NORMALITY OF THE THUE–MORSE SEQUENCE ALONG PIATETSKI-SHAPIRO SEQUENCES, II

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Abstract. We prove that the Thue–Morse sequence $\mathbf{t}$ along subsequences indexed by $\lfloor n^c \rfloor$ is normal, where $1 < c < 3/2$. That is, for c in this range and for each $\omega \in \{0,1\}^L$, where $L \geq 1$, the set of occurrences of ω as a factor (contiguous finite subsequence) of the sequence $n \mapsto \mathbf{t}_{\lfloor n^c \rfloor}$ has asymptotic density 2^{-L}. This is an improvement over a recent result by the second author, which handles the case $1 < c < 4/3$.

In particular, this result shows that for $1 < c < 3/2$ the sequence $n \mapsto \mathbf{t}_{\lfloor n^c \rfloor}$ attains both of its values with asymptotic density $1/2$, which improves on the bound $c < 1.4$ obtained by Mauduit and Rivat (who obtained this bound in the more general setting of *q-multiplicative functions*, however) and on the bound $c \leq 1.42$ obtained by the second author.

In the course of proving the main theorem, we show that $2/3$ is an *admissible level of distribution* for the Thue–Morse sequence, that is, it satisfies a Bombieri–Vinogradov type theorem for each exponent $\eta < 2/3$. This improves on a result by Fouvry and Mauduit, who obtained the exponent 0.5924. Moreover, the underlying theorem implies that every finite word $\omega \in \{0,1\}^L$ is contained as an arithmetic subsequence of $\mathbf{t}$.

1. Introduction

The Thue–Morse sequence $\mathbf{t}$ is a well-known infinite sequence on the two symbols $\mathtt{0}$ and $\mathtt{1}$, which can be defined as follows. Starting with the 1-element sequence $\mathbf{t}^{(0)} = (\mathtt{0})$ and constructing $\mathbf{t}^{(n+1)}$ by concatenating $\mathbf{t}^{(n)}$ and its Boolean complement $\neg\mathbf{t}^{(n)}$, the infinite sequence

$$\mathbf{t} = \mathtt{0110100110010110}\ldots$$

is the pointwise limit of these finite sequences. In other words, it is the fixed point, starting with $\mathtt{0}$, of the substitution $\mathtt{0} \mapsto \mathtt{01}$, $\mathtt{1} \mapsto \mathtt{10}$. The sequence $\mathbf{t}$ can therefore be seen as a 2-automatic sequence (see the book [2] by Allouche and Shallit), indeed it is one of the simplest such sequences. Another equivalent definition uses the *binary sum-of-digits function* s, which counts the number of 1s in the binary expansion of a nonnegative integer: we have $\mathbf{t}_n = \mathtt{0}$ if and only if $2 \mid s(n)$. Since $\mathbf{t}$ is an automatic sequence, its factor complexity is bounded above by a linear function, that is, the number $P(k)$ of different factors of length k of $\mathbf{t}$ is bounded by Ck for some constant C. (For the Thue–Morse sequence, we have $\limsup P(k)/k = 10/3$ [4].) More about the Thue–Morse sequence can be found in the article [1] by Allouche and Shallit, which gathers occurrences of the Thue–Morse sequence in different fields of mathematics and offers a good bibliography, and in the article [13] by Mauduit.

Although the sequence $\mathbf{t}$ itself has low complexity, the situation changes completely if we consider the subsequence indexed by the squares $0, 1, 4, 9, 16, \ldots$. Moshe [19] proved that this subsequence has full factor complexity, that is, every block $\{\varepsilon_0, \ldots, \varepsilon_{L-1}\} \in \{\mathtt{0},\mathtt{1}\}^L$, for $L \geq$

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1, occurs as a factor of the sequence $n \mapsto \mathbf{t}_{n^2}$. Drmota, Mauduit and Rivat [7] proved the stronger statement that each of these blocks in fact occurs with density 2^{-L}, in other words, this subsequence is a *normal sequence*. This latter result was the motivation for our research.

The second author [23] recently proved (using an estimate for discrete Fourier coefficients from [7]) an analogous result for subsequences indexed by so-called *Piatetski-Shapiro sequences*, which are sequences of the form $\lfloor n^c \rfloor$.

Theorem 0 (Spiegelhofer 2015). *Assume that $1 < c < 4/3$. Then the sequence $\left(\mathbf{t}_{\lfloor n^c \rfloor}\right)_{n \geq 0}$ is normal.*

The study of Piatetski-Shapiro sequences, "Polynomials of degree c", can be motivated by problems for polynomials of degree 2. For example, while it is unknown whether there are infinitely many primes of the form $n^2 + 1$, the Piatetski-Shapiro Prime Number Theorem [20] states that for $1 < c < 12/11$ the number of primes of the form $\lfloor n^c \rfloor$ behaves asymptotically as one would expect by heuristic arguments. The exponent c has been improved several times, the currently best known bound $c < 2817/2426$ being due to Rivat and Sargos [21].

In a similar way the problem of studying the sum of digits of $\lfloor n^c \rfloor$ can be motivated. It was asked by Gelfond [11] to investigate the distribution in residue classes of the sum of digits of values of polynomials f such that $f(\mathbb{N}) \subseteq \mathbb{N}$. This problem could not be solved even for polynomials of degree two, and so Mauduit and Rivat [14, 15] first proved a nontrivial result on sequences $n \mapsto \varphi\bigl(\lfloor n^c \rfloor\bigr)$, where $1 \leq c < 1.4$ and φ is a *q-multiplicative function* with values on the unit circle S^1. We do not give a definition of this term, we only note that the Thue–Morse sequence in the form $n \mapsto (-1)^{s(n)}$ is a 2-multiplicative function. (The cited result [15] was also transferred to automatic sequences by Deshouillers, Drmota and Morgenbesser [6]. They also proved in that article that for $1 < c < 10/9$ blocks of length 2 in Piatetski-Shapiro subsequences of $\mathbf{t}$ occur with frequency $1/4$.) Dartyge and Tenenbaum [5] made some progress on the original question of Gelfond, and finally Mauduit and Rivat [16] could tackle the sum of digits of squares. (This result was later generalized to compact groups by Drmota and Morgenbesser [8].) However, there remained a gap to be closed—nothing was known on Piatetski-Shapiro subsequences of q-multiplicative functions for c in the range $[1.4, 2)$. In 2014, the second author [22] extended the bound for c in the case of the Thue–Morse sequence, obtaining the result that for $c \leq 1.42$ the sequence $n \mapsto \mathbf{t}_{\lfloor n^c \rfloor}$ attains both of its values with asymptotic density $1/2$ (that is, this sequence is *simply normal*).

In the present paper, we improve on the known results on normality and simple normality of Piatetski-Shapiro subsequences of $\mathbf{t}$. Our main theorem is the following.

Theorem 1. *Let $1 < c < 3/2$. Then the sequence $\mathbf{u} = \left(\mathbf{t}_{\lfloor n^c \rfloor}\right)_{n \geq 0}$ is normal. More precisely, for any $L \geq 1$ there exists an exponent $\eta > 0$ and a constant C such that*

$$\Bigl|\bigl|\bigl\{n < N : \bigl(\mathbf{u}(n), \mathbf{u}(n+1), \ldots, \mathbf{u}(n+L-1)\bigr) = \omega\bigr\}\bigr| - N/2^L\Bigr| \leq C N^{1-\eta}$$

for all $\omega = \bigl(\omega_0, \ldots, \omega_{L-1}\bigr) \in \{\mathtt{0}, \mathtt{1}\}^L$.

The essential innovation provided by this theorem lies in the new bound $3/2 = 1.5$, which replaces the bound $4/3 = 1.\dot{3}$ for (proper) normality as in Theorem 0. For comparison, we note that $4/3$ is the bound that Mauduit and Rivat obtained in the first paper [14] on Piatetski-Shapiro subsequences of q-multiplicative functions, while 1.42 [22] is the most recent improvement on the exponent c, concerning simple normality of Piatetski-Shapiro subsequences of $\mathbf{t}$. The new bound $3/2$ established in our theorem therefore is a significant improvement—not only does it surpass the bound 1.42 for simple normality (which is the case that $L = 1$), it also pushes the bound for proper normality beyond this value. Another improvement on Theorem 0 is the error term $CN^{1-\eta}$, where both the exponent η and the constant can be made completely explicit.

