Normal Toeplitz Matrices

Douglas R. Farenick* and Mark Krupnik†
Department of Mathematics and Statistics,
University of Regina, Regina,
Saskatchewan S4S 0A2, Canada

Naum Krupnik
Department of Mathematics and Computer Science
Bar – Ilan University,
Ramat-Gan, 52900, Israel

Woo Young Lee‡
Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea

Abstract

It is well-known from the work of A. Brown and P.R. Halmos that an infinite Toeplitz matrix is normal if and only if it is a rotation and translation of a Hermitian Toeplitz matrix. In the present article we prove that all finite normal Toeplitz matrices are either generalised circulants or are obtained from Hermitian Toeplitz matrices by rotation and translation.

*Supported in part by an NSERC Research Grant
†Postdoctoral Fellow, supported in part by grants from NSERC
‡Supported in part by KOSEF Project No. 94-0701-02-01-3 and GARC-KOSEF (1995)

1. Introduction.

The purpose of the present article is to describe fully the structure of all finite normal Toeplitz matrices.

The algebraic theory of Toeplitz matrices and Toeplitz operators is now extensive, having been developed over many years. An overview of the theory for finite Toeplitz matrices is given in the monograph [3] of Iohvidov, whereas the classic paper of Brown and Halmos [**?**] contains many of the fundamental results on the algebraic properties of Toeplitz operators. A well-known theorem from that paper states that an infinite Toeplitz matrix (operator) is normal if and only if it is a rotation and translation of a Hermitian Toeplitz matrix. This theorem does not, however, apply to finite matrices: all circulant matrices, for example, are normal Toeplitz matrices. To date very little has been published about the general structure of a finite normal Toeplitz matrix. In fact it appears that the most informative work on this question is a recent article of Ikramov. In [2] Ikramov has shown that a normal Toeplitz matrix (of order at most 4) over the real field must be of one of four types: symmetric Toeplitz, skew-symmetric Toeplitz, circulant, or skew-circulant. A reading of Ikramov's paper suggests that it may be possible to characterise complex normal Toeplitz matrices of all orders, and we do so here. We first identify the two types of normal Toeplitz matrices that arise.

Type I: a rotation and translation of a Hermitian Toeplitz matrix, that is $T = \alpha I + \beta H$, for some complex α and β and for some Hermitian Toeplitz matrix H.

Type II: a generalised circulant, which is to mean a Toeplitz matrix of the form

$$T = \begin{pmatrix} a_0 & a_N e^{i\theta} & \ddots & a_1 e^{i\theta} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_N e^{i\theta} \\ a_N & \ddots & a_1 & a_0 \end{pmatrix},$$

for some fixed real θ.

The main result of this paper is the following theorem.

Theorem 1.1 *Every finite complex normal Toeplitz matrix is a generalised circulant or is a rotation and translation of a Hermitian Toeplitz matrix. In particular, every finite real normal Toeplitz matrix is symmetric Toeplitz, skew-symmetric Toeplitz, circulant, or skew-circulant.*

This paper consists of four sections. In Section 2 we give criteria for a Toeplitz matrix to be normal. The main result is proved in Section 3. Within the proof we use several technical lemmas, which are derived in Section 4.

2. Key Equalities

Let T be a Toeplitz $(N+1) \times (N+1)$ matrix

$$T = \begin{pmatrix} a_0 & a_{-1} & \ddots & a_{-N} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{-1} \\ a_N & \ddots & a_1 & a_0 \end{pmatrix}.$$

Throughout the paper we use the following notation: $b_i := a_{-i}$, for $i = 1, ..., N$. Note that neither the normality nor the form (I) or (II) of a Toeplitz matrix depends on the value of its diagonal entry; therefore we may assume that $a_0 = 0$.

