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NORMAL STRUCTURE COEFFICIENTS FOR BANACH SPACES

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This paper introduces several geometric concepts in Banach spaces related to normal structure and uses these ideas to generalize a recent theorem of Baillon for fixed points of nonexpansive mappings.

1. **Definitions.** The concepts introduced in this paper are phrased in terms of reflexive Banach spaces. This is not a serious restriction, but rather one of technical convenience. All of the concepts discussed here deal with closed bounded convex subsets of a reflexive Banach space, so one could replace "closed bounded and convex" with "weakly compact and convex", if desired.

For the following definitions, X will denote a reflexive Banach space and C will denote a closed bounded convex subset of X .

For each x in C , let $r(x, C) = \sup\{\|x - y\|: y \text{ in } C\}$ and let $R(C)$ denote the Čebyšev radius of C [11, p. 178]:

$$R(C) = \min \{r(x, C): x \text{ in } C\}.$$

This minimum is achieved because of the weak compactness of C . Let $D(C)$ denote the diameter of C , $D(C) = \sup\{\|x - y\|: x, y \text{ in } C\}$.

A space X has *normal structure* [2] provided that for each closed bounded convex subset C of X with more than one member, $R(C) < D(C)$.

The three Banach space coefficients defined in the following paragraphs are the principal objects of study in this paper.

The *normal structure coefficient* of X , denoted by $N(X)$, is the infimum of the set of number $D(C)/R(C)$ taken over all closed convex subsets C of X with more than one member.

The *asymptotic diameter of a bounded sequence* $\{x_n\}$ in X is defined to be $\lim_n \sup\{\|x_m - x_k\|: m \geq n, k \geq n\}$. The *bounded sequence coefficient* of X , denoted $BS(X)$, is the supremum of the set of all numbers M with the property that for each bounded sequence $\{x_n\}$ with asymptotic diameter A , there is some y in the closed convex hull of the sequence such that $M \cdot \lim \sup_n \|x_n - y\| \leq A$.

The *weakly convergent sequence coefficient* of X , denoted by $WCS(X)$, is defined similarly, replacing "bounded" with "weakly convergent".

These three coefficients are related to normal structure in that if any one of the three is greater than 1 (all are between 1 and 2), the space has normal structure; this result is established in § 2. An

example is given at the end of § 2 of a space with normal structure for which all three coefficients are 1, so the converse of the previous statement is false.

The primary focus of this paper is on the coefficient $WCS(X)$ for two reasons. First, because of Theorem 1 below, it appears to be the "weakest" of the three coefficients, and second, it seems to be the easiest of the three to evaluate, at least for the l_p spaces ($1 < p < \infty$) and some of their isomorphs. Intuitively, it seems that the three coefficients should be equal, at least for some limited class of spaces. However, J.-B. Baillon has constructed an example of a reflexive Banach space X such that $N(X) = 1$ and $WCS(X) = 2^{1/2}$ (see § 2).

The paper is organized as follows. Section 2 contains basic theorems and several examples, § 3 contains a theorem relating the coefficients to the Banach-Mazur distance coefficient, and § 4 contains a fixed point theorem for nonexpansive mappings.

2. Examples and basic results. The three coefficients are related by the following theorem.

THEOREM 1. *For a reflexive Banach space X ,*

$$1 \leq N(X) \leq BS(X) \leq WCS(X) \leq 2.$$

Proof. The inequalities $1 \leq N(X)$ and $BS(X) \leq WCS(X)$ follow from the definitions, and it is obvious from the triangle inequality that $WCS(X) \leq 2$.

We shall show that $N(X) \leq BS(X)$. Take a bounded sequence $\{x_n\}$ in X with asymptotic diameter A , and for each n let C_n be the closed convex hull of $\{x_n, x_{n+1}, \dots\}$.

If there is an n such that C_n has only one member, then the sequence $\{x_n\}$ is constant from that point on, and if u is that constant, we have $N(X) \limsup_n \|x_n - u\| = A = 0$.