In the proof of Theorem 1, we reduce the problem of handling Piatetski-Shapiro sequences to the study of *Beatty sequences* $\lfloor n\alpha+\beta\rfloor$, where $\alpha,\beta\in\mathbb{R}$ and n is contained in a small interval in $\mathbb{N}$ of length N, a process that is basically Taylor approximation of degree 1 [22]. This yields sums of the form $\sum_{n\in I}\mathbf{t}(\lfloor n\alpha+\beta\rfloor)$. In order to deal with these sums, we use methods by Mauduit and Rivat [16, 17], introducing the two-fold restricted sum-of-digits function. Afterwards, we eliminate the Beatty sequences $\lfloor n\alpha+\beta\rfloor$ from our expressions by exploiting the usually very small discrepancy (modulo 1) of $n\alpha$-sequences. The resulting expression can be estimated nontrivially with the help of a new estimate (similar to [7, Proposition 1]) for discrete Fourier coefficients related to the sum-of-digits function, which finishes the proof of Theorem 1.

Let us give a few more details on the methods used in the proof. By Taylor approximation, we may approximate $\lfloor n^c\rfloor$ on short intervals I by Beatty sequences in such a way that $\lfloor n^c\rfloor = \lfloor n\alpha+\beta\rfloor$ for most $n\in I$. This method is summarized in Proposition 2.8 and leads to a statement reminiscent of the Bombieri–Vinogradov theorem in prime number theory: in order to prove our theorem, we have to study the distribution of the Thue–Morse sequence on Beatty sequences $\lfloor n\alpha+\beta\rfloor$, where we take an average in α over dyadic intervals $[D,2D]$ (see Theorem 2.4). Of course, greater values of D correspond to greater exponents c in the original problem. Therefore we want to obtain a nontrivial distribution result for given length N of the Beatty sequence, and D as large as possible. This Beatty sequence approach has been followed by the second author [22]. In that article, trigonometric approximation of indicator functions was used in order to dispose of the Beatty sequences, which led to the integral (2.6) in Section 2. An estimate for this integral, taken from [10], yielded the new bound 1.42, which surprisingly beat the formerly best bound 1.4 [15] (concerning simple normality of subsequences of $\mathbf{t}$). However, we also have a lower bound on this integral, which sets a limit for this method. In particular, $3/2$ can not be reached in this way.

In the second part of the proof of Theorem 1 we therefore use a method different from trigonometric approximation in order to handle Beatty subsequences of $\mathbf{t}$. This method is based on the fact that Beatty sequences $\lfloor n\alpha+\beta\rfloor$, for most α, are uniformly distributed in residue classes, their discrepancy being very small. To put it simply, we will have to deal with sums

$$\sum_{n\in I} f(\lfloor n\alpha+\beta\rfloor), \tag{1.1}$$

where f is 2^γ-periodic and $I\subseteq\mathbb{Z}$ is an interval slightly longer than 2^γ, for instance $|I|=2^{\gamma(1+\varepsilon)}$. By a result on the mean discrepancy of $n\alpha$-sequences (Lemma 3.4), we may replace this sum, on average, by

$$\frac{|I|}{2^\gamma}\sum_{n<2^\gamma} f(n) + O\bigl(2^\gamma(\log|I|)^2\bigr).$$

In order to apply this argument, we have to obtain sums of the form (1.1). To this end, we adapt methods by Mauduit and Rivat [16, 17], thereby introducing the two-fold restricted sum-of-digits function $s_{\mu,\lambda}$ (which we define below). The transition from the sum-of-digits function s to the truncated version s_λ is straightforward and can also be carried out for $\lfloor n^c\rfloor$ (see [23]). It is not so clear, however, how to get rid of the lowest μ digits of $\lfloor n^c\rfloor$, that is, how to proceed from $s_\lambda(\lfloor n^c\rfloor)$ to $s_{\mu,\lambda}(\lfloor n^c\rfloor)$. At this point, Beatty sequences $\lfloor n\alpha+\beta\rfloor$ come into play: using rational approximations to α and a generalization of van der Corput's inequality (Lemma 3.2), it is possible to eliminate the lowest μ digits of $\lfloor n\alpha+\beta\rfloor$, so that we are left with only $\gamma := \lambda-\mu$ binary digits of $\lfloor n\alpha+\beta\rfloor$. This further reduction of the number of digits to be taken into account ultimately allows us to achieve the improvement $3/2$ over the bound $4/3$ obtained in [23]. As the last step of the proof, we use Proposition 2.7 from below. This proposition is a new estimate for discrete Fourier coefficients, related to a result by Drmota, Mauduit and Rivat [7, Proposition 1], and allows us not only to deal with general block lengths $L \geq 1$, but also to derive an explicit error term of the form stated in Theorem 1.

The classical Bombieri–Vinogradov Theorem is a statement on the average distribution of prime numbers in arithmetic progressions $nd + j$, where the average is taken in the modulus d. Let

$$\psi(x; d, j) = \sum_{\substack{1 \leq n \leq x \\ n \equiv j \bmod d}} \Lambda(n),$$

where Λ is the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n = p^k$ for some prime p and some $k \geq 1$ and $\Lambda(n) = 0$ otherwise. Then the Bombieri–Vinogradov Theorem [3, Theorem 4] states that for all positive real numbers $A > 0$ there exist $B > 0$ and a constant C such that

$$\sum_{1 \leq d \leq D} \max_{1 \leq y \leq x} \max_{\substack{j \in \mathbb{Z} \\ (j,d)=1}} \left| \psi(y; d, j) - \frac{y}{\phi(d)} \right| \leq C x (\log x)^{-A},$$

where $D = x^{1/2} (\log x)^{-B}$. While no improvement on the exponent $1/2$ is known, the Elliott–Halberstam conjecture [9] asks whether we can choose $D = x^{1-\varepsilon}$ for any $\varepsilon > 0$. In other words, it is conjectured that 1 is an *admissible level of distribution for the primes*, whereas the largest known admissible level (as of 2015) equals $1/2$. In the article [10], Fouvry and Mauduit prove a Bombieri–Vinogradov type theorem for the sum-of-digits function s in base 2. In particular, for the case of the Thue–Morse sequence they set

$$A^{\pm}(x; d, j) = \left| \left\{ n < x : (-1)^{s(n)} = \pm 1, n \equiv j \bmod d \right\} \right|$$

and obtain

$$\sum_{1 \leq d \leq D} \max_{1 \leq y \leq x} \max_{j \in \mathbb{Z}} \left| A^{\pm}(y; d, j) - \frac{y}{2d} \right| \leq C x (\log 2x)^{-A} \tag{1.2}$$

for all real A and $D = x^{0.5924}$. The exponent 0.5924 can therefore be called *admissible level of distribution for the Thue–Morse sequence*. (We note that Fouvry and Mauduit obtain in fact an error term $x^{1-\eta}$ for some $\eta > 0$, which follows from [10, Théorème 2]. We will use this improved estimate in the proof of Corollary 2.2.) Using sieve theory, they apply this result to the study of the sum of digits of almost prime numbers, that is, integers that are the product of at most two prime factors. Later and by different means, Mauduit and Rivat [17] studied the sum of digits of prime numbers, which was not accessible by the Fouvry–Mauduit method.

As we indicated earlier, the backbone of our main result is a Bombieri–Vinogradov type theorem for $\mathbf{t}$. We establish $2/3$ as an admissible level of distribution for the Thue–Morse sequence, improving on the bound established by Fouvry and Mauduit. A Beatty sequence version of this result, combined with linear approximation of $\lfloor n^c \rfloor$, allows us to obtain the improvement 1.5 over the bound 1.42.

