Normality and paracompactness of Pixley–Roy hyperspaces

by

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Abstract. Normality and paracompactness of Pixley–Roy hyperspaces is investigated with a particular emphasis on Pixley–Roy hyperspaces of metrizable spaces, of compact spaces and of spaces of ordinals. Operations on Pixley–Roy hyperspaces are also studied.

In particular, we show that the Pixley–Roy hyperspaces $\mathscr{F}[X]$ of a compact space X is normal iff it is paracompact iff X is scattered.

§ 1. Introduction. In this paper we study (hereditary) normality and paracompactness of Pixley–Roy hyperspaces.

In Section 2 we characterize those spaces X whose Pixley–Roy hyperspaces $\mathscr{F}[X]$ are paracompact or hereditarily paracompact. We also show that $\mathscr{F}[X]$ is (hereditarily) paracompact iff $\mathscr{F}[X]^n$ is (hereditarily) paracompact for all $n<\omega$ iff $\mathscr{F}[X]$ is (hereditarily) collectionwise Hausdorff (Theorems 2.1 and 2.2).

Section 3 contains a number of applications of these results, mostly concerned with the invariance of paracompactness and hereditary paracompactness under continuous mappings and the operations of union and product of spaces. In particular, we show that if X is a σ-locally finite union of closed subspaces whose Pixley–Roy hyperspaces are paracompact, or if every point of X has a neighbourhood whose Pixley–Roy hyperspace is paracompact, then $\mathscr{F}[X]$ is paracompact (Corollaries 3.4 and 3.5). We also show that if X is scattered, then $\mathscr{F}[X]$ is paracompact and that if $\mathscr{F}[X_i]$ is paracompact for $i = 1, 2, \ldots, n$, then $\mathop{\mathbf{P}}_{i=1}^{n} \mathscr{F}[X_i]$ is paracompact (Corollaries 3.6 and 3.8).

Section 4 is devoted to the investigation of Pixley–Roy hyperspaces of compact (or, more generally, locally Čech-complete) spaces. We prove that if X is locally Čech-complete, then $\mathscr{F}[X]$ is (hereditarily) normal iff $\mathscr{F}[X]$ is (hereditarily) paracompact iff X is scattered (scattered and first countable) (Theorems 4.2 and 4.6).

In Section 5 we examine normality (= perfect normality) and metrizability (= paracompactness) of Pixley–Roy hyperspaces $\mathscr{F}[M]$ of metrizable spaces M. Theorem 5.2 asserts that $\mathscr{F}[M]$ is metrizable iff M is σ-discrete. If $\dim M = 0$, then $\mathscr{F}[M]$ is normal iff M is a strong q-set (Theorem 5.9). Moreover, the existence of a normal non-metrizable hyperspace $\mathscr{F}[M]$ is equivalent to the existence of a non-σ-discrete strong q-set (Theorem 5.11).

Section 6 deals with Pixley–Roy hyperspaces of spaces of ordinals. Such hyperspaces $\mathscr{F}[X]$ are always paracompact (Theorem 6.1) and $\mathscr{F}[X]$ is hereditarily normal iff $\mathscr{F}[X]$ is hereditarily paracompact iff characters of all non-isolated points of X coincide (Theorem 6.2). This result leads to some pathological examples of Pixley–Roy hyperspaces (Examples 6.4–6.6).

In the last Section 7 we define and examine iterated Pixley–Roy hyperspaces $\mathscr{F}^n[X]$. In particular, we prove that $\mathscr{F}^n[X]$ is paracompact for all $n \geqslant 2$ and, if X is first countable, then $\mathscr{F}^n[X]$ is metrizable for all $n \geqslant 2$.

The paper contains a number of open problems.

Throughout this paper, all spaces are T_1, all mappings are continuous, τ denotes an infinite cardinal, $\varkappa$ denotes a (von Neumann) ordinal and $c = 2^{\omega}$.

The Pixley–Roy hyperspace of the real line was defined by Pixley and Roy in [PR] and later generalized by van Douwen in [vD]. Pixley–Roy hyperspaces were also applied and investigated in [PT], [L], [BFL$_1$], [BFL$_2$], [Pl], [R] and [B]. The *Pixley–Roy hyperspace* $\mathscr{F}[X]$ *of a space* X (or, briefly, the PR-*hyperspace of* X) is the set of all non-empty finite subsets F of X with the topology generated by *basic open sets* of the form $[F, V] = \{H \in \mathscr{F}[X]: F \subset H \subset V\}$, where $F \in \mathscr{F}[X]$ and V is a neighbourhood of F in X. We shall often use the fact that $[F, V] \cap [H, W] \neq \emptyset$ if and only if $F \subset W$ and $H \subset V$. Following [BFL$_1$] we write

$$\mathscr{F}_m[X] = \{F \in \mathscr{F}[X]: |F| \leqslant m\}$$

for $m < \omega$. Pixley–Roy hyperspaces are always completely regular and zerodimensional (i.e. $\operatorname{ind} \mathscr{F}[X] = 0$).

A completely regular space Z is *strongly zerodimensional* (i.e. $\dim Z = 0$) if any two disjoint functionally closed subsets of Z can be separated by a clopen set. A space Z is *collectionwise Hausdorff* if every discrete collection of points of Z can be separated by a disjoint collection of open sets.

A space is *scattered* if it contains no dense-in-itself subsets. By $X \oplus Y$ we denote the free (i.e. disjoint) *sum of the spaces* X *and* Y. For the undefined notions the reader is referred to [E].

PROPOSITION 1.1. *$\mathscr{F}[X]$ is metrizable if and only if it is first countable and paracompact.*

Proof. $\mathscr{F}[X]$ is a Moore space iff X is first countable iff $\mathscr{F}[X]$ is first countable [vD]. ■

The proof of the following two propositions is an easy exercise.

PROPOSITION 1.2. *If A is a subspace of X, then $\mathscr{F}[A]$ is a closed subspace of $\mathscr{F}[X]$.* ■

PROPOSITION 1.3. *$\mathscr{F}[X]$ is perfectly normal if and only if $\mathscr{F}[X]$ is normal and all points of X are G_δ sets.* ■

We do not know if the assumption of normality of $\mathscr{F}[X]$ in the following proposition is redundant.

PROPOSITION 1.4. *If $\mathscr{F}[X]$ is normal, then* $\dim \mathscr{F}[X] = 0$.

Proof. One easily proves by induction that $\dim F_n[X] = 0$ for all $n < \omega$. Consequently, $\dim \mathscr{F}[X] = 0$ by ([E]; Theorem 7.2.1). ■

PROPOSITION 1.5. *$\mathscr{F}[X \oplus Y]$ is homeomorphic to $\mathscr{F}[X] \oplus \mathscr{F}[Y] \oplus (\mathscr{F}[X] \times \mathscr{F}[Y])$.*

Proof. Clearly, $\mathscr{F}[X \oplus Y]$ is a disjoint union of open subsets

$A = \{F \in \mathscr{F}[X \oplus Y]: F \subset X\}$,

$B = \{F \in \mathscr{F}[X \oplus Y]: F \subset Y\}$ and

$C = \{F \in \mathscr{F}[X \oplus Y]: F \cap X \neq \emptyset \text{ and } F \cap Y \neq \emptyset\}$.

Moreover,

$A \cong \mathscr{F}[X]$,

$B \cong \mathscr{F}[Y]$ and

$C \cong \mathscr{F}[X] \times \mathscr{F}[Y]$. ■

Propositions 1.3 and 1.5 are also proved in [L].

§ 2. Characterization of paracompact and hereditarily paracompact Pixley-Roy hyperspaces.

In this section we shall prove the following two theorems characterizing paracompact and hereditarily paracompact PR-hyperspaces. Applications of these results will be given in the next section.