So suppose that each C_n has more than one member. The definition of $N(X)$ implies that for each n $N(X) \cdot R(C_n) \leq D(C_n)$. For each y in C_1 , let $\beta(y) = \limsup_n \|x_n - y\|$. Since β is continuous and convex and since C_1 is weakly compact, there is some y_0 in C_1 at which β achieves its minimum value.

For each n , there is some z_n in C_n such that $r(z_n, C_n) = R(C_n)$, and for such a point it is clear that

$$\beta(y_0) \leq \beta(z_n) \leq r(z_n, C_n) \leq D(C_n)/N(X).$$

But it is easy to show that $D(C_n) = \sup\{\|x_m - x_k\| : m, k \geq n\}$, so that the sequence $\{D(C_n)\}$ decreases to A . Consequently, $\beta(y_0) \leq$

$A/N(X)$, and $N(X) \leq BS(X)$, as desired.

For a separable reflexive space X , every closed bounded convex subset contains a dense sequence, so in this case, $N(X) = BS(X)$. J.-B. Baillon has noted that if X is taken to be the l_2 -direct sum of the spaces R_∞^n , $n \geq 1$, where R_∞^n is the space R^n with the maximum norm, then X is separable and reflexive and $N(X) = BS(X) = 1 < WCS(X) = 2^{1/2}$, which shows that the coefficients need not be equal.

If any one of these coefficients for a space X is greater than one, then X has normal structure. First, it is obvious from the definitions that X has normal structure if $N(X) > 1$. As far as the other two coefficients are concerned, if X lacks normal structure, then by [9, Proposition 3] there is a diametral sequence in the unit ball of X converging weakly to zero with asymptotic diameter one; that is, there is a sequence $\{x_n\}$ in the unit ball of X such that $\|x_n\| \rightarrow 1$, $x_n \rightarrow 0$, and the distance from x_{n+1} to the convex hull of $\{x_1, \dots, x_n\}$ tends to 1 as $n \rightarrow \infty$. This last condition implies that for each y in the convex hull of the sequence, $\lim_n \|x_n - y\| = 1$, so that $WCS(X) = BS(X) = N(X) = 1$. Therefore, if either $BS(X)$ or $WCS(X)$ is greater than 1, then X has normal structure also.

The fixed point theorem of § 4 is phrased in terms of $WCS(X)$, and fortunately this coefficient seems relatively easy to evaluate for the l_p spaces ($1 < p < \infty$).

THEOREM 2. For $1 < p < \infty$, $WCS(l_p) = 2^{1/p}$.

Proof. Let $\{x_n\}$ be a sequence in l_p with asymptotic diameter A converging weakly to a point z , and let $B = \limsup_n \|x_n - z\|$. By choosing a subsequence $\{x_{n_k}\} = \{w_k\}$ such that $\lim_k \|w_k - z\| = B$, we obtain that for each y in l_p $\lim_k \|w_k - y\|^p = B^p + \|y - z\|^p$. If we denote this last quantity by $\beta(y)^p$, then $\limsup_n \beta(x_n) \leq A$, and $\beta(w_k)^p = B^p + \|w_k - z\|^p$, so that $\limsup_n \beta(x_n) \geq 2^{1/p}B$ and $A \geq 2^{1/p}B$. Consequently, $WCS(l_p) \geq 2^{1/p}$.

For the opposite inequality, the usual Schauder basis $\{e_n\}$ of l_p has asymptotic diameter $2^{1/p}$ and $\lim_n \|e_n - 0\| = 1$, so $WCS(l_p) = 2^{1/p}$.

Theorem 2.6 of [6, p. 112] shows that $N(l_2) = 2^{1/2}$, so that Theorems 1 and 2 above imply that all three constants coincide in the case of l_2 .