Notation. We use the common abbreviations $\mathrm{e}(x) = \exp(2\pi i x)$, $\{x\} = x - \lfloor x \rfloor$, and $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$, where x is a real number. For a prime number p let $\nu_p(n)$ be the exponent of p in the prime factorization of n. We define the *truncated binary sum-of-digits function*

$$s_\lambda(n) = s(\tilde{n}),$$

where $0 \leq \tilde{n} < 2^\lambda$ and $\tilde{n} \equiv n \bmod 2^\lambda$, which only takes into account the digits of n at positions smaller than λ, and for $\mu \leq \lambda$ the *two-fold restricted binary sum-of-digits function*

$$s_{\mu,\lambda}(n) = s_\lambda(n) - s_\mu(n),$$

which only depends on the digits at the positions $\mu,\ldots,\lambda-1$. The functions s_λ and $s_{\mu,\lambda}$ are periodic with period 2^λ. In estimates we use the convenient abbreviation

$$\log^+ x = \max\{1,\log x\}.$$

Summation variables are always assumed to be nonnegative. In particular, we often omit conditions such as $0\le n$ under summation signs. In this article, the symbol $\mathbb{N}$ denotes the set of nonnegative integers. Moreover, constants implied by the symbols $\ll$ and O depend at most on L, that is, on the length of a factor $\omega=(\omega_0,\ldots,\omega_{L-1})$ of the sequences considered.

2. Results and overall structure of the proofs

In the introduction we have already stated our main theorem (Theorem 1), concerning the normality of Piatetski-Shapiro subsequences of $\mathbf{t}$ for exponents $c<3/2$. In the current section we state auxiliary results used for proving this theorem: approximation of $\lfloor n^c\rfloor$ by Beatty sequences (Proposition 2.8), a Beatty–Bombieri–Vinogradov theorem for $\mathbf{t}$ (Theorem 2.4 and its precursor, Proposition 2.6), and an estimate for discrete Fourier coefficients (Proposition 2.7). Moreover, we state results analogous to Theorem 2.4 and Proposition 2.6, concerning arithmetic progressions (Theorem 2.1 and Proposition 2.5), which follow from the same method of proof.

Let α,β,y and z be nonnegative real numbers such that $\alpha\ge 1$, and $\omega=(\omega_0,\ldots,\omega_{L-1})\in\{0,1\}^L$, where $L\ge 1$ is an integer. We define

$$A_\omega(y,z;\alpha,\beta)=\bigl|\bigl\{y\le m<z : \exists n\in\mathbb{Z} \text{ such that } m=\lfloor n\alpha+\beta\rfloor \text{ and } s(\lfloor (n+\ell)\alpha+\beta\rfloor)\equiv\omega_\ell \bmod 2 \text{ for } 0\le\ell<L\bigr\}\bigr|.$$

Note that for $L=1$, $\alpha,\beta\in\mathbb{Z}$ and $y=0$ this yields the sets $A^{\pm}(x;d,j)$ that occur in [10] (in that article, however, general moduli $q\ge 2$ are handled. In the present article, we are only concerned with the case $q=2$, that is, the Thue–Morse sequence.) The first auxiliary result, a Bombieri–Vinogradov type theorem, is an average result on the sets $A_\omega(y,z;d,j)$ and might also be of independent interest.

Theorem 2.1. *Let $L\ge 1$ be an integer and $\omega=(\omega_0,\ldots,\omega_{L-1})\in\{0,1\}^L$. Assume that*

$$0<\delta_1\le\delta_2<2/3.$$

There exist $\eta>0$ and a constant C such that

$$\sum_{D<d\le 2D}\max_{\substack{y,z\\ 0\le y\le z\\ z-y\le x}}\max_{j\in\mathbb{Z}}\left|A_\omega(y,z;d,j)-\frac{z-y}{2^L d}\right|\le Cx^{1-\eta}$$

for all x and D such that $x\ge 1$ and $x^{\delta_1}\le D\le x^{\delta_2}$.

(Note that the maximum is well-defined by a finiteness argument. The same holds true for the subsequent results.) This theorem differs in several aspects from [10, Corollary 3]. The most important novelty is the exponent $2/3$, which improves on the exponent 0.5924. Moreover, the left endpoint of the interval $[y,z)$ may be an arbitrary nonnegative real number (which works well in the sum-of-digits setting, but fails for prime numbers). Finally, this theorem handles consecutive elements of arithmetic subsequences of the Thue–Morse sequence $\mathbf{t}$. This latter feature necessitates a nontrivial lower bound for D, since factors of length 2 of $\mathbf{t}$ do not appear with frequency $1/4$, therefore the contribution of $d=1$ would already be too large.

Setting $L=1$ and using the above-cited result (with the improved error term $x^{1-\eta}$) in order to handle small step lengths d, we obtain the following corollary.

Corollary 2.2. *For real $y \geq 0$ and integers $d \geq 1$ and $j \geq 0$ set*

$$A(y; d, j) = \left|\{m < y : s(m) \equiv 0 \bmod 2, m \equiv j \bmod d\}\right|.$$

For all $\delta \in (0, 2/3)$ there exist $\eta > 0$ and C such that

$$\sum_{1 \leq d \leq D} \max_{y \leq x} \max_{j \in \mathbb{Z}} \left| A(y; d, j) - \frac{y}{2d} \right| \leq C x^{1-\eta}$$

for $x \geq 1$ and $D = x^{\delta}$.

A simple but interesting consequence of Theorem 2.1 is the following result.

Corollary 2.3. *Every finite sequence on the symbols* `0` *and* `1` *appears as an arithmetic subsequence of the Thue–Morse sequence.*

An adaptation of the proof of Theorem 2.1 yields the following Beatty sequence analogue.

Theorem 2.4. *Let $L \geq 1$ be an integer, $\omega = (\omega_0, \ldots, \omega_{L-1}) \in \{0,1\}^L$ and $0 < \delta_1 \leq \delta_2 < 2/3$. There exist $\eta > 0$ and C such that*

$$\int_D^{2D} \max_{\substack{y,z \\ 0 \leq y \leq z \\ z-y \leq x}} \max_{\beta \geq 0} \left| A_\omega(y, z; \alpha, \beta) - \frac{z-y}{2^L \alpha} \right| \mathrm{d}\alpha \leq C x^{1-\eta}$$

for all x and D such that $x \geq 1$ and $x^{\delta_1} \leq D \leq x^{\delta_2}$.

As we announced in the introduction, this theorem can be used to obtain Theorem 1.

The proofs of Theorems 2.1 and 2.4 are based on exponential sum estimates. In order to establish Theorem 2.1, it is sufficient to prove the following proposition.

Proposition 2.5. *Let $L \geq 1$ be an integer, $a = (a_0, \ldots, a_{L-1}) \in \{0,1\}^L$ and $a \neq (0, \ldots, 0)$. For real numbers $N, D \geq 1$ and ξ set*

$$S_1 = S_1(N, D, \xi) = \sum_{D \leq d < 2D} \max_{j \geq 0} \left| \sum_{n < N} \mathrm{e}\left(\frac{1}{2} \sum_{\ell < L} a_\ell s\big((n+\ell)d + j\big) \right) \mathrm{e}(n\xi) \right|.$$

Let $0 < \rho_1 \leq \rho_2 < 2$. There exists an $\eta > 0$ and a constant C such that

$$\frac{S_1}{ND} \leq C N^{-\eta} \tag{2.1}$$

holds for all $\xi \in \mathbb{R}$ and all real numbers $N, D \geq 1$ satisfying $N^{\rho_1} \leq D \leq N^{\rho_2}$.

For $L = 1$ this result intuitively states that for most step lengths d slightly smaller than the square of the length of the sum, we have a nontrivial estimate for sums over the Thue–Morse sequence on arithmetic progressions.

Analogously, Theorem 2.4 is based on the following result.