THEOREM 2.1. *The following conditions are equivalent:*

(i) *$\mathscr{F}[X]$ is paracompact;*

(ii) *$\mathscr{F}[X]^n$ is paracompact for all $n < \omega$;*

(iii) *$\mathscr{F}[X]$ is collectionwise Hausdorff;*

(iv) *for every non-empty finite subset F of X one can choose a neighbourhood V_F so that the inclusions $F \subset V_H$ and $H \subset V_F$ imply $F \cap H \neq \emptyset$.*

THEOREM 2.2. *The following conditions are equivalent:*

(i) *$\mathscr{F}[X]$ is hereditarily paracompact;*

(ii) *$\mathscr{F}[X]^n$ is hereditarily paracompact for all $n < \omega$;*

(iii) *$\mathscr{F}[X]$ is hereditarily collectionwise Hausdorff;*

(iv) *for every non-empty finite subset F of X one can choose a neighbourhood W_F so that the inclusions $F \subset W_H$ and $H \subset W_F$ imply $F \subset H$ or $H \subset F$.*

Remark. M. G. Tkačenko [T] introduced, for quite a different purpose, the notion of a weakly separated space: X is *weakly separated* if for every point $x \in X$ one can choose a neighbourhood V_x so that if $y \in V_x$ and $x \in V_y$, then $x = y$ (cf. also [A$_1$] and [HJ]). One easily sees that $\mathscr{F}_2[X]$ is (hereditarily) paracompact if and only if X is weakly separated. Thus conditions (iv) in Theorems 2.1 and 2.2 are natural strengthenings of this notion. ■

PROBLEM 1. Find similar characterizations of (hereditarily) normal PR-hyperspaces.

PROBLEM 2. Suppose that $\mathscr{F}[X]$ is paracompact. Is $\mathscr{F}[X]^\omega$ paracompact? Is $\mathscr{F}[X^2]$ paracompact?

Before proving Theorem 2.1 we shall need the following definition and two lemmas. Let $m<\omega$. We say that a family of open subsets of $\mathscr{F}[X]$ is *m-proper* if it covers $\mathscr{F}_m[X]$ and consists of mutually disjoint sets of the form $[F, V]$, where $|F|\leqslant m$.

LEMMA 2.3. *Every m-proper family is discrete.*

Proof. Let $\mathscr{U} = \{[F_s, W_s]:\ s\in S\}$ be m-proper. Since $\mathscr{U}$ is disjoint and consists of clopen sets it suffices to show that every $F\in\mathscr{F}[X]$ has a neighbourhood $[F, W]$ intersecting finitely many elements of $\mathscr{U}$.

For every $x\in F$ put

$$W(x) = \bigcup\{W_t:\ F_t\subset F,\ x\in W_t \text{ and } t\in S\}$$

and

$$W = \bigcup\{W(x):\ x\in F\}.$$

We shall show that if $[F_s, W_s]\cap[F, W]\neq\emptyset$, then $F_s\subset F$, which will complete the proof. We have $F_s\subset W$, $F\subset W_s$ and $|F_s|\leqslant m$. Thus there exists an $H\subset F$, with $|H|\leqslant m$, such that

$$F_s\subset\bigcup\{W(x):\ x\in H\}.$$

There exists a $t\in S$ such that $H\in[F_t, W_t]$, hence $F_t\subset H\subset W_t$. We infer that $\bigcup\{W(x):\ x\in H\}\subset W_t$ and therefore $F_s\subset W_t$ and $F_t\subset H\subset F\subset W_s$. Thus $[F_s, W_s]\cap\cap[F_t, W_t]\neq\emptyset$ which implies $s=t$ and $F_s\subset F$. ■

LEMMA 2.4. *Let $\mathscr{G}$ be an open covering of $\mathscr{F}[X]$. If for all $m<\omega$ every m-proper family refining $\mathscr{G}$ can be extended to an $(m+1)$-proper family refining $\mathscr{G}$, then $\mathscr{G}$ has a disjoint refinement consisting of basic open sets.*

Proof. Put $\mathscr{U}_0=\emptyset$ and let $\mathscr{U}_{m+1}$ be an $(m+1)$-proper extension of $\mathscr{U}_m$ refining $\mathscr{G}$. The family $\mathscr{U}=\bigcup_{m<\omega}\mathscr{U}_m$ clearly has the desired properties. ■

Proof of Theorem 2.1. The implications (ii) → (i) and (i) → (iii) are obvious.

(iii) → (iv). Let $\mathscr{U}$ be an m-proper family in $\mathscr{F}[X]$. By 2.3 $\mathscr{U}$ is discrete and $\mathscr{A}=\bigcup\mathscr{U}$ is clopen. Let $\mathscr{B}=\{F\in\mathscr{F}[X]:\ |F|=m+1 \text{ and } F\notin\mathscr{A}\}$. Clearly $\mathscr{B}$ is a closed discrete subset of $\mathscr{F}[X]$. Let $\mathscr{V}=\{[F, W_F]:\ F\in\mathscr{B}\}$ be a disjoint family of basic open sets separating points of $\mathscr{B}$. We may obviously assume that $\mathscr{A}\cap\bigcup\mathscr{V}=\emptyset$. The family $\mathscr{U}\cup\mathscr{V}$ is an $(m+1)$-proper extension of $\mathscr{U}$.

Put $\mathscr{G}=\{\mathscr{F}[X]\}$. From 2.4 we infer that there exists a disjoint covering $\mathscr{W}$ of $\mathscr{F}[X]$ consisting of basic open sets. Let $\mathscr{W}=\{[F_s, W_s]:\ s\in S\}$.

For every $F\in\mathscr{F}[X]$ there is exactly one $s\in S$ such that $F\in[F_s, W_s]$. Put $V_F=W_s$. One easily checks that the family $\{V_F:\ F\in\mathscr{F}[X]\}$ satisfies (iv).

(iv) → (i). Let $\mathscr{G}$ be an open covering of $\mathscr{F}[X]$. By 2.4 it suffices to show that every m-proper family $\mathscr{U}$ refining $\mathscr{G}$ has an $(m+1)$-proper extension refining $\mathscr{G}$.

Let $\mathscr{U}=\{[F_s, W_s]:\ s\in S\}$, $\mathscr{A}=\bigcup\mathscr{U}$ and $\mathscr{B}=\{F\in\mathscr{F}[X]:\ |F|=m+1$ and $F\notin\mathscr{A}\}$. Since by 2.3 $\mathscr{A}$ is clopen it suffices to show that points of $\mathscr{B}$ can be separated by disjoint basic open sets (we can always require that those sets refine $\mathscr{G}$ and are disjoint from $\mathscr{A}$).

Take $F\in\mathscr{B}$ and for every $x\in F$ put

$$W_F(x) = V_F\cap\bigcap\{W_s:\ F_s\subset F,\ x\in W_s \text{ and } s\in S\}$$

and

$$W_F = \bigcap\{W_F(x):\ x\in F\}.$$

We claim that the sets $\{[F, W_F]:\ F\in\mathscr{B}\}$ are disjoint. Suppose that $[F, W_F]\cap\cap[K, W_K]\neq\emptyset$, $F, K\in\mathscr{B}$ and $F\neq K$. By (iv) we have $H=F\cap K\neq\emptyset$ and naturally $|H|\leqslant m$. There exists an $s\in S$ such that $H\in[F_s, W_s]$. We get

$$\bigcup\{W_F(x):\ x\in F\cap W_s\}\subset W_s \quad\text{and}\quad \bigcup\{W_K(x):\ x\in K\cap W_s\}\subset W_s.$$

Since $F, K\notin\mathscr{A}$ the sets $A=F\setminus W_s$ and $B=K\setminus W_s$ are non-empty and $A\cap B\subset(F\setminus H)\cap(K\setminus H)=\emptyset$. There exist $t, u\in S$ such that $A\in[F_t, W_t]$ and $B\in[F_u, W_u]$. Since $A\cap B=\emptyset$ we have $t\neq u$ and also

$$\bigcup\{W_F(x):\ x\in A\}\subset W_t \quad\text{and}\quad \bigcup\{W_K(x):\ x\in B\}\subset W_u.$$

This implies

$$A\subset F\subset W_K\subset W_s\cup W_u \quad\text{and}\quad B\subset K\subset W_F\subset W_s\cup W_t$$

and thus $A\subset W_u$ and $B\subset W_t$ which shows that $F_t\subset W_u$ and $F_u\subset W_t$. Thus $[F_t, W_t]\cap[F_u, W_u]\neq\emptyset$. This contradiction completes the proof of the equivalence of (i), (iii) and (iv).