Theorem 2.1 *A Toeplitz matrix T of the form*

$$T = \begin{pmatrix} 0 & b_1 & \ddots & b_N \\ a_1 & 0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & b_1 \\ a_N & \ddots & a_1 & 0 \end{pmatrix}$$

is normal if and only if for each p and q the following equalities hold

$$\overline{a}_q a_p + a_{N-q+1}\overline{a}_{N-p+1} = b_q\overline{b}_p + \overline{b}_{N-q+1}b_{N-p+1}\,. \tag{1}$$

Proof:

The matrix T is normal if and only if $R = TT^* - T^*T = 0$. Let $R = \{r_{i,j}\}_0^N$. Let us write down the condition $r_{p,q} = 0$ for some $p \geq 0, q \geq 0$ in terms of the entries of the matrix T. Suppose first that $p \leq q$. Then

$$[a_p\overline{a}_q + a_{p-1}\overline{a}_{q-1} + ... + a_1\overline{a}_{q-p+1}] + (b_1\overline{a}_{q-p-1} + ... + b_{q-p-1}\overline{a}_1) + \tag{2}$$

$$+b_{q-p+1}\overline{b}_1 + ... + b_{N-p}\overline{b}_{N-q} = [\overline{b}_p b_q + ... + \overline{b}_1 b_{q-p+1}] +$$

$$+(b_1\overline{a}_{q-p-1} + ... + b_{q-p-1}\overline{a}_1) + a_1\overline{a}_{q-p+1} + ... + a_{N-q}\overline{a}_{N-p}\,.$$

(We suppose here that the expressions in ring brackets equal to 0 for $p = q-1$ and $p = q$, and that the expressions in square brackets equal to 0 when $p = 0$.) Let $p \leq N - q$, then after the simplification, we obtain

$$a_{p+1}\overline{a}_{q+1} + ... + a_{N-q}\overline{a}_{N-p} = \overline{b}_{p+1}b_{q+1} + ... + b_{N-p}\overline{b}_{N-q}. \tag{3}$$

Now the condition $r_{p-1,q-1} - r_{p,q} = 0$ applied to the previous equalities with $1 \leq p \leq q$ gives

$$a_p \overline{a}_q + a_{N-q+1} \overline{a}_{N-p+1} = b_q \overline{b}_p + b_{N-p+1} \overline{b}_{N-q+1}. \tag{4}$$

It remains to show that these equalities hold for all p and q, without the restrictions $p \leq q$ and $p + q < N + 1$.

1. If $p > q$, it is enough to interchange in (**??**) p by q and consider conjugated equalities.

2. If $p + q > N + 1$, then denote $s := N - q + 1$,$r := N - p + 1$ and we come to the same equalities with respect to r and s, with the conditions $s + r = 2N + 2 - p - q < 2N + 2 - N - 1 = N + 1$.

3. If, finally, $N - q = p - 1$, then from (**??**) it follows that $a_p \overline{a}_q = b_q \overline{b}_p$, which is a particular case of (**??**), corresponding to the choice $q + p = N + 1$.

The proof that equation (1) implies normality will not be given, for it is a consequence of our main theorem on the structure of normal Toeplitz matrices. All subsequent work requires only the implication that normal Toeplitz matrices satisfy equation (1). ∎

Remark 2.2 *If we consider (**??**) with $\tilde{p} = N - q + 1, \tilde{q} = p, N - \tilde{p} + 1 = q, N - \tilde{q} + 1 = N - p + 1$ we obtain that for each $\tilde{p}, \tilde{q}$ the following equalities hold,*

$$\overline{a}_{N-\tilde{p}+1} a_{\tilde{q}} + a_{\tilde{p}} \overline{a}_{N-\tilde{q}+1} = b_{N-\tilde{p}+1} \overline{b}_{\tilde{q}} + \overline{b}_{\tilde{p}} b_{N-\tilde{q}+1}\,. \tag{5}$$

The following observation will be put to use in the proof of the main theorem.