Proposition 2.6. *Let $L \geq 1$ be an integer, $a = (a_0, \ldots, a_{L-1}) \in \{0,1\}^L$ and $a \neq (0, \ldots, 0)$. For real numbers $D, N \geq 1$ and ξ set*

$$\tilde{S}_1 = \tilde{S}_1(N, D, \xi) = \int_D^{2D} \max_{\beta \geq 0} \left| \sum_{n < N} \mathrm{e}\left(\frac{1}{2} \sum_{\ell < L} a_\ell s\big(\lfloor (n+\ell)\alpha + \beta \rfloor\big) \right) \mathrm{e}(n\xi) \right| \mathrm{d}\alpha.$$

Let $0 < \rho_1 \leq \rho_2 < 2$. There exist $\eta > 0$ and a constant C such that

$$\frac{\tilde{S}_1}{ND} \leq C N^{-\eta} \tag{2.2}$$

holds for all real numbers $D, N \geq 1$ satisfying $N^{\rho_1} \leq D \leq N^{\rho_2}$ and for all $\xi \in \mathbb{R}$.

The proofs of Propositions 2.5 and 2.6 in turn rely on an estimate for discrete Fourier coefficients related to the sum-of-digits function. These Fourier coefficients have been used as an essential tool in the article [7] on the normality of the Thue–Morse sequence along the sequence of squares. For nonnegative integers d and λ, for sequences $i:\mathbb{N}\rightarrow\mathbb{N}$ and $a:\mathbb{N}\rightarrow\mathbb{Z}$, where a has finite support, and for $h\in\mathbb{Z}$ we define

$$G_\lambda^{i,a}(h,d)=\frac{1}{2^\lambda}\sum_{u<2^\lambda}\mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{N}}a_\ell s_\lambda\bigl(u+\ell d+i_\ell\bigr)-\frac{hu}{2^\lambda}\right). \tag{2.3}$$

The function $h\mapsto G_\lambda^{i,a}(h,d)$ is the discrete Fourier transform of the 2^λ-periodic sequence

$$n\mapsto \mathrm{e}\left(\frac{1}{2}\sum_{\ell\geq 0}a_\ell s_\lambda\bigl(n+\ell d+i_\ell\bigr)\right).$$

We have the following important technical estimate for these Fourier terms.

Proposition 2.7. *Let $L\geq 1$ be an integer and choose $m\geq 5$ such that $2^{m-5}\leq L<2^{m-4}$. Assume that $(a_\ell)_{\ell\in\mathbb{Z}}$ is a sequence such that $a_0=1$, $a_1,\ldots,a_{L-1}\in\{0,1\}$ and $a_\ell=0$ for $\ell\notin[0,L)$. For $r\geq 1$ and $\ell\geq 0$ let $b_\ell^r=a_{\ell-r}-a_\ell$. There exist $\eta>0$ and C such that for all $\lambda\geq 0$ and $r\geq 1$ satisfying $2m\leq\nu_2(r)\leq\lambda/4$, and for all sequences $(i_\ell)_{\ell\in\mathbb{Z}}$ satisfying $i_0=0$ and $0\leq i_{\ell+1}-i_\ell\leq 1$ for $0\leq\ell<L+r-1$ we have*

$$\frac{1}{2^\lambda}\sum_{d<2^\lambda}\max_{h<2^\lambda}\left|G_\lambda^{i,b^r}(h,d)\right|^2\leq C2^{-\eta\lambda}.$$

This result differs from [7, Proposition 1] in two aspects. First, the maximum over h is *inside* the sum over d; second, the sequence b^r consists of two identical blocks (modulo 2) spaced by r. The constant as well as the exponent do not depend on the shift r, which will allow us to prove the quantitative normality result as stated in Theorem 1.

Remark. Combining Proposition 2.7 with the method of proof from [23], we could obtain a quantitative version of Theorem 0 that differs from our main theorem only in the (worse) bound $c<4/3$.

Finally, we state the following result, which allows replacing $\lfloor n^c\rfloor$ by $\lfloor n\alpha+\beta\rfloor$.

Proposition 2.8. *Assume that $\varphi:\mathbb{N}\rightarrow\Omega$ is a function into a finite set Ω and assume that $\omega=(\omega_0,\ldots,\omega_{L-1})\in\Omega^L$, where $L\geq 1$ is an integer. We write $f(x)=x^c$, where $1<c<2$ is a real number. Let $\delta\in[0,1]$ and define, for real $N,K>0$,*

$$J(N,K)=\frac{1}{f'(2N)-f'(N)}\int_{f'(N)}^{f'(2N)}\max_{f(N)<\beta\leq f(2N)}\left|\frac{1}{K}\bigl|\bigl\{n<K:\ \varphi\bigl(\lfloor(n+\ell)\alpha+\beta\rfloor\bigr)=\omega_\ell \text{ for } 0\leq\ell<L\bigr\}\bigr|-\delta\right|\mathrm{d}\alpha. \tag{2.4}$$

There exists a constant C such that for all $N\geq 2$ and $K>0$ we have

$$\left|\frac{1}{N}\bigl|\bigl\{n\in(N,2N]:\varphi\bigl(\lfloor f(n+\ell)\rfloor\bigr)=\omega_\ell \text{ for } 0\leq\ell<L\bigr\}\bigr|-\delta\right| \leq C\left(f''(N)K^2+\frac{(\log N)^2}{K}+J(N,K)\right). \tag{2.5}$$

Results on the distribution of values of an arithmetic function $\varphi:\mathbb{N}\rightarrow\Omega$ on *Beatty* sequences can therefore be used for proving statements concerning φ on *Piatetski-Shapiro* sequences $n\mapsto\lfloor n^c\rfloor$, at least in cases where the shift β does not cause problems. (This is the case for our problem concerning $\mathbf{t}$, however, this proposition cannot be used for the original Piatetski-Shapiro problem, since our knowledge on primes in short intervals is not sufficient.) Proposition 2.8 is a modification of [22, Proposition 1], which, together with the statement by Fouvry and Mauduit [10, Théorème 3 and inequality (1.5)] asserting that

$$\int_0^1 \prod_{j<k} \left|\sin\left(2^j\pi\theta\right)\right| \mathrm{d}\theta \sim \kappa\lambda^k \tag{2.6}$$

for some $\kappa \in \mathbb{R}$ and some $\lambda \in (0.6543, 0.6632)$, enabled the second author to obtain simple normality of $n \mapsto \mathbf{t}(\lfloor n^c \rfloor)$ for $c \le 1.42$.

The plan of the paper is as follows. In Section 3 we state a number of lemmas. In Section 4 we show how to prove Theorems 2.1, 2.4 and 1 from Propositions 2.5 and 2.6. Section 5 is concerned with the proof of Proposition 2.5, while Section 6, which is shorter, proves Proposition 2.6 in a way that is to a large extent analogous. (This section is shorter because some parts that have been treated in detail in the first proof have been left out. We also note that it would be possible to unify to a large extent the proofs of Propositions 2.5 and 2.6 by rewriting some sums as integrals with respect to some measure. However, we refrained from doing so since we wanted to keep the presentation clear.)

The last two sections are dedicated to the proofs of Propositions 2.7 and 2.8.

3. Lemmas

We begin with the following elementary facts about the functions $\lfloor\cdot\rfloor$, $\|\cdot\|$ and $\langle\cdot\rangle$, where $\langle x\rangle = \lfloor x + 1/2\rfloor$ (the "nearest integer" to x). The (easy) proofs are left to the reader.

Lemma 3.1. *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.*

(i) If $\|a\| < \varepsilon$ and $\|b\| \ge \varepsilon$, then $\lfloor a + b\rfloor = \langle a\rangle + \lfloor b\rfloor$.
(ii) $\|na\| \le n\,\|a\|$.
(iii) If $\|a\| < \varepsilon$ and $2n\varepsilon < 1$, then $\langle na\rangle = n\,\langle a\rangle$.

An essential tool in our proofs is the following generalization of van der Corput's inequality (see [16, Lemme 17]).

