)\,. \end{array}$$

Via multiplication with a SL_2-matrix, these rows transform to

$$\begin{array}{lllllll} (\,0 & \cdots & 0 & 0 & 0 \cdots 0 & 0 & 1\,)\,, \\ (\,0 & \cdots & 0 & \mu'_{jr_j-1} & 0 \cdots 0 & \mu'_{jr_j-1} v'_{jr_j-1} & 0\,)\,. \end{array}$$

Hence, we have $\det M_{\mathcal{Z}} = \det M_{\mathcal{Z}'}$. But $\det M_{\mathcal{Z}'} = 1$ holds by induction. □

For the case $Y = \boldsymbol{P}^1$ we obtain a complete classification. Now, $z_1, \dots, z_s$ are points in $\boldsymbol{P}^1$. Without loss of generality, we may assume that the support of $\mathcal{D}$ consists of at least 3 points. Otherwise X would be toric, and this implies $X = \mathrm{A}^n$. By applying an isomorphism of $\boldsymbol{P}^1$, we may assume $z_1 = \infty$, $z_2 = 0$ and $z_3, \dots, z_s \in k^*$. Via $K(\boldsymbol{P}^1) \cong k(t)$ we get $\operatorname{div}(t) = [0] - [\infty] = z_2 - z_1$.

COROLLARY 6.3 ([HHS11, Th. 1.9]). *Every normal k-algebra A of dimension n admitting a (positive) grading by $\boldsymbol{N}^{n-1}$ such that $A_0 = k$ is factorial if and only if it is isomorphic to a free algebra over some*

$$A[\mathcal{Z}] = k\big[T_{ij}; 0 \leq i \leq s, 1 \leq j \leq r_i\big] / \big(T_i^{\mu_i} + T_2^{\mu_2} - z_i T_1^{\mu_1}, 3 \leq i \leq s\big),$$

such that the integers $\gcd(\mu_i)$ *are pairwise coprime. Here, we define $T_i^{\mu_i} := \prod_j T_{ij}^{\mu_{ij}}$.*

In particular, every such k-algebra is a complete intersection of dimension $2 + \sum_i (r_i - 1)$.

REMARK 6.4. For the cases of dimension two and three, this result was obtained in [Mor77] and [Ish77], respectively.

PROOF. The corresponding polyhedral divisor $\mathcal{D}$ by Theorem 4.4 necessarily lives on $\boldsymbol{P}^1$. Since $X = \operatorname{Spec} A = X[\mathcal{D}]$ is factorial, the Cox ring $\mathbf{Cox}(X) := \bigoplus_{D \in \mathrm{Cl}\, X} \mathcal{O}\big(X, \mathcal{O}(D)\big)$ equals A. Now [HS10, Cor. 4.9] implies that A is of the desired form.

For the other direction, the construction 6.1 provides a polyhedral divisor $\mathcal{D}$ being factorial by Theorem 6.2 with $A = A[\mathcal{D}]$. $\square$

REMARK 6.5. We can easily identify the log-terminal singularities of the form $A[\mathcal{Z}]$. Namely, by Theorem 4.9 $A[\mathcal{Z}]$ is log-terminal if and only if $\max \mu_i > 1$ for at most three $1 \leq i_1 < i_2 < i_3 \leq s$ and $(\max \mu_{i_1}, \max \mu_{i_2}, \max \mu_{i_3})$ is one of the platonic triples $(1, p, q)$, $(2, 2, q)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ with $1 \leq p \leq q$.

In the case of complexity one, we are also able to characterize *isolated* factorial singularities. Every (normal) factorial surface singularity is of course isolated. For the remaining cases we provide the following theorem.

THEOREM 6.6. *Every factorial T-variety of complexity one and dimension at least three having an isolated singularity at the vertex is one of the following.*

(i) *A* cA_q *threefold singularity of the form*

$$k[T_1, \dots, T_4]/(T_1T_2 + T_3^{q+1} + T_4^r)\,,$$

with $0 < q + 1 < r$ *being coprime.*

(ii) *A fourfold singularity which is stably equivalent to* A_q

$$k[T_1, \dots, T_5]/(T_1T_2 + T_3T_4 + T_5^{q+1})\,.$$

(iii) *A fivefold singularity which is stably equivalent to* A_1

$$k[T_1, \dots, T_6]/(T_1T_2 + T_3T_4 + T_5T_6)\,.$$

PROOF. By Corollary 6.3, the variety is given by equations of the form

$$\prod_{j=1}^{r_i} T_{ij}^{\mu_{ij}} + \prod_{j=1}^{r_2} T_{2j}^{\mu_{2j}} - z_i \prod_{j=1}^{r_1} T_{1j}^{\mu_{1j}} \quad \text{with} \quad 3 \leq i \leq s\,.$$

Now, we consider the Jacobian matrix of these equations.

$$\begin{pmatrix}
-z_3 f_{11} & \cdots & -z_3 f_{1r_1} & f_{21} & \cdots & f_{2r_2} & f_{31} & \cdots & f_{3r_3} & & & & & & \\
-z_4 f_{11} & \cdots & -z_4 f_{1r_1} & f_{21} & \cdots & f_{2r_2} & & & & f_{41} & \cdots & f_{4r_4} & & & \\
\vdots & & \vdots & \vdots & & \vdots & & & & & & & \ddots & & \\
-z_s f_{11} & \cdots & -z_r f_{1r_1} & f_{21} & \cdots & f_{2r_2} & & & & & & & f_{s1} & \cdots & f_{sr_s}
\end{pmatrix}$$