Remark 2.3 *If $N = 2n - 1$ and $a_n = b_n = 0$, then using the notation $\tilde{a}_p = a_p$, $\tilde{b}_p = b_p$, for $p < n$ and $\tilde{a}_p = a_{p+1}$, $\tilde{b}_p = b_{p+1}$, for $p > n$, we again come to the equalities of the form (**??**) for $4(n-1)$ variables $\tilde{a}_p$ and $\tilde{b}_p$.*

3. Main Result

Let

$$T = \begin{pmatrix} 0 & b_1 & \ddots & b_N \\ a_1 & 0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & b_1 \\ a_N & \ddots & a_1 & 0 \end{pmatrix}$$

be a normal Toeplitz matrix and let $\{m_i\} \subseteq \{1, ..., N\}$ be a set of positive integers. We say that a set of pairs of diagonals of the matrix T with the indices m_i is *co-connected* (with argument θ) if $b_{m_i} = \overline{a}_{m_i} e^{i\theta}$ and *contra-connected* (with argument θ) if $b_{m_i} = a_{N-m_i+1} e^{i\theta}$. Now using these definitions, we cast the statement of the main theorem in the following equivalent form.

Theorem 3.1 *If a finite complex Toeplitz matrix T of trace zero is normal, then there exists a single argument $0 \le \theta \le 2\pi$ such that with respect to θ either all pairs of diagonals of T are co-connected or all pairs of diagonals of T are contra-connected.*

The proof is based on the following principal idea: we prove first that each two pairs of diagonals with the indices p and $2n+1-p$, $1 \le p \le n$ (in case $N = 2n$) or each three pairs of diagonals with the indices $n-k, n, n+k$, $1 \le k \le n$ (in the case $N = 2n-1$) are either co-connected or contra-connected, or are simultaneously co- and contra-connected. Then we show that for all sets of pairs of diagonals there is a unique common argument θ, such that all pairs of diagonals are either co-connected or contra-connected, or simultaneously co- and contra-connected with the same argument.

Proof: We split the proof of this theorem into three parts.

Part I. $N = 2n - 1$ with $a_n \neq 0$.

Take two natural numbers m and k, such that $0 \leq m, k \leq n-1$ and apply Theorem 2.1 with $p = n - k$ and $q = n + m$. The following equalities

$$a_{n-m}\overline{a}_{n+k} + a_{n-k}\overline{a}_{n+m} = b_{n+m}\overline{b}_{n-k} + b_{n+k}\overline{b}_{n-m} \tag{6}$$

hold for 2 arbitrary pairs of diagonals with indices $[n-m, n+m]$ and $[n+k, n-k]$.

Consider then the following system of equalities

$$a_n\overline{a}_{n+k} + a_{n-k}\overline{a}_n = b_n\overline{b}_{n-k} + b_{n+k}\overline{b}_n, \quad (m = 0) \tag{7}$$

$$a_{n-k}\overline{a}_{n+k} = b_{n+k}\overline{b}_{n-k}, \quad (m = k \neq 0) \tag{8}$$

$$|a_n| = |b_n|, \quad (m = k = 0) \tag{9}$$

for the three pairs of diagonals with the indices $n-k, n, n+k$.

Taking into the account (**??**) one can suppose, without loss of generality, that $a_n = |a_n|, b_n = a_n e^{i\theta}$; then from (**??**) it follows that

$$\overline{a}_{n+k} + a_{n-k} = \overline{b}_{n-k}e^{i\theta} + b_{n+k}e^{-i\theta}$$

and from (**??**) it follows

$$\overline{a}_{n+k}a_{n-k} = (\overline{b}_{n-k}e^{i\theta})(b_{n+k}e^{-i\theta}).$$

Therefore at least one of the following two pairs of equations holds for all k:

$$\overline{a}_{n+k} = \overline{b}_{n-k}e^{i\theta}, \qquad a_{n-k} = b_{n+k}e^{-i\theta} \tag{10}$$

or

$$\overline{a}_{n+k} = b_{n+k}e^{-i\theta}, \qquad a_{n-k} = \overline{b}_{n-k}e^{i\theta} \tag{11}$$

Validity of (**??**) implies

$$b_{n-k} = a_{n+k}e^{i\theta},\ b_n = a_n e^{i\theta},\ b_{n+k} = a_{n-k}e^{i\theta} \tag{12}$$

which means that pairs of diagonals $[n-k, n, n+k]$ are contra-connected. If (**??**) holds, then

$$b_{n-k} = \overline{a}_{n-k}e^{i\theta},\ b_n = \overline{a}_n e^{i\theta},\ b_{n+k} = \overline{a}_{n+k}e^{i\theta} \tag{13}$$

and these diagonals are co-connected.