Lemma 3.2. *Let I be a finite interval containing N integers and let a_n be a complex number for $n \in I$. For all integers $K \ge 1$ and $R \ge 1$ we have*

$$\left|\sum_{n\in I} a_n\right|^2 \le \frac{N + K(R-1)}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \sum_{\substack{n\in I\\ n+Kr\in I}} a_{n+Kr}\overline{a_n}.$$

In particular, the right hand side is a nonnegative real number.

The following simple lemma will help us to remove the expression $\lfloor n\alpha + \beta\rfloor$, which appears in our calculations via linear approximation. It introduces the *discrepancy* of a sequence (modulo 1) instead.

Lemma 3.3. *Let J be an interval in $\mathbb{R}$ containing N integers and let α and β be real numbers. Assume that t, T, k and K are integers such that $0 \le t < T$ and $0 \le k < K$. Then*

$$\left|\left\{n \in J : \frac{t}{T} \le \{n\alpha + \beta\} < \frac{t+1}{T}, \lfloor n\alpha + \beta\rfloor \equiv k \bmod K\right\}\right| = \frac{N}{KT} + O\Big(N D_N\Big(\frac{\alpha}{K}\Big)\Big),$$

where

$$D_N(\alpha)=\sup_{\substack{0\le x\le 1\\ y\in\mathbb{R}}}\left|\frac{1}{N}\sum_{n<N} c_{[0,x)+y+\mathbb{Z}}(n\alpha)-x\right|.$$

Proof. We set $I=\big[(Tk+t)/(KT),(Tk+t+1)/(KT)\big)$, which is a subinterval of $[0,1)$ of length $1/(KT)$. The two conditions $t/T\le\{n\alpha+\beta\}<(t+1)/T$ and $\lfloor n\alpha+\beta\rfloor\equiv k \bmod K$ are satisfied if and only if $\{n\alpha/K+\beta/K\}\in I$ and the lemma follows by inserting the definition of $D_N(\alpha)$. $\square$

In order to handle the discrepancy (of $n\alpha$-sequences) thus introduced, we will use average results as in the following lemma.

Lemma 3.4. *Let J be a finite interval in $\mathbb{R}$ containing N integers. Then*

$$\sum_{k<2^\rho}\left|\sum_{j\in J}\mathrm{e}\left(\frac{jmk}{2^\rho}\right)\right|\ll 2^{\nu_2(m)}N+2^\rho\log^+N \tag{3.1}$$

for all integers $\rho\ge 0$ and $m\ne 0$. For integers $\mu\ge 0$ and $N\ge 1$ we have

$$\sum_{d<2^\mu} D_N\left(\frac{d}{2^\mu}\right)\ll\frac{N+2^\mu}{N}\big(\log^+N\big)^2. \tag{3.2}$$

Moreover, the estimate

$$\int_0^1 D_N(\alpha)\,\mathrm{d}\alpha\ll\frac{\big(\log^+N\big)^2}{N} \tag{3.3}$$

holds. The implied constants in these three estimates are absolute.

Proof. We prove the first claim. The estimate is trivial for $N\le 1$. We assume therefore that $N\ge 2$. Let $0<a\le b$. We have

$$\begin{aligned}\sum_{a\le k<b}\frac{1}{k}&=\sum_{\lfloor a\rfloor+1\le k<\lfloor b\rfloor}\frac{1}{k}+O(1)=\log(\lfloor b\rfloor)-\log(\lfloor a\rfloor+1)+O(1)\\&\le\log b-\log a+O(1).\end{aligned}$$

Therefore we get for all integers $\rho\ge 1$

$$\begin{aligned}\sum_{k<2^\rho}\min\Big\{N,\|k/2^\rho\|^{-1}\Big\}&=2\sum_{k<2^{\rho-1}}\min\Big\{N,|k/2^\rho|^{-1}\Big\}\\&\ll N\big|\big\{k<2^{\rho-1}:k<2^\rho/N\big\}\big|+2^\rho\sum_{2^\rho/N\le k<2^{\rho-1}}\frac{1}{k}\\&\ll N(1+2^\rho/N)+2^\rho\big(1+\log 2^\rho-\log(2^\rho/N)\big)\\&\ll N+2^\rho\log^+N.\end{aligned}$$

This estimate is also valid for $\rho=0$. Let $2^\eta\mid m$ and $2^{\eta+1}\nmid m$, that is, $\nu_2(m)=\eta$. If $\eta\le\rho$, we have

$$\begin{aligned}\sum_{k<2^\rho}\min\Big\{N,\|km/2^\rho\|^{-1}\Big\}&=2^\eta\sum_{k<2^{\rho-\eta}}\min\Big\{N,\big\|k/2^{\rho-\eta}\big\|^{-1}\Big\}\\&\ll 2^\eta N+2^\rho\log^+N.\end{aligned}$$

Note that this estimate holds trivially for $\eta>\rho$. The statement (3.1) follows therefore from the inequality

$$\left|\sum_{j\in J}\mathrm{e}\big(jmk/2^\rho\big)\right|\le\min\Big\{N,\|km/2^\rho\|^{-1}\Big\}.$$

In order to prove the first result on the average discrepancy, we use the Erdős–Turán inequality and (3.1) and obtain

$$\begin{aligned} N\sum_{d<2^\mu} D_N\left(\frac{d}{2^\mu}\right) &\ll 2^\mu + \sum_{1\le h\le N}\frac{1}{h}\sum_{d<2^\mu}\left|\sum_{n<N}\mathrm{e}\left(\frac{hnd}{2^\mu}\right)\right| \\ &\ll 2^\mu + \sum_{\rho\le\frac{\log N}{\log 2}}\sum_{\substack{1\le h\le N\\ \nu_2(h)=\rho}}\frac{1}{h}\left(2^\rho N + 2^\mu\log^+ N\right) \\ &\ll 2^\mu + \log^+ N\sum_{\rho\le\frac{\log N}{\log 2}}\frac{1}{2^\rho}\left(2^\rho N + 2^\mu\log^+ N\right) \\ &\ll \left(N+2^\mu\right)\left(\log^+ N\right)^2. \end{aligned}$$

The proof of the last statement is analogous. □

The following lemma concerning the discrete Fourier transform can easily be proved using orthogonality relations.

Lemma 3.5. *Assume that $M\ge 1$ is an integer and that $f:\mathbb{Z}\to\mathbb{C}$ is an M-periodic function. Then*

$$\frac{1}{M}\sum_{n<M} f(n+t)\overline{f(n)} = \sum_{h<M}\left|\hat{f}(h)\right|^2\mathrm{e}\left(ht/M\right), \tag{3.4}$$

where

$$\hat{f}(h) = \frac{1}{M}\sum_{u<M} f(u)\,\mathrm{e}\left(-hu/M\right).$$

We also need the following *carry propagation lemma* for the sum-of-digits function. Statements of this type were used in the articles [16, 17] by Mauduit and Rivat on the sum of digits of primes and squares.

Lemma 3.6. *Let r,L,N,λ be nonnegative integers and $\alpha>0,\beta\ge 0$ real numbers. Assume that I is an interval containing N integers. Then*

$$\begin{aligned} \big|\big\{n\in I : \exists \ell\in[0,L) \text{ such that } & s(\lfloor(n+\ell+r)\alpha+\beta\rfloor) - s(\lfloor(n+\ell)\alpha+\beta\rfloor) \\ &\ne s_\lambda(\lfloor(n+\ell+r)\alpha+\beta\rfloor) - s_\lambda(\lfloor(n+\ell)\alpha+\beta\rfloor)\big\}\big| \\ &\le (r+L)(N\alpha/2^\lambda+2). \end{aligned}$$

Proof. Let $E=(r+L)\alpha$. The statement is trivial for $E\ge 2^\lambda$. We assume therefore that $E<2^\lambda$. Moreover, we may assume that $L\ge 1$, since the estimate is trivial for $L=0$. We first note that if

$$n\alpha+\beta\in[0,2^\lambda-E)+2^\lambda\mathbb{Z}, \tag{3.5}$$

then

$$\begin{aligned} s(\lfloor(n+\ell+r)\alpha+\beta\rfloor) &- s(\lfloor(n+\ell)\alpha+\beta\rfloor) \\ &= s_\lambda(\lfloor(n+\ell+r)\alpha+\beta\rfloor) - s_\lambda(\lfloor(n+\ell)\alpha+\beta\rfloor) \end{aligned}$$

for all $\ell<L$. This follows easily by studying the binary representation of the arguments: if hypothesis (3.5) is satisfied, then $(n+0)\alpha+\beta,\ldots,(n+L-1+r)\alpha+\beta$ are contained in an interval $[k2^\lambda,(k+1)2^\lambda-1)$, therefore the digits of $\lfloor(n+\ell+r)\alpha+\beta\rfloor$ and $\lfloor(n+\ell)\alpha+\beta\rfloor$ with indices $\ge\lambda$ are the same, for all $\ell<L$. It remains to count the number of exceptions to (3.5).