Unfortunately, the normal structure coefficients for $L_p[0, 1] = L_p$ seem to be difficult to determine, and no precise values have been obtained. By considering various sequences in L_p , it is relatively easy to show that $WCS(L_p) \leq \min\{2^{1-1/p}, 2^{1/p}\}$, so that when $1 < p < 2$, $WCS(L_p)$ and $WCS(l_p)$ must differ. The bound given above is a likely candidate for the actual value of $WCS(L_p)$, but so far noth-

ing definite has been obtained.

The question of how these coefficients relate to the modulus of convexity of a space is a natural one to ask, and the following theorem gives a partial answer.

THEOREM 3. *If X is a reflexive Banach space with modulus of convexity δ , then $N(X) \geq (1 - \delta(1))^{-1}$.*

Proof. Let C be a closed bounded convex subset of X with more than one member, and let $\varepsilon > 0$.

Choose x and y in C such that $\|x - y\| \geq D(C) - \varepsilon$, let $w = (x + y)/2$, and choose z in C such that $\|z - w\| \geq r(w, C) - \varepsilon$. Then by the definition of δ

$$\|z - w\| \leq D(C)(1 - \delta((D(C) - \varepsilon)/D(C)))$$

and by the definition of $R(C)$

$$R(C) \leq r(w, C) \leq \|z - w\| + \varepsilon.$$

Combining these two inequalities with the continuity of δ [10, p. 145], we obtain that $R(C) \leq (1 - \delta(1))D(C)$, from which the desired result follows.

This theorem shows that if $\delta(1) > 0$, then $N(X) > 1$, and consequently, X has normal structure. This implication was first noted in [8] with essentially the same argument as above.

Note that by Theorem 1, $\delta(1) > 0$ also implies that both $WCS(X)$ and $BS(X)$ are greater than one. The space $l_{p,1}$ (defined in [3]), which is simply l_p renormed with the norm described below, is an interesting space to consider now, because it is not uniformly convex (its modulus of convexity is zero on the interval $[0, 2^{1/p}]$), yet it has normal structure. Moreover, as we shall see in Theorem 4 below, $WCS(l_{p,1}) = 2^{1/p}$.

The norm for $l_{p,1}$ is defined as follows (see [3]): for x in l_p and for each n , let $x^+(n) = \max\{x(n), 0\}$, and let $x^- = (-x)^+$. If $\|\cdot\|$ denotes the usual l_p norm, then define the norm $|\cdot|$ for $l_{p,1}$ by $|x| = \|x^+\| + \|x^-\|$.

THEOREM 4. *For $1 < p < \infty$, $WCS(l_{p,1}) = 2^{1/p}$.*

Proof. Without loss of generality, we can consider a sequence $\{x_n\}$ converging weakly to 0. By choosing as many subsequences as necessary, we can obtain a subsequence $\{x_{n_k}\} = \{z_k\}$ such that $z_k \rightarrow 0$, $|z_k| \rightarrow \limsup_n |x_n|$, $\|z_k^+\| \rightarrow B$, and $\|z_k^-\| \rightarrow C$ as $k \rightarrow \infty$.

We shall show that for each y in $l_{p,1}$,

$$(1) \quad \|(z_k - y)^+\|^p \longrightarrow B^p + \|y^-\|^p$$

and

$$(2) \quad \|(z_k - y)^-\|^p \longrightarrow C^p + \|y^+\|^p.$$

Let $\{e_n\}$ denote the usual Schauder basis of l_p , and for each x in $l_{p,1}$ and each m , let $U_mx = x(1)e_1 + \dots + x(m)e_m$ and $V_mx = x - U_mx$.

We shall show that (1) holds. Fix m and note that $\|U_m(z_k - y)^+\| \rightarrow \|U_m(-y^+)\| = \|U_my^-\|$ because $z_k \rightarrow 0$ as $k \rightarrow \infty$.