(iv) → (ii). From Proposition 1.5 and simple induction it follows that $\mathscr{F}[\bigoplus_{i=1}^{n} X_i]$, where $X_i=X$ for $i=1,2,\ldots,n$, contains $\mathscr{F}[X]^n$ as a clopen subset. One easily checks that if X satisfies (iv) then so does $Z=\bigoplus_{i=1}^{n} X_i$ (cf. Corollary 3.5). Hence, by (i), $\mathscr{F}[Z]$ is paracompact and thus $\mathscr{F}[X]^n$ is paracompact. ■

Remark. From the above proof it follows that each of the conditions (i)–(iv) in Theorem 2.1 is equivalent to the existence of a disjoint covering of $\mathscr{F}[X]$ consisting of basic open sets. ■

Proof of Theorem 2.2. The implications (ii) → (i) and (i) → (iii) are obvious.

(iii) → (iv). Suppose that sets W_F satisfying (iv) have been already assigned to all sets F of cardinality $\leqslant m$. The set $\mathscr{B}=\{F\in\mathscr{F}[X]:\ |F|=m+1\}$ is closed and discrete in $\mathscr{F}[X]\setminus\mathscr{F}_m[X]$, hence by (iii) there exist disjoint basic open sets $\{[F, G_F]:\ F\in\mathscr{B}\}$.

For $x\in F\in\mathscr{B}$ put

$$W(x) = G_F\cap\bigcap\{W_H:\ x\in H\subsetneq F\} \quad\text{and}\quad W_F=\bigcup\{W(x):\ x\in F\}.$$

Suppose that $F\in\mathscr{B}$, $|H|\leqslant m+1$, $F\neq H$, $F\subset W_H$ and $H\subset W_F$. From the construction of the W_F's it follows that $|H|\leqslant m$. There exists a $K\subset F$, $|K|=m$ such that $H\subset\bigcup\{W(x):\ x\in K\}\subset W_K$. We have $K\subset F\subset W_H$ and $H\subset W_K$, which by the inductive assumption, implies that either $K\subset H$ or $H\subset K$. But $|K| = m$ and $|H|\leqslant m$, thus $H\subset K\subset F$, which completes the inductive construction.

(iv) $\to$ (i). Let U be an open subset of $\mathcal{F}[X]$ and let $\mathcal{G}$ be an open covering of U. We can clearly assume that for every $F\in U$ there exists a $G\in\mathcal{G}$ such that $[F, W_F]\subset G$.

Let us put

$$\mathcal{V}_0 = \emptyset,$$

$$\mathcal{V}_{m+1} = \{[F, W_F]\setminus\bigcup_{i\leqslant m}\bigcup\mathcal{V}_i\colon |F| = m+1 \text{ and } F\in U\},$$

and

$$\mathcal{V} = \bigcup_{m<\omega}\mathcal{V}_m.$$

Clearly $\mathcal{V}$ is a covering of U refining $\mathcal{G}$. In order to show that $\mathcal{V}$ is open and locally finite in U it suffices to verify that for all $m<\omega$:

$(*)_m$ the family $\mathcal{V}_m$ is locally finite in U and consists of clopen subsets of U.

Clearly $(*)_0$ is true. If $(*)_i$ holds for all $i\leqslant m$, then clearly the elements of $\mathcal{V}_{m+1}$ are clopen in U. It is enough to check that $\mathcal{V}_{m+1}$ is locally finite in U. Let $K\in U$. If $K\in\bigcup_{i\leqslant m}\bigcup\mathcal{V}_i = W$, then W is a neighbourhood of K intersecting no elements of $\mathcal{V}_{m+1}$. Otherwise, $|K|\geqslant m+1$ and if $[K, W_K]\cap[F, W_F]\neq\emptyset$, for some F of cardinality $m+1$ then, by (iv), either $F\subset K$ or $K\subset F$. But $|F| = m+1$ and $|K|\geqslant m+1$, hence $F\subset K$. Thus $[K, W_K]$ intersects only finitely many elements of $\mathcal{V}_{m+1}$ which proves $(*)_{m+1}$ and completes the proof of the equivalence of (i), (iii) and (iv).

(iv) $\to$ (ii). As in the proof of the implication (iv) $\to$ (ii) in Theorem 2.1, one easily sees that it suffices to show that if X satisfies (iv) then so does $Z = \bigoplus_{i=1}^{n} X_i$, where $X_i = X$, for $i = 1, 2, \ldots, n$. This, in turn, can be reduced by induction to the case of $Z = X_1\oplus X_2$, where $X_1 = X_2 = X$.

For a subset A of X by A_1 and A_2 we shall denote the corresponding subsets of X_1 and X_2, respectively. Let $K, L\subset X$ and $P = K_1\cup L_2$ be an arbitrary element of $\mathcal{F}[Z]$. We shall assign to P a neighbourhood W_P in $Z = X_1\oplus X_2$ so that (iv) is satisfied.

Let us put

$$W_P = (W_{K\cup L}\setminus(L\setminus K))_1\cup(W_{K\cup L}\setminus(K\setminus L))_2.$$

Suppose that $K^*, L^*\subset X$, $P^* = K_1^*\cup L_2^*$ and that $P\subset W_{P^*}$ and $P^*\subset W_P$. Then

(1) $$K^*\subset W_{K\cup L}\setminus(L\setminus K);\quad L^*\subset W_{K\cup L}\setminus(K\setminus L);$$

(2) $$K\subset W_{K^*\cup L^*}\setminus(L^*\setminus K^*)\quad\text{and}\quad L\subset W_{K^*\cup L^*}\setminus(K^*\setminus L^*).$$

This implies that $K^*\cup L^*\subset W_{K\cup L}$ and $K\cup L\subset W_{K^*\cup L^*}$, thus, by (iv), either $K\cup L\subset K^*\cup L^*$ or $K^*\cup L^*\subset K\cup L$. Suppose e.g. $K\cup L\subset K^*\cup L^*$ and assume that $P\not\subset P^*$. Then either $K\setminus K^*\neq\emptyset$ or $L\setminus L^*\neq\emptyset$. Suppose e.g. $K\setminus K^*\neq\emptyset$. This implies that $\emptyset\neq K\setminus K^*\subset L^*$, hence $K\cap(L^*\setminus K^*)\neq\emptyset$. But $K\cap(L^*\setminus K^*) = \emptyset$ by (2); contradiction. All other cases are dealt with similarly. This completes the proof. ∎

Remark. From the above proof it follows that each of the conditions (i)–(iv) in Theorem 2.2 is equivalent to the following condition:

(iv)′ for every $m<\omega$ and every m-element subset F of X one can choose a neighbourhood $V_F(m)$ so that inclusions $F\subset V_H(m)$ and $H\subset V_F(m)$ imply $F = H$. ∎

§ 3. Applications. This section contains a number of applications of Theorems 2.1 and 2.2 mostly concerned with the invariance of paracompactness and hereditary paracompactness of PR-hyperspaces under continuous mappings and the operations of union and product of spaces.

PROPOSITION 3.1. *Let $(S, <)$ be a partially ordered set and suppose that $X = \bigcup\{X_s\colon s\in S\}$, where $X_s\cap X_t = \emptyset$ for $s\neq t$ and $\bigcup\{X_t\colon t\leqslant s\}$ is open in X for every $s\in S$.*

If $\mathcal{F}[X_s]$ is paracompact for every $s\in S$, then $\mathcal{F}[X]$ is paracompact.

Proof. By Theorem 2.1 for every $s\in S$ and every non-empty finite subset F of X_s one can assign a neighbourhood V_F^s of F in X_s so that if $F\subset V_H^s$ and $H\subset V_F^s$, then $F\cap H\neq\emptyset$.