Here, f_{ij} denotes the partial derivative $\partial T_i^{\mu_i}/\partial T_{ij}$. From Corollary 6.3, we know that the variety has dimension $2 + \sum_i (r_i - 1)$. Since we consider varieties of dimension at least three, we must have $r_l > 1$ for at least one l. Then for

$$T_{ij} = \begin{cases} 1 & \text{if } (i, j) = (l, 1)\,, \\ 0 & \text{otherwise,} \end{cases}$$

all but one column vanish. Hence, we are in the case of a hypersurface. Now, one easily checks that a multi-exponent $\mu_i > (1, 1)$ automatically leads to partial derivatives f_{ij} which jointly vanish even if one of the terms T_{ij} does not vanish. Hence, the singular locus has dimension at least one. $\square$

REFERENCES

[AH06] K. ALTMANN AND J. HAUSEN, Polyhedral divisors and algebraic torus actions, Math. Ann. 334 (2006), 557–607.

[AHS08] K. ALTMANN, J. HAUSEN AND H. SÜß, Gluing affine torus actions via divisorial fans, Transform. Groups 13 (2008), 215–242.

[AW11] K. ALTMANN, J. Wiśniewski Polyhedral divisors of Cox rings, Mich. Math. 60 (2011), 463–480.

[Art66] M. ARTIN, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966), 129–136.

[Bro11] M. V. BROWN, Singularities of Cox rings of Fano varieties, arXiv:1109.6368, 2011.

[Dai02] D. I. DAIS, Resolving 3-dimensional toric singularities, In Geometry of toric varieties, 155–186, Sémin. Congr. 6, Soc. Math. France, Paris, 2002.

[Dem70] M. DEMAZURE, Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. École Norm. Sup. (4) 3 (1970), 507–588.

[Eis95] D. EISENBUD, Commutative algebra, Graduate Texts in Mathematics, No. 150, Springer-Verlag, New York, 1995.

[Elk78] R. ELKIK, Singularités rationnelles et déformations, Invent. Math. 47 (1978), 139–147.

[Fuj03] O. FUJINO, Notes on toric varieties from Mori theoretic viewpoint, Tohoku Math. J. (2) 55 (2003), 551–564.

[FZ03] H. FLENNER AND M. ZAIDENBERG, Normal affine surfaces with $\mathbb{C}^*$-actions, Osaka J. Math. 40 (2003), 981–1009.

[GOST12] Y. GONGYO, S. OKAWA, A. SANNAI AND S. TAKAGI, Characterization of varieties of Fano type via singularities of Cox rings, arXiv:1201.1133, 2012.

[Har77] R. HARTSHORNE, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[HHS11] J. HAUSEN, E. HERPPICH AND H. SÜß, Multigraded factorial rings and Fano varieties with torus action, Doc. Math. 16 (2011), 71–109.

[HS10] J. HAUSEN AND H. SÜß, The Cox ring of an algebraic variety with torus action, Adv. Math. 225 (2010), 977–1012.

[Ish77] M.-N. ISHIDA, Graded factorial rings of dimension 3 of a restricted type, J. Math. Kyoto Univ. 17 (1977), 441–456.

[KO12] Y. KAWAMATA AND S. OKAWA, Mori dream spaces of Calabi-Yau type and the log canonicity of the Cox rings, arXiv:1202.2696, 2012.

[KKMS73] G. KEMPF, F. F. KNUDSEN, D. Mumford and B. Sain-Donat, Toroidal embeddings. I, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973.

[Kov99] S. J. KOVÁCS, Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink, Compositio Math. 118 (1999), 123–133.

[Lau77] H. B. LAUFER, On minimally elliptic singularities, Amer. J. Math. 99 (1977), 1257–1295.

[Lie10] A. LIENDO, Affine T-varieties of complexity one and locally nilpotent derivations, Transform. Groups 15 (2010), 389–425.

[Mor77] S. MORI, Graded factorial domains, Japan. J. Math. (N.S.) 3 (1977), 223–238.

[Oda88] T. ODA, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, volume 15 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1988.

[PS11] L. PETERSEN AND H. SÜß, Torus invariant divisors, Israel J. Math. 182 (2011), 481–505.

[Sam64] P. SAMUEL, Lectures on unique factorization domains, Notes by M. Pavman Murthy. Tata Institute of Fundamental Research Lectures on Mathematics, No. 30, Tata Institute of Fundamental Research, Bombay, 1964

[SS84] G. SCHEJA AND U. STORCH, Zur Konstruktion faktorieller graduierter Integritätsbereiche, Arch. Math. (Basel) 42 (1984), 45–52.

[SS10] K. SCHWEDE AND K. E. SMITH, Globally F-regular and log Fano varieties, Adv. Math. 224 (2010), 863–894.

[Sum74] H. SUMIHIRO, Equivariant completion, J. Math. Kyoto Univ. 14 (1974), 1–28.

[Tim08] D. TIMASHËV, Torus actions of complexity one, In Toric topology, 349–364, Contemp. Math. 460, Amer. Math. Soc., Providence, RI, 2008.

[Wat81] K. WATANABE, Some remarks concerning Demazure's construction of normal graded rings, Nagoya Math. J. 83 (1981), 203–211.

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