For $k \in \mathbb{Z}$ let $a_k = \min\{n : k2^\lambda \le n\alpha + \beta\}$ and $b_k = \min\{n : (k+1)2^\lambda - E \le n\alpha + \beta\}$. Then for $a_k \le n < b_k$ we have $n\alpha + \beta \in [0, 2^\lambda - E) + 2^\lambda\mathbb{Z}$. It is therefore sufficient to count the number of $n \in I$ such that $b_k \le n < a_{k+1}$ for some k.

Clearly we have $a_{k+1} - b_k = r + L$. Assume that $I = [a, b)$ and choose k in such a way that $k2^\lambda \le a\alpha + \beta < (k+1)2^\lambda$. Then $a_k \le a$. Moreover $(b-1)\alpha + \beta < a\alpha + \beta + (b-a-1)\alpha < (k+1+(b-a-1)\alpha/2^\lambda)2^\lambda < (k + \lfloor N\alpha 2^{-\lambda}\rfloor + 2)2^\lambda$, therefore $b - 1 < a_{k+\lfloor N\alpha 2^{-\lambda}\rfloor+2}$. It follows that the exceptional indices n are contained in one of $\lfloor N\alpha 2^{-\lambda}\rfloor + 2$ intervals of length $r+L$. □

The following standard lemma allows us to extend the range of a summation in exchange for a controllable factor.

Lemma 3.7. *Let $x \le y \le z$ be real numbers and $a_n \in \mathbb{C}$ for $x \le n < z$. Then*

$$\left|\sum_{x \le n < y} a_n\right| \le \int_0^1 \min\left\{y - x + 1, \|\xi\|^{-1}\right\} \left|\sum_{x \le n < z} a_n \operatorname{e}(n\xi)\right| \mathrm{d}\xi.$$

Proof. Since $\int_0^1 \operatorname{e}(k\xi)\, \mathrm{d}\xi = \delta_{k,0}$ for $k \in \mathbb{Z}$ we have

$$\sum_{x \le n < y} a_n = \sum_{x \le n < z} a_n \sum_{x \le m < y} \delta_{n-m,0} = \int_0^1 \sum_{x \le m < y} \operatorname{e}(-m\xi) \sum_{x \le n < z} a_n \operatorname{e}(n\xi)\, \mathrm{d}\xi,$$

from which the statement follows. □

Let $\mathcal{F}_n$ be the *Farey series of order* n, by which we understand the set of rational numbers p/q such that $1 \le q \le n$. It is easy to see that each $a \in \mathcal{F}_n$ has two *neighbours* $a_L, a_R \in \mathcal{F}_n$, satisfying $a_L < a < a_R$ and $(a_L, a) \cap \mathcal{F} = (a, a_R) \cap \mathcal{F} = \emptyset$. We have the following elementary lemma concerning this set, which follows from the theorems in chapter 3 of the book [12] by Hardy and Wright.

Lemma 3.8. *Assume that a/b, c/d are reduced fractions such that $b, d > 0$ and $a/b < c/d$. Then $a/b < (a+c)/(b+d) < c/d$. If a/b and c/d are neighbours in $\mathcal{F}_n$, then $bc - ad = 1$ and $b + d > n$, moreover*

$$(a+c)/(b+d) - a/b < \frac{1}{bn}$$

and

$$c/d - (a+c)/(b+d) < \frac{1}{dn}.$$

We will also use the large sieve inequality, which we state here in the form provided by Selberg (see for example Montgomery [18, Theorem 3]).

Lemma 3.9 (Selberg)**.** *Let $N \ge 1, R \ge 1$ and M be integers, $\alpha_1, \ldots, \alpha_R \in \mathbb{R}$ and $a_{M+1}, \ldots, a_{M+N} \in \mathbb{C}$. Assume that $\|\alpha_r - \alpha_s\| \ge \delta$ for $r \ne s$, where $\delta > 0$. Then*

$$\sum_{r=1}^{R} \left|\sum_{n=M+1}^{M+N} a_n \operatorname{e}(n\alpha_r)\right|^2 \le (N - 1 + \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

4. Reduction of the problem

Before proving Propositions 2.5 and 2.6, we want to show that these results imply Theorems 2.1 and 2.4 respectively.

4.1. **Reducing Theorems 2.1 and 2.4 to Propositions 2.5 and 2.6.** We only deduce Theorem 2.1 from Proposition 2.5. The second implication can be shown in an analogous way.

Assume that the statement of Proposition 2.5 holds for some ρ_1, ρ_2 such that $0 < \rho_1 \le \rho_2$. Note that we allow $\rho_i \ge 2$ to keep the proof more general. We want to show that in Theorem 2.1 we may choose $\delta_1 = \rho_1/(\rho_1+1)$ and $\delta_2 = \rho_2/(\rho_2+1)$. Choosing ρ_2 close to 2, justified by Proposition 2.5, we see that δ_2 approaches $2/3$.

For real numbers $x, D \ge 1$ we define

$$S_0 = S_0(x,D) = \sum_{D<d\le 2D} \max_{\substack{y,z\\ 0\le y\le z\\ z-y\le x}} \max_{j\in\mathbb{Z}} \left| A_\omega(y,z;d,j) - \frac{z-y}{2^L d}\right|. \tag{4.1}$$

Let $x \ge 1$ and D be real numbers such that $x^{\delta_1} \le D \le x^{\delta_2}$. We rewrite the difference appearing in S_0 to exponential sums, using orthogonality relations:

$$\begin{aligned} &A_\omega(y,z;d,j) - \frac{z-y}{2^L d}\\ &\quad= \sum_{\substack{m,n\\ y\le m<z\\ m=nd+j}} \left\{\begin{array}{ll} 1 & \text{if } s((n+\ell)d+j) \equiv \omega_\ell \bmod 2 \text{ for } \ell < L\\ 0 & \text{otherwise}\end{array}\right\} - \frac{z-y}{2^L d}\\ &\quad= \sum_{n<\frac{z-y}{d}} \left\{\begin{array}{ll} 1 & \text{if } s((n+\ell)d+j+\lfloor (y-j)/d\rfloor\, d) \equiv \omega_\ell \bmod 2 \text{ for } \ell < L\\ 0 & \text{otherwise}\end{array}\right\}\\ &\qquad - \frac{z-y}{2^L d} + O(1)\\ &\quad= \frac{1}{2^L} \sum_{\substack{a\in\{0,1\}^L\\ a\ne(0,\ldots,0)}} \mathrm{e}\left(-\frac12 (a_0\omega_0 + \cdots + a_{L-1}\omega_{L-1})\right)\\ &\qquad\times \sum_{n<(z-y)/d} \mathrm{e}\left(\frac12 \sum_{\ell<L} a_\ell s((n+\ell)d+j+\lfloor (y-j)/d\rfloor\, d)\right) + O(1). \end{aligned}$$

It follows that

$$S_0 \le \frac{1}{2^L} \sum_{\substack{a\in\{0,1\}^L\\ a\ne(0,\ldots,0)}} \sum_{D\le d<2D} \max_{u\le x} \max_{j\ge 0} \left| \sum_{n<u/d} \mathrm{e}\left(\frac12 \sum_{\ell<L} a_\ell s((n+\ell)d+j)\right)\right| + O(D).$$