For every subset F of X put $F_s = F\cap X_s$ and define

$$V_F = \bigcup\{V_{F_s}^s\cup\bigcup_{t<s}X_t\colon s\in S \text{ and } F_s\neq\emptyset\}$$

for every non-empty finite subset of X. One easily checks that V_F is a neighbourhood of F in X. Suppose that $F\subset V_H$ and $H\subset V_F$. We have to show that $F\cap H\neq\emptyset$.

Let $A = \{s\in S\colon F_s\neq\emptyset \text{ or } H_s\neq\emptyset\}$. The set A is finite and thus A contains a maximal element s_0. Suppose e.g. $F_{s_0}\neq\emptyset$. Since $F_{s_0}\subset V_H$, there exists an s such that $H_s\neq\emptyset$ and $F_{s_0}\cap(V_{H_s}^s\cup\bigcup_{t<s}X_t)\neq\emptyset$. From the maximality of s_0 and the inequality $s_0\leqslant s$ we infer that $s = s_0$ and that $F_{s_0}\subset V_{H_{s_0}}^{s_0}$. Similarly, we prove that $H_{s_0}\subset V_{F_{s_0}}^{s_0}$. Therefore, $F\cap H\supset F_{s_0}\cap H_{s_0}\neq\emptyset$. ∎

COROLLARY 3.2 [BFL$_2$]. *Suppose that X can be partially ordered by $<$ in such a way that $\{y\colon y\leqslant x\}$ is open in X for every $x\in X$. Then $\mathcal{F}[X]$ is paracompact.* ∎

COROLLARY 3.3. *Suppose that $X = \bigcup_{\alpha<\varkappa}A_\alpha$ and the sets $K_\alpha = \bigcup\{A_\beta\colon \beta<\alpha\}$ are open (or closed) for every $\alpha<\varkappa$.*

If $\mathcal{F}[A_\alpha]$ is paracompact for every $\alpha<\varkappa$, then $\mathcal{F}[X]$ is paracompact.

Proof. Putting $A' = A_\alpha\setminus\bigcup_{\beta<\alpha}A_\beta$ and using Proposition 1.2 we can always make the sets A_α disjoint. If the sets K_α are open then also the sets $\bigcup\{A_\beta\colon \beta\leqslant\alpha\} = K_{\alpha+1}$

are open. If the sets K_α are closed, then the sets $\bigcup\{A_\beta\colon \beta\geqslant\alpha\}$ are open. Now, it suffices to apply Proposition 3.1. ■

Corollary 3.4. *If X is a σ-locally finite union of closed subspaces whose PR-hyperspaces are paracompact, then $\mathscr{F}[X]$ is paracompact.* ■

Corollary 3.5. *If every point of X has a neighbourhood whose PR-hyperspace is paracompact, then $\mathscr{F}[X]$ is paracompact.* ■

Corollary 3.6. *If X is scattered, then $\mathscr{F}[X]$ is paracompact.*

Proof. Clearly, for some $\varkappa$, $X=\bigcup_{\alpha<\varkappa}(X^{(\alpha)}\setminus X^{(\alpha+1)})$ ([1]) and the sets $\bigcup_{\beta<\alpha}(X^{(\beta)}\setminus X^{(\beta+1)})=X\setminus X^{(\alpha)}$ are open for every $\alpha<\varkappa$. Our assertion follows now from Corollary 3.3. ■

The following corollary generalizes one of the results obtained by M. E. Rudin [R] for X being a subspace of a Suslin line.

Corollary 3.7. *Suppose that $X=\bigcup_{\alpha<\omega_1}A_\alpha$, where sets A_α are countable and the set $S=\{\alpha\colon \bigcup_{\beta<\alpha}A_\beta$ is not closed$\}$ is not stationary. Then $\mathscr{F}[X]$ is paracompact.*

Proof. Let C be a closed unbounded subset of ω_1 disjoint from S and put $B_\beta=\bigcup_{\alpha<\beta}A_\alpha$, for $\beta\in C$. Then sets B_β are countable, $X=\bigcup_{\beta\in C}B_\beta$ and $\bigcup_{\beta<\gamma}B_\beta$ is closed for every $\gamma\in C$. By 3.3 $\mathscr{F}[X]$ is paracompact. ■

Our next corollary concerns products of PR-hyperspaces.

Corollary 3.8. *If $\mathscr{F}[X_i]$ is paracompact for $i=1,2,\ldots,n$, then $\mathop{\mathrm{P}}_{i=1}^{n}\mathscr{F}[X_i]$ is paracompact.*

Proof. By Proposition 1.5 the space $\mathop{\mathrm{P}}_{i=1}^{n}\mathscr{F}[X_i]$ is a closed subspace of $\mathscr{F}[Z]$, where $Z=\bigoplus_{i=1}^{n}X_i$. By 3.5, $\mathscr{F}[Z]$ is paracompact. ■

Remark. No one of the above results 3.1–3.8 is valid for hereditary paracompactness. There exist spaces X and Y such that $\mathscr{F}[X]$ and $\mathscr{F}[Y]$ are hereditarily paracompact, but $\mathscr{F}[X\oplus Y]$ and $\mathscr{F}[X]\times\mathscr{F}[Y]$ are not (see Example 6.6). ■

With regard to the (inverse) invariance of paracompactness under continuous mappings we have the following corollaries to Theorems 2.1 and 2.2.

Corollary 3.9. *Let $f\colon X\to Y$ be a continuous bijection. If $\mathscr{F}[Y]$ is (hereditarily) paracompact, then $\mathscr{F}[X]$ is (hereditarily) paracompact.*

Proof. Use (iv) of Theorems 2.1 and 2.2, respectively. ■

Corollary 3.10. *Let $f\colon X\to Y$ be a closed finite-to-one mapping of X onto Y. If $\mathscr{F}[X]$ is (hereditarily) paracompact, then $\mathscr{F}[Y]$ is (hereditarily) paracompact.*

([1]) $X^{(\alpha)}$ denotes the αth derivative of X; sets $X^{(\alpha)}\setminus X^{(\alpha+1)}$ are discrete.

Proof. Suppose that X is paracompact. By Theorem 2.1 for every finite subset K of X one can choose an open neighbourhood V_K so that (iv) is satisfied. Let us put

$$V_F=Y\setminus f(X\setminus V_{f^{-1}(F)})$$

for every non-empty finite subset F of Y. Clearly the sets V_F are open neighbourhoods of F in Y. If $F\subset V_H$ and $H\subset V_F$, then $f^{-1}(F)\subset V_{f^{-1}(H)}$ and $f^{-1}(H)\subset V_{f^{-1}(F)}$, hence $f^{-1}(F)\cap f^{-1}(H)\neq\emptyset$. Therefore $F\cap H\neq\emptyset$.

The proof for hereditary paracompactness is analogous. ■

Problem 3. Let $f\colon X\to Y$ be a perfect mapping of X onto Y. Is $\mathscr{F}[Y]$ (hereditarily) paracompact if $\mathscr{F}[X]$ is such?

This problem has an affirmative answer if X is either locally Čech-complete (see Theorems 4.2 and 4.6) or metrizable (see Theorem 5.2 and also Problem 9).

As it follows from the next result, hereditary paracompactness of $\mathscr{F}[X]$ in many cases implies that $\mathscr{F}[X]$ is perfectly paracompact.

Corollary 3.11. *If X contains a countable non-discrete subset and if $\mathscr{F}[X]$ is hereditarily paracompact, then $\mathscr{F}[X]$ is perfectly paracompact.*

Proof. By 2.2 $\mathscr{F}[X]^2$ is hereditarily paracompact. It is easy to see that $\mathscr{F}[X]$ also contains a countable non-discrete subset. Now, it suffices to apply Katětov's theorem (see [E]; Problem 2.7.15). ■

Also the relation between normality and paracompactness of PR-hyperspaces is delicate. Assuming MA+¬CH there exists a separable metric space M such that $\mathscr{F}[M]$ is normal but not paracompact [PT]. On the other hand the following corollary is an immediate consequence of Theorems 2.1, 2.2 and a theorem of Fleissner [F].