If it happens that both (**??**) and (**??**) hold for all $k = 1, ..., n-1$, then the proof of Part I of the theorem is complete. Otherwise let us suppose that for some k only one of (**??**), (**??**) holds, say (**??**), i.e. $\overline{a}_{n-k} \neq a_{n+k}$. We will show that the equalities (**??**) hold for all m. Assume that for some m (**??**) holds; we show that (**??**) is valid too. Substitute (**??**) to (**??**), then

$$(\overline{a}_{n+m} - a_{n-m})(a_{n-k} - \overline{a}_{n+k}) = 0$$

This means that $\overline{a}_{n+m} = a_{n-m}$, and therefore

$$b_{n-m} = a_{n+m}e^{i\theta},\ b_n = a_n e^{i\theta},\ b_{n+m} = a_{n-m}e^{i\theta}.$$

Part II. $N = 2n$.

Set $N = 2n$ and $q = p$ in equation (1). We obtain

$$|a_p|^2 + |a_{2n-p+1}|^2 = |b_p|^2 + |b_{2n-p+1}|^2 . \tag{14}$$

If we set $q = 2n + 1 - p$ in (1), we will have

$$a_p \overline{a}_{2n+1-p} = b_{2n+1-p} \overline{b}_p. \tag{15}$$

The system of equations (**??**), (**??**), as in Part I, possesses two representations, namely

$$b_p = \overline{a}_p e^{i\theta_p}, \qquad b_{2n+1-p} = \overline{a}_{2n+1-p} e^{i\theta_p} \tag{16}$$

(which means that the pairs of diagonals $[p, N - p + 1]$ are co-connected) or

$$b_p = a_{2n+1-p} e^{i\gamma_p}, \qquad b_{2n+1-p} = a_p e^{i\gamma_p} \tag{17}$$

(these pairs of diagonals are contra-connected). In other words, for each $p = 1, ..., n$ at least one of the two equations above holds.

Now we consider two possibilities:

1) $a_p \neq 0$ for each p.

We have to consider three cases:

Case 1. All pairs of diagonals are co-connected , i.e. $b_p = \overline{a}_p e^{i\beta_p}$.

If all β_p are equal, then we have finished the proof. Let $\beta_1 \neq \beta_2$. By Lemma (**??**) there exists α such that $b_1 = a_{2n} e^{i\alpha}$, $b_{2n} = a_1 e^{i\alpha}$, $b_2 = a_{2n-1} e^{i\alpha}$, $b_{2n-1} = a_2 e^{i\alpha}$.

Now take arbitrary $p > 2$. $b_p = \overline{a}_p e^{i\beta_p}$, $b_{2n-p+1} = \overline{a}_{2n-p+1} e^{i\beta_p}$. It is important here that $\beta_p \neq \beta_k$, either for $k = 1$ or for $k = 2$ (because $\beta_1 \neq \beta_2$). Applying Lemma (**??**) for $[b_p, b_{2n-p+1}, b_k, b_{2n-k+1}]$, we obtain $b_p = a_{2n-p+1}e^{i\alpha}$ and $b_{2n-p+1} = a_p e^{i\alpha}$.

Case 2. Assume now that all pairs of diagonals are contra-connected , i.e $b_p = a_{2n-p+1}e^{i\gamma_p}$ for all p.

Using the same arguments as in Case 1, we obtain that either all γ_p are equal to each other, or for some γ, $b_p = \overline{a}_p e^{i\gamma}$ for all p.