In order to dispose of the maximum over u, we apply Lemma 3.7. We exchange the appearing integral with the maximum over j and with the sum over d:

$$\begin{aligned} S_0(x,D) \le \frac{1}{2^L} &\sum_{\substack{a\in\{0,1\}^L\\ a\ne(0,\ldots,0)}} \int_0^1 \min\left\{\frac{x}{D}+1, \lVert\xi\rVert^{-1}\right\}\\ &\times \sum_{D\le d<2D} \max_{j\ge 0} \left| \sum_{n<x/D} \mathrm{e}\left(\frac12 \sum_{\ell<L} a_\ell s((n+\ell)d+j)\right) \mathrm{e}\left(n\xi\right)\right| \mathrm{d}\xi + O(D). \end{aligned} \tag{4.2}$$

Proposition 2.5 therefore implies that there exist a constant C and an exponent $\eta > 0$ such that for all $a_0, \ldots, a_{L-1} \in \{0,1\}$, not all equal to zero, for all real numbers $x, D \ge 1$ satisfying

$$\left(\frac{x}{D}\right)^{\rho_1} \le D \le \left(\frac{x}{D}\right)^{\rho_2},$$

and for all $\xi \in \mathbb{R}$ we have

$$\sum_{D\le d<2D} \max_{j\ge 0} \left| \sum_{n<x/D} \mathrm{e}\left(\frac{1}{2}\sum_{\ell<L} a_\ell s((n+\ell)d+j)\right) \mathrm{e}\left(n\xi\right)\right| \le CD\left(\frac{x}{D}\right)^{1-\eta}. \tag{4.3}$$

The condition on D can be rewritten as $x^{\delta_1} \le D \le x^{\delta_2}$. By the estimate

$$\int_0^1 \min\left\{A, \|\xi\|^{-1}\right\} \mathrm{d}\xi = 2\left(\int_0^{1/A} A\,\mathrm{d}\xi + \int_{1/A}^{1/2} \xi^{-1}\,\mathrm{d}\xi\right) \ll \log A,$$

which holds for $A \ge 2$, and by (4.2) and (4.3), there exists some $\eta_1 > 0$ and constants C and C_1 such that for all $x, D \ge 1$ satisfying $x^{\delta_1} \le D \le x^{\delta_2}$ we have

$$S_0(x,D) \le Cx\left(\frac{D}{x}\right)^{\eta} \log^+ x + O(D) \le C_1 x^{1-\eta_1},$$

which proves the theorem.

4.2. **Proving Theorem 1 from Theorem 2.4 and Proposition 2.8.** We show the more general implication that if the statement of Theorem 2.4 holds for some $0 < \delta_1 \le \delta_2 < 1$, then we may choose $c < 2/(2-\delta_2)$ in Theorem 1. Choosing δ_2 close to $2/3$ yields the desired statement. We have to find an estimate for $J(N,K)$ defined in (2.4). Therefore we calculate:

$$\begin{aligned}
&\left|\{n<K : s(\lfloor (n+\ell)\alpha+\beta\rfloor) \equiv \omega_\ell \bmod 2 \text{ for } \ell<L\}\right| \\
&\quad= \big|\{\lfloor\beta\rfloor \le m < \lfloor K\alpha+\beta\rfloor : \exists n\in\mathbb{Z} : m = \lfloor n\alpha+\beta\rfloor, \\
&\qquad\qquad s(\lfloor (n+\ell)\alpha+\beta\rfloor) \equiv \omega_\ell \bmod 2 \text{ for } \ell<L\}\big| \\
&\quad= \big|\{\beta \le m < K\alpha+\beta : \exists n\in\mathbb{Z} : m = \lfloor n\alpha+\beta\rfloor, \\
&\qquad\qquad s(\lfloor (n+\ell)\alpha+\beta\rfloor) \equiv \omega_\ell \bmod 2 \text{ for } \ell<L\}\big| + O(1) \\
&\quad= A_\omega(\beta, K\alpha+\beta; \alpha,\beta) + O(1).
\end{aligned}$$

We use the definition (2.4), where we set $\delta = 2^{-L}$, and define $D = f'(N)$. Noting that $f'(2N) = 2^{c-1}D \le 2D$, we obtain

$$\begin{aligned}
J(N,K) &\le \frac{1}{f'(2N)-f'(N)}\frac{1}{K}\int_{f'(N)}^{f'(2N)} \max_{\beta\ge 0}\bigg| A_\omega(\beta, K\alpha+\beta;\alpha,\beta) \\
&\qquad\qquad - \frac{K\alpha+\beta-\beta}{2^L\alpha}\bigg| \,\mathrm{d}\alpha + O(1/K) \\
&\le \frac{1}{(2^{c-1}-1)DK}\int_D^{2D} \max_{\substack{y,z\\ 0\le y\le z\\ z-y\le 2DK}} \max_{\beta\ge 0}\bigg| A_\omega(y,z;\alpha,\beta) \\
&\qquad\qquad - \frac{z-y}{2^L\alpha}\bigg| \,\mathrm{d}\alpha + O(1/K).
\end{aligned}$$

It follows from Theorem 2.4 that for $(2DK)^{\delta_1} \le D \le (2DK)^{\delta_2}$, that is, for $\frac{1}{2}D^{1/\delta_2-1} \le K \le \frac{1}{2}D^{1/\delta_1-1}$, we have

$$J(N,K) \le \frac{C}{DK}(2DK)^{1-\eta} + O(1/K)$$

for some $\eta > 0$ and C depending on c and L. Setting $K = \frac{1}{2}D^{1/\delta_2-1}$, we obtain

$$J(N,K) \le CD^{-\eta/\delta_2} + 2D^{1-1/\delta_2}.$$

By Proposition 2.8 we get

$$\begin{aligned}\left|\frac{1}{N}\left|\left\{n\in(N,2N]: s(\lfloor(n+\ell)^c\rfloor)=\omega_\ell \text{ for } 0\le\ell<L\right\}\right|-\frac{1}{2^L}\right|\\ \le C_1\left(f''(N)K^2+\frac{(\log N)^2}{K}+J(N,K)\right)\\ \le C_2\left(N^{c-2+2(c-1)(1/\delta_2-1)}+\frac{(\log N)^2}{N^{(c-1)(1/\delta_2-1)}}+N^{-\eta(c-1)/\delta_2}\right).\end{aligned}$$

All of the occurring exponents of N are negative by the conditions $c<2/(2-\delta_2)$ and $0<\delta_2<1$, which proves Theorem 1.

5. Proof of Proposition 2.5

Assume that $L\ge 1$ is an integer, $a=(a_0,\ldots,a_{L-1})\in\{0,1\}^L$ and that $a_\ell=1$ for some ℓ. It is easy to see, using the shift j, that we may assume $a_0=1$. We also assume that $N\ge 1$ is an integer; the general statement follows from the estimate $S_1(N,D,\xi)-S_1(\lfloor N\rfloor,D,\xi)\ll D$. Moreover, it is sufficient to prove the statement

$$\frac{S_1(N,2^\nu,\xi)}{N2^\nu}\le CN^{-\eta}$$

for all integers $N,\nu\ge 1$ and real numbers $D\ge 1$ such that $N^{\rho_1}\le D\le N^{\rho_2}$ and $D<2^\nu\le 2D$. This can be seen by considering sums (in d) over the intervals $[2^{\nu-1},2^\nu)$ and $[2^\nu,2^{\nu+1})$ and using the estimate $S_1(N,2^{\nu-1},\xi)\le S_1(N,2^\nu,\xi)$, which follows from the identity $s(2m)=s(m)$. Choose η and C according to Proposition 2.7. Moreover, let $\tau=2m$, where $2^{m-5}\le L<2^{m-4}$, and $\lambda\ge 0$. By the Cauchy–Schwarz inequality we have

$$|S_1(N,2^\nu,\xi)|^2\le 2^\nu\sum_{2^\nu\le d<2^{\nu+1}}\max_{j\ge 0}S_2(N,d,j,\xi), \tag{5.1}$$

where

$$S_2=S_2(N,d,j,\xi)=\left|\sum_{n<N}\mathrm{e}\left(\frac12\sum_{\ell<L}a_\ell s((n+\ell)d+j)\right)\mathrm{e}(n\xi)\right|^2.$$