Corollary 3.12. (V = L). *If $\mathscr{F}[X]$ is (hereditarily) normal and its character is $\leqslant c$, then $\mathscr{F}[X]$ is (hereditarily) paracompact.* ■

Problem 4. Give a "real" example of a normal non-paracompact Pixley–Roy hyperspace $\mathscr{F}[X]$.

It follows from Theorem 4.2 and Corollary 3.12 that X can be neither locally Čech-complete nor first countable.

§ 4. Pixley–Roy hyperspaces of compact spaces.

In this section we investigate (hereditary) normality and (hereditary) paracompactness of PR-hyperspaces of locally Čech-complete (in particular, compact) spaces.

Lemma 4.1. *If X has a closed irreducible mapping onto the Cantor set C, then $\mathscr{F}[X]$ is not normal.*

Proof. Let $f\colon X\to C$ be such a mapping and suppose that $\mathscr{F}[X]$ is normal. Let $C=A\cup B$, where $A\cap B=\emptyset$ and both A and B are of the second category in every non-empty open subset of C (see e.g. [E]; Problem 5.5.4). Put $K=f^{-1}(A)$ and $L=f^{-1}(B)$. Since K and L are disjoint subsets of X, there exist disjoint open subsets U and V of $\mathscr{F}[X]$ such that $\{\{x\}\colon x\in K\}\subset U$ and $\{\{x\}\colon x\in L\}\subset V$. For every $x \in X$ choose a neighbourhood V_x such that $[\{x\}, V_x]$ is either contained in U or in V.

Let us notice that sets V_x have the following property:

(1) if $x \in K$ and $z \in L$ then either $x \notin V_z$ or $z \notin V_x$.

Indeed, if $x \in K$, $z \in L$, $x \in V_z$ and $z \in V_x$, then $\{x, z\} \in [\{x\}, V_x] \cap [\{z\}, V_z] \subset U \cap V = \emptyset$, which is impossible.

For every $y \in C$ find a neighbourhood U_y of y such that

(2) $$f^{-1}(U_y) \subset \bigcup \{V_x \colon x \in f^{-1}(y)\}.$$

For every open subset W of X let $\tilde{W} = \bigcup \{f^{-1}(y) \colon f^{-1}(y) \subset W\}$. Since f is closed $f(\tilde{W})$ is open in C and $\tilde{W} = f^{-1}f(\tilde{W})$. Since f is irreducible, for every non-empty W, we have $\tilde{W} \neq \emptyset$.

Put $W_y = U_y \cap \bigcup \{f(\tilde{V}_x) \colon x \in f^{-1}(y)\}$. The set W_y is open and contained in U_y. Let us observe that

(3) W_y is dense in U_y.

Indeed, let G be non-empty and open in U_y. Then, by (2),
$$f^{-1}(G) \subset \bigcup \{V_x \colon x \in f^{-1}(y)\}.$$
There exists an x such that $W = f^{-1}(G) \cap V_x \neq \emptyset$. We have
$$\emptyset \neq f(\tilde{W}) \subset G \cap f(\tilde{V}_x) \subset G \cap W_y.$$

Let us put $A_n = \{a \in A \colon B(a, 1/n) \subset U_a\}$, where $B(a, 1/n)$ denotes a ball in C of radius $1/n$ and center at a. Since A is of the second category, there exists a k such that A_k is not nowhere dense. Find an open subset G of C such that
$$\emptyset \neq G \subset \bar{A}_k$$
and let Q be a countable dense subset of A_k. Hence $Q \subset A_k \subset \bar{Q} = \bar{A}_k$. By (3) the set
$$T = \bigcup \{U_a \setminus W_a \colon a \in Q\}$$
is of the first category. Therefore there exists a $b \in (B \setminus T) \cap G$.

We have $b \in B(b, 1/k) \cap G \cap U_b \neq \emptyset$, hence by (3), $B(b, 1/k) \cap G \cap W_b \neq \emptyset$. Since Q is dense in G, there exists an $a \in Q \cap B(b, 1/k) \cap G \cap W_b$. We have $a \in A_k$, hence $b \in U_a$, but $b \notin T \supset U_a \setminus W_a$, hence $b \in W_a$.

Finally, we obtain $a \in W_b$ and $b \in W_a$. Therefore there exist $x \in f^{-1}(a)$ and $z \in f^{-1}(b)$ such that $a \in f(\tilde{V}_z)$ and $b \in f(\tilde{V}_x)$. We have $z \in f^{-1}(b) \subset V_x$ and $x \in f^{-1}(a) \subset V_z$, $x \in K$ and $z \in L$, which is impossible by (1). ■

THEOREM 4.2. *For a locally Čech-complete space X the following conditions are equivalent:*

(i) *$\mathcal{F}[X]$ is normal,*

(ii) *$\mathcal{F}[X]$ is paracompact,*

(iii) *X is scattered.*

Proof. Implication (iii) → (ii) follows from Corollary 3.6 and implication (ii) → (i) is obvious.

(i) → (iii) (cf. [A_2]). Suppose that X is not scattered. Then X contains a dense-in-itself, closed and Čech-complete subspace, which implies that there exists a perfect mapping $f \colon Y \to C$ of some (closed) subspace Y of X onto the Cantor set. We can assume that f is irreducible (cf. [E]; Exercise 3.1.C). By Proposition 1.2 $\mathcal{F}[Y]$ is normal, which contradicts Lemma 4.1. ■

LEMMA 4.3. *Let X be an uncountable compact space with exactly one non-isolated point x_0. Then $\mathcal{F}[X]$ is not hereditarily normal.*

Proof. Let A be an arbitrary countably infinite subset of X not containing x_0. We claim that disjoint closed subsets $K = \{\{x_0, a\} \colon a \in A\}$ and $L = \{\{x_0, b\} \colon b \in B = X \setminus (A \cup \{x_0\})\}$ of $\mathcal{F}[X] \setminus \mathcal{F}_1[X]$ cannot be separated by disjoint open subsets U and V of $\mathcal{F}[X]$. Suppose the contrary. For every $x \in A \cup B$ find a neighbourhood $V(x)$ of x_0 such that the set $[\{x_0, x\}, V(x) \cup \{x\}]$ is either contained in U or in V. Clearly if $a \in A$ and $b \in B$ then either $a \notin V(b)$ or $b \notin V(a)$, since otherwise we would have
$$U \cap V \supset [\{x_0, a\}, V(a) \cup \{a\}] \cap [\{x_0, b\}, V(b) \cup \{b\}] \supset \{\{x_a, a, b\}\} \neq \emptyset.$$

Clearly the complement of every $V(x)$ is finite, thus there exists a $b \in B \cap \bigcap \{V(a) \colon a \in A\}$. Since $|X \setminus V(b)| < \omega$, there exists an $a \in A$ such that $a \in V(b)$. We have $b \in V(a)$ and $a \in V(b)$, which is a contradiction. ■

Remark. There exists an uncountable Lindelöf space X with exactly one non-isolated point such that $\mathcal{F}[X]$ is hereditarily paracompact (see Example 6.5). ■

THEOREM 4.4 *For a compact space X the following conditions are equivalent:*

(i) *$\mathcal{F}[X]$ is hereditarily normal,*

(ii) *$\mathcal{F}[X]$ is hereditarily paracompact,*

(iii) *$\mathcal{F}[X]$ is metrizable,*

(iv) *X is countable.*

Proof. The implications (iii) → (ii) and (ii) → (i) are obvious. The implication (iv) → (iii) follows from Proposition 1.1, because $\mathcal{F}[X]$ is then countable (hence Lindelöf) and X is first countable.