For each k ,

$$\|z_k^+\| - \|U_mz_k^+\| - \|V_my^+\| \leq \|V_mz_k^+\| - \|V_my^+\| \leq \|V_m(z_k - y)^+\|$$

and

$$\|V_m(z_k - y)^+\| \leq \|V_mz_k^+\| + \|V_my^-\| \leq \|z_k^+\| + \|V_my^-\|.$$

Because $z_k \rightarrow 0$, it follows that $\lim_k \|U_mz_k^+\| = 0$, and thus

$$\liminf_k \|V_m(z_k - y)^+\| \geq B - \|V_my^+\|$$

and

$$\limsup_k \|V_m(z_k - y)^+\| \leq B + \|V_my^-\|.$$

Consequently, for each m :

$$\limsup_k \|(z_k - y)^+\|^p \leq \|U_my^-\|^p + (B + \|V_my^-\|)^p$$

and

$$\liminf_k \|(z_k - y)^+\|^p \geq \|U_my^-\|^p + |B - \|V_my^+\||^p.$$

By taking limits as $m \rightarrow \infty$, we obtain the result sought, namely statement (1) above. The proof of statement (2) is similar.

Consequently, for each fixed n ,

$$\lim_k |z_k - x_n| = (B^p + \|x_n^-\|^p)^{1/p} + (C^p + \|x_n^+\|^p)^{1/p}.$$

Setting $n = n_k$ in this expression and letting $k \rightarrow \infty$, we obtain finally that $2(B^p + C^p)^{1/p}$ does not exceed the asymptotic diameter A of the sequence $\{x_n\}$. On the other hand, $\limsup_n |x_n - 0| = B + C$, and by the Hölder inequality, $B + C \leq 2^{1-1/p}(B^p + C^p)^{1/p}$. Consequently, $2^{1/p} \limsup_n |x_n - 0| \leq A$, and so $WCS(l_{p,1}) \geq 2^{1/p}$.

To obtain the reverse inequality, let $x_n = e_{2n} - e_{2n+1}$. The asymptotic diameter of this sequence is $2^{1+1/p}$ and $\lim_n |x_n - 0| = 2$, so that $WCS(l_{p,1}) \leq 2^{1/p}$, and at last the proof is complete.

We conclude this section with another example: a space X with normal structure such that $N(X) = BS(X) = WCS(X) = 1$.

Let X be the l_2 -direct sum of the sequence of spaces $\{l_n\}_{n \geq 2}$; i.e.,

let X be the collection of all sequences $x = \{x_n\}_{n \geq 2}$ such that x_n is in l_n for each n and $|x|^2 = \sum_n \|x_n\|^2 < \infty$. This space is reflexive and is easily shown to be uniformly convex in every direction (cf. [7]), so it has normal structure. On the other hand, $WCS(X) \leq WCS(l_n) = 2^{1/n}$ for each n , so all three constants for this space are 1.

3. The normal structure coefficients and the Banach-Mazur distance coefficient. For isomorphic Banach spaces X and Y , the *Banach-Mazur distance from X to Y* , denoted $d(X, Y)$, is defined to be the infimum of $\|U\| \|U^{-1}\|$ taken over all bicontinuous linear operators U from X onto Y .

When any one of the normal structure coefficients is greater than one, this condition is contagious. More precisely, we have the following theorem:

THEOREM 5. *If X and Y are isomorphic Banach spaces, then $N(X) \leq d(X, Y)N(Y)$, $BS(X) \leq d(X, Y)BS(Y)$, and $WCS(X) \leq d(X, Y)WCS(Y)$.*

Proof. Each of these inequalities has a similar proof. We shall show here that $N(X) \leq d(X, Y)N(Y)$.

Let C be a closed bounded convex subset of Y with more than one member. If U is an isomorphism of Y onto X , then $U(C)$ is a subset of X with the same properties as those of C . It follows easily from the definitions that

$$\begin{aligned} R_Y(C) &\leq \|U^{-1}\| R_X(U(C)) \leq \|U^{-1}\| D_X(U(C))/N(X) \\ &\leq \|U^{-1}\| \|U\| D_Y(C)/N(X). \end{aligned}$$

Consequently, $R_Y(C) \leq d(X, Y)D_Y(C)/N(X)$, from which the desired result follows.