Case 3. Let now some pairs of diagonals be co-connected and some pairs of diagonals be contra-connected , but not all of them be co-connected . Without loss of generality, we can assume that

$$b_1 = a_{2n}e^{i\theta_1}, b_{2n} = a_1 e^{i\theta_1}, \tag{18}$$

and $[b_1, b_{2n}, a_1, a_{2n}]$ are not co-connected . Take arbitrary p. Then

$$b_p = \overline{a}_p e^{i\theta_p}, b_{2n-p+1} = a_{2n-p+1}e^{i\theta_p} \tag{19}$$

According to Lemma (**??**) either all of pairs of diagonals

$$[b_1, b_{2n}, a_1, a_{2n}, b_p, b_{2n-p+1}, a_p, a_{2n-p+1}]$$

are co-connected or contra-connected with the same argument. But because $[b_1, b_{2n}, a_1, a_{2n}]$ are not co-connected , then

$$[b_1, b_{2n}, a_1, a_{2n}, b_p, b_{2n-p+1}, a_p, a_{2n-p+1}]$$

are contra-connected . We have thus reduced this case to Case 2.

2) $a_p = 0$ for some p.

Consider again three Cases:

Case 1.

$$a_{2n-p+1} \neq 0, \quad b_{2n-p+1} = \overline{a}_{2n-p+1} e^{i\theta},\ b_p = \overline{a}_p e^{i\theta} \tag{20}$$

Then $b_p = 0$ and for each q, substituting (**??**) in (5), we obtain

$$\overline{a}_{N-q+1} a_p + a_q \overline{a}_{N-p+1} = b_{N-q+1} \overline{b}_p + b_{N-p+1} \overline{b}_q,$$

but $a_p = b_p = 0$ and $a_{2n-p+1} \neq 0$, so $a_q = \overline{b}_q e^{i\theta}$ or $b_q = \overline{a}_q e^{i\theta}$ for each q.

Case 2.

$$a_{2n-p+1} \neq 0, \quad b_{2n-p+1} = a_p e^{i\theta},\ b_p = a_{2n-p+1} e^{i\theta} \tag{21}$$

In this case $b_{2n-p+1} = 0$ and using the same arguments as in Case 1, we obtain $a_q = b_{2n-q+1} e^{-i\theta}$ or $b_{2n-q+1} = a_q e^{i\theta}$ for each q, i.e. $b_m = a_{2n-m+1} e^{i\theta}$ for each m.

Case 3. $a_p = a_{2n-p+1} = b_p = b_{2n-p+1} = 0$.

In this case we reduce the order by 2 and consider the Toeplitz matrix of order $N - 2$ without these four zero diagonals. To this normal Toeplitz matrix the results of the previous cases apply and yield the desired conclusion.

Part III.

$N = 2n - 1, \quad a_n = 0$ (and so $b_n = 0$ by (9)). In the light of Remark (**??**) of Theorem (**??**) the equalities (**??**) hold for an even number of diagonals. This part of the proof, therefore, can be reduced to Part II. ∎

4. The Technical Lemmas

This section contains the technical lemmas used in the proof of Theorem 3.1. Throughout this section we assume that $N = 2n$ and that T_N is normal.

Lemma 4.1 *If there exist $p, q, \theta, \gamma, \theta \neq \gamma$, such that*

$$b_p = \overline{a}_p e^{i\theta}, \quad b_{2n-p+1} = \overline{a}_{2n-p+1} e^{i\theta} \tag{22}$$

and

$$b_q = \overline{a}_q e^{i\gamma}, \quad b_{2n-q+1} = \overline{a}_{2n-q+1} e^{i\gamma}, \tag{23}$$

then there exists α, such that $b_m = a_{2n-m+1} e^{i\alpha}$ for $m \in \{p, q, 2n-p+1, 2n-q+1\}$.

Proof:

Substitute (**??**) and (**??**) into (**??**) to obtain

$$\overline{a}_q a_p + a_{2n-q+1}\overline{a}_{2n-p+1} = \overline{a}_q e^{i\gamma} a_p e^{-i\theta} + \overline{a}_{2n-p+1} e^{i\theta} a_{2n-q+1} e^{-i\gamma},$$

but $\theta \neq \gamma$, so

$$\overline{a}_q a_p = \overline{a}_{2n-p+1} a_{2n-q+1} e^{i(\theta-\gamma)}. \tag{24}$$

Analogously, substitute (**??**) and (**??**) into (**??**) to get

$$\overline{a}_{2n-p+1} a_q = a_p \overline{a}_{2n-q+1} e^{-i(\theta-\gamma)}. \tag{25}$$

Consider now a product of (**??**) and (**??**); we obtain

$$\overline{a}_{2n-p+1} |a_q|^2 a_p = \overline{a}_{2n-p+1} |a_{2n-q+1}|^2 a_p\,. \tag{26}$$

1. If $a_p = a_{2n-p+1} = 0$, we take $\alpha = \gamma$.

2. If $a_p = 0, a_{2n-p+1} \neq 0$, then (**??**) implies $a_q = 0$ and (**??**) implies $a_{2n-q+1} = 0$. In this case we take $\alpha = \theta$.