(i) → (iv). Since X is compact, it suffices to show that X is locally countable. Let α be the first ordinal such that there exists an $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$ with no countable neighbourhood (Theorem 4.2 implies that X is scattered). Find a neighbourhood V of x such that $\bar{V} \cap X^{(\alpha)} = \{x\}$. For every neighbourhood W of x in $\bar{V}$ the set $\bar{V} \setminus W$ is compact and contained in $X \setminus X^{(\alpha)}$. From our assumption on α we infer that $\bar{V} \setminus W$ is countable for any such W. If $\bar{V} \setminus W$ is finite for all neighbourhoods W of x, then, applying Lemma 4.3, we infer that $\mathcal{F}[\bar{V}]$ is not hereditarily normal, which is a contradiction.

If there exists a W such that $\bar{V} \setminus W$ is infinite, then $\bar{V} \setminus W$ is compact, metrizable and infinite, hence it contains a non-trivial convergent sequence. Using the fact that a complement of every neighbourhood of x in $\overline{V}$ is countable, one shows, as in the proof of the implication (i) → (iii) in Theorem 6.2, that $\mathscr{F}[\overline{V}]$ is not hereditarily normal, which again is a contradiction. ■

Remark. In fact, Theorem 4.4 is valid for Lindelöf Čech-complete spaces. ■

PROPOSITION 4.5. *Let X be a space of point-countable type (e.g. a p-space). If $\mathscr{F}[X]$ is hereditarily normal, then X is first countable and hence $\mathscr{F}[X]$ is perfectly normal.*

Proof. By the definition of a space of point-countable type for every point $x \in X$ there exists a compact set $K \ni x$ of countable character in X. Since $\mathscr{F}[K]$ is hereditarily normal, K is countable by Theorem 4.4. Thus the point x has a countable character in X (cf. [E]; Exercise 3.1.E). Hence $\mathscr{F}[X]$ is perfectly normal by Proposition 1.3. ■

THEOREM 4.6. *For a locally Čech-complete space X the following conditions are equivalent:*

(i) *$\mathscr{F}[X]$ is hereditarily normal,*

(ii) *$\mathscr{F}[X]$ is hereditarily paracompact,*

(iii) *$\mathscr{F}[X]$ is metrizable,*

(iv) *X is first countable and scattered.*

Proof. The implications (iii) → (ii) and (ii) → (i) are obvious. The implication (i) → (iv) follows from Proposition 4.5 and Theorem 4.2. The implication (iv) → (iii) follows from Proposition 1.1 and Theorem 4.2. ■

Remarks. 1. There exist first countable, paracompact, scattered and Čech-complete spaces which are not locally countable, e.g. the real line with all non-zero points isolated.

2. Since normality and paracompactness in PR-hyperspaces of metric spaces do not, in general, coincide [PT], Theorems 4.2 and 4.6 are not valid for (locally) p-spaces. ■

§ 5. Pixley–Roy hyperspaces of metrizable spaces. In this section we investigate the normality (= perfect normality) and metrizability (= paracompactness) of PR-hyperspaces of metrizable spaces. Throughout this section *M always denotes a metrizable space.*

PROPOSITION 5.1. *If $\mathscr{F}_2[M]$ is collectionwise Hausdorff, then M is σ-discrete.*

Proof. For every $x \in M$ find an $n(x) < \omega$ such that the family

$$\{[\{x\}, B(x, 1/n(x)] \cap \mathscr{F}_2[M]\}_{x \in M}$$

is disjoint. One easily checks that sets $A_n = \{x \in M\colon n(x) = n\}$ are closed and discrete. ■

Metrizability of the hyperspaces $\mathscr{F}[M]$ is fully characterized by the following theorem, which is a consequence of Proposition 5.1 and Corollary 3.4.

THEOREM 5.2. *$\mathscr{F}[M]$ is metrizable if and only if M is* σ-discrete. ■

Theorem 5.2 as well as Proposition 5.1 are valid for semimetric spaces. As for the normality of hyperspaces $\mathscr{F}[M]$ the following result is known (the "if" part proved in [PT] and the "only if" part in [R]).

THEOREM 5.3. *If M is separable, then $\mathscr{F}[M]$ is normal if and only if M is a strong Q-set.* ■

Recall that M is a *Q-set* if M is separable and all subsets of M are G_δ's and that M is a *strong Q-set* if all finite powers of M are Q-sets.

THEOREM 5.4. *The following conditions are equivalent:*

(i) *there exists and uncountable Q-set;*

(ii) *there exists an uncountable strong Q-set;*

(iii) *there exists a separable M such that $\mathscr{F}[M]$ is normal but not metrizable.*

Proof. By [P_1] the existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set. Thus it suffices to apply Theorems 5.2 and 5.3. ■

In an attempt to generalize Theorems 5.3 and 5.4 for the non-separable case it is natural to introduce non-separable Q- and strong Q-sets. However, as we shall see, the generalization will not be complete. We say that a metric space M is a *q-set* if all subsets of M are G_δ's and that M is a *strong q-set* if all finite powers of M are q-sets. Thus, Q-sets are separable q-sets.

PROPOSITION 5.5. *$\mathscr{F}_2[M]$ is normal if and only if M is a q-set.* ■

COROLLARY 5.6. [Re_1] (V = L). *Every q-set is σ-discrete.*

Proof. Let M be a q-set. By 5.5 $\mathscr{F}_2[M]$ is normal and hence, by [F], $\mathscr{F}_2[M]$ is collectionwise Hausdorff. Thus 5.1 implies that M is σ-discrete. ■

LEMMA 5.7. *If M is a strong q-set and $\dim M = 0$, then $\mathscr{F}[M]$ is normal.*

Proof. By a result of Herrlich [H], the space M is linearly ordered. Now the proof on page 294 of [PT] applies with obvious modifications. ■

LEMMA 5.8. *If $\mathscr{F}[M]$ is normal and if $f\colon X \to M$ is a one-to-one continuous mapping of a metric space X with $\dim X = 0$ onto M, then $\mathscr{F}[X]$ is normal and X is a strong q-set.*

Lemma 5.8 will be proved at the end of this section.

THEOREM 5.9. *If $\dim M = 0$, then $\mathscr{F}[M]$ is normal if and only if M is a strong q-set.*

Proof. This is an immediate consequence of Lemmas 5.7 and 5.8. ■

PROBLEM 5. Can the assumption that $\dim M = 0$ in Theorem 5.9 be omitted?

PROBLEM 6. Is every q-set strongly zerodimensional?

Remark. From Proposition 5.5 it follows that a positive answer to Problem 6 implies a positive answer to Problem 5. Since every Q-set is obviously strongly zerodimensional, Theorem 5.9 in fact generalizes Theorem 5.3. ■

LEMMA 5.10. *For every non σ-discrete metric space M there exists a non-σ-discrete metric space X with* $\dim X = 0$ *and a one-to-one continuous mapping* $f\colon X \to M$ *of X onto M. Moreover, if M is a (strong) q-set, then also X is such.*

Proof. By ([E], Exercise 4.4.J) there exists a strongly zerodimensional metric space Z and a perfect mapping $g\colon Z \to M$ of Z onto M. Let X be a subspace of Z such that $f = g|X\colon X \to M$ is one-to-one and onto. Suppose that $X = \bigcup_{n<\omega} A_n$ is a union of countably many discrete subspaces A_n. For every n, A_n is an open subspace of $\mathrm{Cl}_Z A_n$, hence $A_n = \bigcup_{m<\omega} F_{n,m}$ is a union of countably many closed discrete subspaces of Z. Thus $M = \bigcup_{n,m<\omega} g(F_{n,m})$ is σ-discrete, which is impossible. The last assertion is obvious. ■

The following theorem (partially) generalizes Theorem 5.4.

THEOREM 5.11. *The following two conditions are equivalent*:

(i) *there exists a non-σ-discrete strong q-set*;

(ii) *there exists an M such that* $\mathcal{F}[M]$ *is normal but not metrizable.*

Proof. If (i) holds, then by 5.10 there exists a strongly zerodimensional non-σ-discrete strong q-set M. By Theorems 5.9 and 5.2, $\mathcal{F}[M]$ is normal but not metrizable.