The following observation is a corollary of Theorem 5: if X is a reflexive Banach space such that $N(X) > 1$ and if Y is a Banach space such that $d(X, Y) < N(X)$, then $N(Y) > 1$ also. A combination of this observation with Theorems 2 and 3 subsumes the results announced in [5].

The space $l_{p,\infty}$, defined in [3], is interesting with regard to the previous paragraph. The space $l_{p,\infty}$ is simply l_p renormed with the norm $|x| = \max \{\|x^+\|, \|x^-\|\}$ (see § 2). The space $l_{p,\infty}$ lacks normal structure (see [3]) and $d(l_{p,\infty}, l_p) = 2^{1/p}$, yet by the remark in the previous paragraph if $d(l_p, Y) < 2^{1/p}$, then Y has normal structure, so that the space $l_{p,\infty}$ is a sort of "limiting case" of spaces with normal structure.

This space is also interesting with regard to the concept of *asymptotic normal structure* introduced in [13]. A Banach space X

has asymptotic normal structure provided that for each closed bounded subset C of X with more than one member and each sequence $\{x_n\}$ in C with the property that $\lim_n \|x_n - x_{n+1}\| = 0$, there is some y in C such that $\liminf_n \|x_n - y\| < D(C)$. The space $l_{p,\infty}$ lacks asymptotic normal structure also (see [4]), and since normal structure implies asymptotic normal structure, the space $l_{p,\infty}$ is a sort of "limiting case" of spaces with asymptotic normal structure as well.

4. **A fixed point theorem for nonexpansive mappings.** The following theorem is a generalization of a recent fixed point theorem of Baillon [1] (see the Corollary below).

THEOREM 6. *If X is a uniformly convex Banach space and if Y is a Banach space such that $d(X, Y) \leq WCS(X)$, then each nonexpansive selfmap of a closed bounded convex subset of Y has a fixed point.*

Proof. Let C be a closed bounded convex subset of Y and let T be a fixed-point-free nonexpansive selfmap of C . Let K be a minimal closed convex subset of C which is selfmapped by T . Let $\|\cdot\|$ denote the norm on X and $|\cdot|$ the norm on Y .

By a result of Karlovitz [12] if $\{x_n\}$ is a sequence in K such that $\lim_n |x_n - Tx_n| = 0$, then for each y in K , $\lim_n |x_n - y| = D_Y(K) = D$. By replacing $\{x_n\}$ with one of its subsequences, we can assume that $\{x_n\}$ converges weakly to some point of K .

Choose $0 < b < 1$. By the definition of $d(X, Y)$ there is an isomorphism U of Y onto X such that $\|U^{-1}\| = 1$ and $b\|U\| < d(X, Y) \leq WCS(X)$. If A is the asymptotic diameter of $\{Ux_n\}$, then since $\{Ux_n\}$ converges weakly in X there is some w in K with the property that Uw is in the closed hull of $\{Ux_n\}$ and $\beta(Uw) = \limsup_n \|Ux_n - Uw\| \leq A/WCS(X)$. But for each n , $|x_n - w| \leq \|Ux_n - Uw\|$, so by the previous paragraph,

$$(3) \quad D \leq \beta(Uw) \leq A/WCS(X) < A/(b\|U\|) .$$

Since it is easy to show that $A \leq \|U\|D$, we obtain the following results from (3):

$$(4) \quad bD\|U\| < A$$

$$(5) \quad \beta(Uw) < D/b .$$

For simplicity in what follows, let $M = D\|U\|$.

By the definition of A and by (4) above, there exist m and n as large as desired such that

$$(6) \quad bM < \|Ux_m - Ux_n\|.$$

Note that $\|Ux_m - Ux_n\| \leq \|U\| |x_m - x_n|$, so that inequality (6) implies that $|x_m - x_n| > bD$. Since $\lim_n |x_n - Tx_n| = 0$, it follows that $\lim_n \|Ux_n - UTx_n\| = 0$ also, and thus we can find m and n as large as desired such that the inequalities $|x_m - Tx_m| < 1 - b$, $|x_n - Tx_n| < 1 - b$, and $bM < \|UTx_m - UTx_n\|$ hold in addition to inequality (6).