3. Case $a_p \neq 0, a_{2n-p+1} = 0$ is the same as 2.

4. Let $a_p a_{2n-p+1} \neq 0$. Then (**??**) implies $|a_q| = |a_{2n-q+1}|$. Then if $a_q = 0$, then let $\alpha = \theta$. If $a_q \neq 0$, there are $s > 0$ and $0 \leq \delta, \beta < 2\pi$ such that $a_q = se^{i\delta}$ and $a_{2n-q+1} = se^{i\beta}$. Now substituting these into (**??**) we have

$$a_{2n-p+1} = \overline{a}_p e^{i(\theta+\delta+\beta-\gamma)}$$

or

$$a_p = \overline{a}_{2n-p+1} e^{i(\theta+\delta+\beta-\gamma)}\,.$$

We come to such a system of equalities:

$$b_p = a_{2n-p+1} e^{-i(\theta+\delta+\beta-\gamma)} e^{i\theta} = a_{2n-p+1} e^{-i(\delta+\beta-\gamma)}$$

$$b_{2n-p+1} = a_p e^{-i(\theta+\delta+\beta-\gamma)} e^{i\theta} = a_p e^{-i(\delta+\beta-\gamma)}$$

$$b_q = se^{-i\delta} e^{i\gamma} = a_{2n-q+1} e^{-i(\delta+\beta-\gamma)}$$

$$b_{2n-q+1} = se^{-i\beta} e^{i\gamma} = a_q e^{-i(\delta+\beta-\gamma)}\,.$$

Denoting now $\delta + \beta - \gamma = -\alpha$, we obtain
$b_m = a_{2n-m+1} e^{i\alpha}$ for $m \in \{p, 2n-p+1, q, 2n-q+1\}$. ■

The proofs of next two lemmas use the same ideas as that of Lemma 4.1.

Lemma 4.2 *If there exist* $p, q, \theta, \gamma, \theta \neq \gamma$, *such that*

$$b_p = a_{2n-p+1} e^{i\theta}, \quad b_{2n-p+1} = a_p e^{i\theta}$$

and

$$b_q = a_{2n-q+1}e^{i\gamma}, \quad b_{2n-q+1} = a_q e^{i\gamma},$$

then there exists α *such that* $b_m = \overline{a}_m e^{i\alpha}$ *for* $m \in \{p, q, 2n-p+1, 2n-q+1\}$.

Lemma 4.3 *If there exist* $p, q, \theta, \gamma, \theta \neq \gamma$*, such that*

$$b_p = a_{2n-p+1}e^{i\theta}, \quad b_{2n-p+1} = a_p e^{i\theta}$$

and

$$b_q = \overline{a}_q e^{i\gamma}, \quad b_{2n-q+1} = \overline{a}_{2n-q+1}e^{i\gamma},$$

then there exists β *such that one of the equalities holds for all* $m \in \{p, q, 2n - p+1, 2n-q+1\}$*:* $b_m = \overline{a}_m e^{i\beta}$ *or* $b_m = a_{2n-m+1}e^{i\beta}$.

References

[1] A. Brown and P.R. Halmos, Algebraic properties of Toeplitz operators, J. Reine Angew. Math. 213 (1963/1964), 89-102.

[2] Kh.D. Ikramov, Describing normal Toeplitz matrices, Zh. Vychisl. Mat. Mat. Fiz. 34 (1994), 473-479; English translation in Comp. Maths. Math. Phys. 34 (1994), 399-404.

[3] I.S. Iohvidov, "Hankel and Toeplitz matrices and forms," Birkhäuser, Boston, 1982.