If (ii) holds, then by 5.2 M is not σ-discrete and by 5.10 there exists a strongly zerodimensional non-σ-discrete metric space X and a continuous one-to-one mapping $f\colon X \to M$ of X onto M. By 5.8 X is a strong q-set. ■

PROBLEM 7. Is the existence of a non-σ-discrete q-set equivalent to the existence of a non-σ-discrete strong q-set?

PROBLEM 8 [1]. Is the existence of a non-σ-discrete q-set equivalent to the existence of an uncountable Q-set?

A comparison of Theorems 4.2 and 5.2 suggests the following problem (see also Problem 3).

PROBLEM 9. Let X be a paracompact p-space. Is $\mathcal{F}[X]$ paracompact if and only if X is σ-scattered?

Proof of Lemma 5.8. By Lemma 5.7 it is enough to show that X is a strong q-set. Take $n<\omega$ and assume that we have already proved that X^n is a q-set. Let $\leqslant$ be a linear order on X generating the topology of X [H]. Let

$$Z = \{(x_1, \ldots, x_{n+1}) \in X^{n+1}\colon x_1 < \ldots < x_{n+1}\}.$$

One easily sees that X^{n+1} is a finite union of its G_δ subspaces which are either homeomorphic to Z or to some X^k, for $k \leqslant n$. Thus in order to show that X^{n+1} is a q-set it suffices to show that Z is a q-set. Since f is one-to-one we can identify the points of X and M. Let $A \subset Z$, $B = Z \setminus A$ and let $g\colon Z \to \mathcal{F}[M]$ be defined by

$$g(x_1, \ldots, x_{n+1}) = \{x_1, \ldots, x_{n+1}\}.$$

Clearly the sets $g(A)$ and $g(B)$ are disjoint closed subsets of $\mathcal{F}[M] \setminus \mathcal{F}_n[M]$, hence there exist open and disjoint subsets U and V of $\mathcal{F}[M]$ such that $U \supset g(A)$ and $V \supset g(B)$. Define $A_m = \{z \in A\colon [g(z), B(g(z), 1/m)] \subset U\}$, where $B(F, 1/m) = \bigcup_{y\in F} B(y, 1/m)$ and $B(y, 1/m)$ denotes a ball in M of radius $1/m$.

Since $A = \bigcup_{m=1}^{\infty} A_m$, it suffices to prove that $\mathrm{Cl}_Z A_m \subset A$. Suppose the contrary and let $z^* \in \mathrm{Cl}_Z A_m \cap B$. There exists a $k \geqslant m$ such that $[g(z^*), B(g(z^*), 1/k)] \subset V$ and a $z \in A_m$ such that $z \in \mathrm{P}_{i=1}^{n+1} B(z_i^*, 1/k)$, i.e. $z_i \in B(z_i^*, 1/k)$. Thus $z_i^* \in B(z_i, 1/k) \subset B(z_i, 1/m)$, for all $i \leqslant n+1$. This implies, however, that

$$g(z) \cup g(z^*) \subset B(g(z), 1/m) \cap B(g(z^*), 1/k)$$

and hence

$$\emptyset \neq [g(z), B(g(z), 1/m)] \cap [g(z^*), B(g(z^*), 1/k)] \subset U \cap V,$$

which is a contradiction and completes the proof. ■

(1) Recently G. M. Reed answered this problem in the negative by showing that the existence of non-σ-discrete q-sets is consistent with GCH [Re$_2$].

§ 6. Pixley–Roy hyperspaces of spaces of ordinals. In this section we shall consider PR-hyperspaces of spaces of ordinals. By a *space of ordinals* we mean an arbitrary subspace of some ordinal. From Corollary 3.2 immediately follows

THEOREM 6.1 ([BFL$_2$], [P$_2$]). *If X is a space of ordinals, then* $\mathcal{F}[X]$ *is paracompact.* ■

The following theorem, however, sharply contrasts with Theorems 4.4, 4.6 and 5.2 which may suggest that hereditary normality (hereditary paracompactness) of $\mathcal{F}[X]$ is always equivalent to perfect normality (metrizability) of $\mathcal{F}[X]$.

THEOREM 6.2. *For a space of ordinals X the following conditions are equivalent*:

(i) $\mathcal{F}[X]$ *is hereditarily normal*;

(ii) $\mathcal{F}[X]$ *is hereditarily paracompact*;

(iii) *characters of all non-isolated points of X coincide*;

(iv) $\mathcal{F}[X]$ *is τ-metrizable* [1] *for some τ.*

(1) A regular space Z is *τ-metrizable* (see [Ha], [S]) if it has a τ-locally finite base (i.e. a base being a union of τ locally finite families) and if the intersection of less than τ open subsets of Z is open. A space is ω-metrizable if and only if it is metrizable. All τ-metrizable spaces are hereditarily paracompact.

Theorem 6.2 can be generalized in a natural way: the equivalence of conditions (i)–(iv) takes place in every space X such that $\mathcal{F}[X]$ is paracompact and all points have a well-ordered by inclusion base of neighbourhoods.

COROLLARY 6.3 (cf. also 4.6). *For an ordinal* $\varkappa$, *the hyperspace* $\mathscr{F}[\varkappa]$ *is hereditarily normal if and only if* $\varkappa \leqslant \omega_1$. ∎

EXAMPLE 6.4. Consider $T = \{\alpha\colon \alpha<\omega_2 \text{ and } \operatorname{cf}(\alpha) = \omega_1\}$ as a subspace of ω_2. By 6.2, $\mathscr{F}[T]$ is hereditarily paracompact, ω_1-metrizable and not perfect. Moreover, T contains ω_2 non-isolated points. ∎

EXAMPLE 6.5. Consider $Y = \{\alpha\colon \omega<\alpha\leqslant\omega_1 \text{ and } \alpha \text{ is isolated or } \alpha = \omega_1\}$ as a subspace of ω_1+1. Then, Y is Lindelöf with exactly one non-isolated point, $\mathscr{F}[Y]$ is hereditarily paracompact by 6.2 and $\mathscr{F}[Y]$ is not perfect by 1.3. ∎

EXAMPLE 6.6. The space $Z = X\oplus Y$, where $X = \omega+1$, is a Lindelöf subspace of ω_1+1 with exactly two non-isolated points and Y is as in 6.5, $\mathscr{F}[X]$ is metrizable and countable, $\mathscr{F}[Y]$ is ω_1-metrizable and hereditarily paracompact, but $\mathscr{F}[Z]$ is not hereditarily normal, by 6.2. Thus, by 1.5, $\mathscr{F}[X]\times\mathscr{F}[Y]$ is not hereditarily normal (cf. Corollary 3.8). ∎

Proof of Theorem 6.2. The implications (iv) → (ii) and (ii) → (i) are obvious.

(i) → (iii). Suppose that there exist non-isolated points $x, y\in X$, say $x<y$, such that $\chi(x) = \tau \neq \varkappa = \chi(y)$, where $\chi(x)$ denotes the character of x in X. Thus there exist sequences $\{x_\alpha\}_{\alpha<\tau}$ and $\{y_\gamma\}_{\gamma<\varkappa}$ such that $x_\alpha<x_\beta<x<y_\gamma<y_\delta<y$, for all $\alpha<\beta$ and $\gamma<\delta$, $x = \sup\{x_\alpha\colon \alpha<\tau\}$ and $y = \sup\{y_\gamma\colon \gamma<\varkappa\}$. Of course, both τ and $\varkappa$ must be regular.

Suppose that $\tau<\varkappa$ and let us put $Y = A\cup B$, where $A = \{x\}\cup\{x_\alpha\}_{\alpha<\tau}$ and $B = \{y\}\cup\{y_\gamma\}_{\gamma<\varkappa}$. Then Y is a subspace of X and $Y = A\oplus B$. From (i) and Proposition 1.5 it follows that the space $\mathscr{F}[A]\times\mathscr{F}[B]$ is hereditarily normal but this contradicts Katetov's theorem since $\{y\}$ is not an intersection of τ open subsets of $F[B]$ (cf. [E]; Problem 2.7.15).