Now if $f = UTx_m - U(T(x_m + x_n)/2)$ and if $g = U(T(x_m + x_n)/2) - UTx_n$, then $\|f\| < M/2$, $\|g\| < M/2$, and by the previous sentence and the uniform convexity of X we obtain:

$$bM < \|f + g\| < M(1 - \delta(2\|f - g\|/M)).$$

From this, the strict monotonicity of δ , and the choice of U , we obtain finally that:

$$\begin{aligned} |x_m + x_n - 2T((x_m + x_n)/2)| &\leq |x_m - Tx_m| + |x_n - Tx_n| + \|f - g\| \\ &< 2(1 - b) + (M/2)\delta^{-1}(1 - b) \\ &< 2(1 - b) + (D/2b)d(X, Y)\delta^{-1}(1 - b). \end{aligned}$$

Since the right side of the last inequality approaches zero as b approaches one, it follows that there exist subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with these two properties:

$$(7) \quad \text{if } z_k = (x_{m_k} + x_{n_k})/2, \text{ then } \lim_k |z_k - Tz_k| = 0.$$

$$(8) \quad \lim_k |x_{m_k} - x_{n_k}| = D.$$

Now again choose some b in $(0, 1)$, choose an isomorphism U as before and obtain a w in K such that (5) holds; namely, $\limsup_n \|Ux_n - Uw\| < D/b$.

By Karlovitz's result [12] again, we obtain from (7) that $D = \lim_k |z_k - w| \leq \liminf_k \|Uz_k - Uw\|$. But for k sufficiently large, we know that $\|Ux_{m_k} - Uw\|$ and $\|Ux_{n_k} - Uw\|$ are both less than D/b because of (5). By the uniform convexity of X once more, we find that

$$\begin{aligned} \|Uz_k - Uw\| &\leq (D/b)(1 - \delta(b\|Ux_{m_k} - Ux_{n_k}\|/D)) \\ &\leq (D/b)(1 - \delta(b|x_{m_k} - x_{n_k}|/D)). \end{aligned}$$

By letting $k \rightarrow \infty$ in this inequality, we obtain from (8) and the continuity of δ that $D \leq (D/b)(1 - \delta(b))$. Since b is arbitrary in $(0, 1)$, we obtain the long-sought contradiction: $D \leq D(1 - \delta(1))$. Therefore, T has a fixed point.

The following result is a corollary of Theorems 2 and 6.

COROLLARY (*Baillon* [1]). *Let X be a Banach space having*

norms $|\cdot|_2$ and $|\cdot|_\infty$ such that $(X, |\cdot|_2)$ is a Hilbert space and $|\cdot|_\infty \leq |\cdot|_2$. If $\|\cdot\|$ is the norm for X defined by $|\cdot| = \max\{|\cdot|_2, 2^{1/2}|\cdot|_\infty\}$, then $(X, \|\cdot\|)$ has the nonexpansive fixed point property.

This corollary gives an alternate proof to the result of Karlovitz [12] that the James space has the nonexpansive fixed point property. In this connection, note that $d(l_{p,\infty}, l_p) = 2^{1/p}$, so that Theorems 2 and 6 show that the space $l_{p,\infty}$ described in §3 has the nonexpansive fixed point property as well. Since this space lacks asymptotic normal structure, the fixed point theorem of [13] does not apply in this case.

Although uniform convexity is used extensively in the proof of Theorem 6, it is not clear whether uniform convexity can be dropped from the hypothesis; that is, this question remains open: if $d(X, Y) \leq WCS(X)$ and $WCS(X) > 1$, does Y have the nonexpansive fixed point property?

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