(iii) → (iv). Suppose that τ is such that $\chi(x) = \tau$ for every non-isolated point $x\in X$. That the intersection of less than τ open subsets of $\mathscr{F}[X]$ is open follows immediately from the analogous property of X. It suffices to show that $\mathscr{F}[X]$ has a τ-locally finite base. For every non-isolated $x\in X$ fix a sequence $\{x_\alpha\}_{\alpha<\tau}$ of points of X such that $x_\alpha<x_\beta<x$, for all $\alpha<\beta<\tau$ and $x = \sup\{x_\alpha\colon \alpha<\tau\}$. Define

$$U_\alpha(x) = \begin{cases}\{x\}, & \text{if } x \text{ is isolated},\\ (x_\alpha, x], & \text{otherwise},\end{cases}$$

and

$$\mathscr{V}_\alpha = \{V_\alpha(F)\colon F\in\mathscr{F}[X]\}, \quad \text{where} \quad V_\alpha(F) = [F, \bigcup\{U_\alpha(x)\colon x\in F\}].$$

Let $\mathscr{B}_\alpha$ be a locally finite open refinement of $\mathscr{V}_\alpha$ existing by virtue of 6.1. One easily checks that the family $\mathscr{B} = \bigcup_{\alpha<\tau}\mathscr{B}_\alpha$ is a base of $\mathscr{F}[X]$. ∎

§ 7. Iterated Pixley–Roy hyperspaces. In this section we define iterated PR-hyperspaces and examine their properties. Let us define $\mathscr{F}^1[X] = \mathscr{F}[X]$ and $\mathscr{F}^{n+1}[X] = \mathscr{F}[\mathscr{F}^n[X]]$.

THEOREM 7.1. *If* $n\geqslant 2$ *then* $\mathscr{F}^n[X]$ *is paracompact.*

Proof. From $n\geqslant 2$ it follows that $\mathscr{F}^n[X] = \mathscr{F}[\mathscr{F}^{n-1}[X]]$. Since $\mathscr{F}^{n-1}[X]$ is a PR-hyperspace it is a union of countably many discrete subspaces $\{A_m\}_{m=1}^\infty$ such that $K_m = \bigcup_{i\leqslant m} A_i$ is closed for every $m<\omega$. Now, paracompactness of $\mathscr{F}^n[X]$ follows from Corollary 3.3. ∎

COROLLARY 7.2. *If X is first countable, then $\mathscr{F}^n[X]$ is metrizable for every* $n\geqslant 2$. ∎

Let $D(\tau)$ denote a discrete space of cardinality τ and let $p\in D(\tau)$. By $Q(\tau)$ we shall denote the subspace of $D(\tau)^\omega$ consisting of those points of $D(\tau)^\omega$ all coordinates of which, except for finitely many, coincide with p. Clearly, $Q(\tau)$ does not depend on the choice of $p\in D(\tau)$.

The following theorem is a natural generalization of the well-known fact about the space of rational numbers Q and has a standard proof.

THEOREM 7.3. *Every non-empty σ-discrete metric space, all non-empty open subsets of which have weight (or cardinality) τ, is homeomorphic to $Q(\tau)$.*

For the sake of completeness we include a short proof of Theorem 7.3 at the end of this section.

COROLLARY 7.4. *The space Q of the rationals is homeomorphic to $Q(\omega)$.* ∎

COROLLARY 7.5. *The space $Q(\tau)$ is universal in the class of σ-discrete metric spaces of weight (cardinality) τ.*

Proof. Let M be a σ-discrete metric space of weight (cardinality) τ. By 7.3, $M\times Q(\tau)$ is homeomorphic to $Q(\tau)$ and contains M. ∎

COROLLARY 7.6. *Every metrizable Pixley–Roy hyperspace $\mathscr{F}[X]$ is a subspace of $Q(\tau)$, where $\tau = |X|$. Moreover, $\mathscr{F}[X]$ is homeomorphic to $Q(\tau)$ if and only if all non-empty open subsets of X have cardinality τ.*

Proof. Every first countable Pixley–Roy hyperspace is σ-discrete and clearly $w(\mathscr{F}[X]) = |X|$. Non-empty open subsets of X have cardinality τ if and only if non-empty open subsets of $\mathscr{F}[X]$ have cardinality τ. ∎

COROLLARY 7.7. *If M is metrizable, then $\mathscr{F}[M]$ is homeomorphic to $Q(\tau)$ if and only if M is homeomorphic to $Q(\tau)$.*

Proof. This is an immediate consequence of 7.3, 7.6 and 5.2. ∎

From Corollaries 3.2, 7.6 and 7.7 it follows that $\mathscr{F}^n[S]\cong Q(c)$, for all $n\geqslant 1$, where S is the Sorgenfrey line.

COROLLARY 7.8. *If X is first countable, then $\mathscr{F}^2[X]$ is a subspace of $Q(\tau)$, where $\tau = |X|$. Moreover, $\mathscr{F}^2[X]$ is homeomorphic to $Q(\tau)$ if and only if all non-empty open subsets of X have cardinality τ.* ∎

From Corollary 7.8 is follows that $\mathscr{F}^n[R]\cong Q(c)$, for every $n\geqslant 2$, where R denotes the real line. The space $\mathscr{F}[R]$, although not metrizable, also contains $Q(c)$ homeomorphically [Pl].

PROBLEM 10. [vD]. Is the square of $\mathscr{F}[R]$ homeomorphic to $\mathscr{F}[R]$?

Proof of Theorem 7.3. Let M be a non-empty metric space all non-empty open subsets of which have weight (or cardinality) τ and let $M = \bigcup_{n<\omega} D_n$, where the sets D_n are closed and discrete and $D_0 = \emptyset$. Assume also that M is well-ordered.

Put $\mathscr{U}_0 = \{X\}$, fix $n<\omega$ and suppose that an open partition $\mathscr{U}_n$ of X has been defined in such a way that no two distinct elements of $\bigcup_{i\leqslant n} D_i$ belong to the same element of $\mathscr{U}_n$. We shall define $\mathscr{U}_{n+1}$.

Fix $U\in\mathscr{U}_n$ and let $m = \min\{k\colon U\cap D_k \neq \emptyset\}$ and let x_0 be the first element of D_m belonging to U. Let $\{U_\alpha\}_{\alpha<\tau}$ be a partition of U consisting of non-empty open sets of diameter less than $1/(n+1)$ and such that no two distinct points of $\bigcup_{i\leqslant n+1} D_i$ belong to the same element U_α. Additionally we require that $x_0 \in U_0$. One easily sees that such a partition exists. Put $\mathscr{U}_{n+1} = \{U_\alpha\colon \alpha<\tau,\ U\in\mathscr{U}_n\}$.

For every $n\geqslant 1$ let $f_n\colon M\to D(\tau)$ be a continuous function defined by

$$f_n(x) = \alpha \quad \text{iff} \quad x\in U_\alpha, \text{ for some } U\in\mathscr{U}_{n-1}$$

where $D(\tau)$ is identified with the set $\tau = \{\alpha\colon \alpha<\tau\}$.

It is routine to check that the diagonal

$$f = \mathop{\triangle}_{n=1}^{\infty} f_n\colon M\to D(\tau)^\omega$$

of mappings f_n is a homeomorphic embedding of M onto $Q(\tau)$. ■

Added in proof. A different characterization of normal Pixley–Roy hyperspaces and other interesting results involving PR-hyperspaces were obtained independently by M. G. Bell ("Hyperspaces of finite subsets", to appear). Second part of Problem 2 has been solved by H. Tanaka ("Paracompactness of Pixley–Roy hyperspaces I and II", to appear) who showed that $\mathscr{F}[X]$ is (hereditarily) paracompact iff $\mathscr{F}[X^m]^n$ is (hereditarily) paracompact for all $m, n<\omega$. H. Tanaka also proved that the assumption, that $\dim M = 0$ in Lemma 5.7 is redundant.

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