We apply Lemma 3.2 (the generalized inequality of van der Corput) with $K=2^\tau$. Let $R\ge 1$ be an integer. Then

$$\begin{aligned}S_2\le\frac{N+2^\tau(R-1)}{R}\sum_{|r|<R}\left(1-\frac{|r|}{R}\right)\mathrm{e}(r2^\tau\xi)\\ \times\sum_{0\le n,n+r2^\tau<N}\mathrm{e}\left(\frac12\sum_{\ell<L}a_\ell\big(s((n+\ell+r2^\tau)d+j)-s((n+\ell)d+j)\big)\right).\end{aligned}$$

Using Lemma 3.6 and treating the summand $r=0$ separately, moreover omitting the condition $0\le n+r2^\tau<N$, we obtain for all $\lambda\ge 0$

$$\begin{aligned}S_2\ll O\left(\frac{N^2}{R}+NR2^\tau+N^2\frac{R2^\tau d}{2^\lambda}\right)\\ +\frac{N}{R}\sum_{1\le r<R}\left|\sum_{n<N}\mathrm{e}\left(\frac12\sum_{\ell<L}a_\ell\big(s_\lambda((n+\ell+r2^\tau)d+j)-s_\lambda((n+\ell)d+j)\big)\right)\right|\end{aligned}$$

with an implied constant depending only on the block length L. Note that we also replaced $N+(R-1)2^\tau$ by N; this is clearly admissible if $R2^\tau\le N$, otherwise we use the trivial estimate $|S_2|\le N^2$ and note the presence of the error term $O(NR2^\tau)$.

We set a_ℓ to 0 for $\ell \notin \{0,\ldots,L-1\}$ and define $b_\ell^r = a_{\ell-r2^\tau} - a_\ell$. Applying the Cauchy–Schwarz inequality twice and using (5.1) gives

$$|S_1(N,2^\nu,\xi)|^2 \ll (2^\nu N)^2 O\left(\frac{1}{R}+\frac{R2^\tau 2^\nu}{2^\lambda}+\frac{R2^\tau}{N}\right) + 2^{3\nu/2}N\left(\frac{1}{R}\sum_{1\le r<R} S_3(N,r,\lambda,\nu)\right)^{1/2}, \tag{5.2}$$

where

$$S_3(N,r,\lambda,\nu) = \sum_{2^\nu\le d<2^{\nu+1}} \max_{j\ge 0} \left|\sum_{n<N} \mathrm{e}\left(\frac12 \sum_{\ell\in\mathbb{Z}} b_\ell^r s_\lambda((n+\ell)d+j)\right)\right|^2. \tag{5.3}$$

Applying Lemma 3.2 for $K=2^\mu$, we obtain for all integers $M\ge 1$ and $\mu\ge 0$

$$\begin{aligned}\left|\sum_{n<N} \mathrm{e}\left(\frac12 \sum_{\ell\in\mathbb{Z}} b_\ell^r s_\lambda((n+\ell)d+j)\right)\right|^2 &\le \frac{N+2^\mu(M-1)}{M}\sum_{|m|<M}\left(1-\frac{|m|}{M}\right)\\ &\times \sum_{0\le n,n+m2^\mu<N} \mathrm{e}\left(\frac12\sum_{\ell\in\mathbb{Z}} b_\ell^r\big(s_{\mu,\lambda}((n+\ell+m2^\mu)d+j) - s_{\mu,\lambda}((n+\ell)d+j)\big)\right)\\ &\ll N\,|S_4| + 2^\mu MN,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}S_4 = S_4(N,M,d,j,r,\mu,\lambda) &= \frac{1}{M}\sum_{|m|<M}\left(1-\frac{|m|}{M}\right)\\ &\times \sum_{n<N} \mathrm{e}\Bigg(\frac12\sum_{\ell\in\mathbb{Z}} b_\ell^r\Bigg(s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)d+j}{2^\mu}\right\rfloor + md\right)\\ &\qquad - s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)d+j}{2^\mu}\right\rfloor\right)\Bigg)\Bigg).\end{aligned}$$

The replacement of s_λ by the two-fold restricted sum-of-digits function $s_{\mu,\lambda}$, which we performed in (5.4), is admissible since the arguments differ by a multiple of 2^μ and therefore the difference does not depend on the lower digits. We obtain

$$S_3 \ll N \sum_{2^\nu\le d<2^{\nu+1}} \max_{j\ge 0} |S_4| + 2^{\mu+\nu}MN. \tag{5.5}$$

The rough idea at this point is to estimate S_4 by a nonnegative real number independent of j, which will allow us to remove the maximum over j and the absolute value appearing in (5.5). In the following we will work out the details of this process. We want to split the summation over N into T parts, according to the fractional part of $(nd+j)2^{-\mu}$. Let t,T be integers such that $0\le t<T$. We define

$$\begin{aligned}S_5 &= S_5(N,T,a,d,j,r,m,t,\mu,\lambda)\\ &= \sum_{\substack{n<N\\ \frac{t}{T}\le\left\{\frac{nd+j}{2^\mu}\right\}<\frac{t+1}{T}}} \mathrm{e}\Bigg(\frac12\sum_{\ell\in\mathbb{Z}} b_\ell^r\Bigg(s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)d+j}{2^\mu}\right\rfloor + md\right)\end{aligned}$$

$$-s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)d+j}{2^\mu}\right\rfloor\right)\Bigg)\Bigg).$$

We will see that, for most values of d, the values of the floor function distribute evenly modulo $2^{\lambda-\mu}$ as n runs through the set defined by the two conditions under the summation sign. For this to hold, we have to assure that $N > 2^{\lambda-\mu}$. Inspecting the error terms in (5.2) and (5.5), we see that we also need $2^\mu < N$ and $\nu < \lambda$ in order to get a nontrivial estimate. These observations ultimately lead to the restriction $\rho_2 < 2$ in Proposition 2.5.

The idea behind the decomposition into T subintervals $[t/T,(t+1)/T)$ of $[0,1)$ is the following. Let A_t be the set of n such that $\{(nd+j)/2^\mu\}$ lies in the t-th interval. Then the differences

$$\left\lfloor\frac{(n+1)d+j}{2^\mu}\right\rfloor-\left\lfloor\frac{nd+j}{2^\mu}\right\rfloor,\ldots,\left\lfloor\frac{(n+L-1)d+j}{2^\mu}\right\rfloor-\left\lfloor\frac{nd+j}{2^\mu}\right\rfloor$$

should not depend on $n \in A_t$. This will in fact be the case for most, but not for all, t, so that we have to take out some t. We define a set of "good" values,

$$G = G(T,d,r,\mu) = \left\{t<T : \left[\frac{t}{T}+\frac{\ell d}{2^\mu},\frac{t+1}{T}+\frac{\ell d}{2^\mu}\right)\cap\mathbb{Z}=\emptyset\right.$$
$$\left.\text{for all } \ell\in[0,L)\cup[r2^\tau,r2^\tau+L)\right\}.$$

We have

$$|G| \ge T-2L, \tag{5.6}$$

since the intervals in the definition of $G(d,T,R,\mu)$ are disjoint and cover an interval of length 1, therefore we have to exclude at most one integer t for each ℓ.

We differentiate between the cases $t \in G$ and $t \notin G$. For $t \notin G$ we estimate the sum in S_5 trivially, that is, we count the number of summands, using Lemma 3.3. We apply this lemma for $K = 2^{\lambda-\mu}$ (note that we could also take $K = 1$, however, our choice spares us the separate treatment of an error term) and multiply with $2^{\lambda-\mu}$, which accounts for the $2^{\lambda-\mu}$ residue classes we have to collect. We obtain

$$S_5 \ll \frac{N}{T} + 2^{\lambda-\mu} N D_N\left(\frac{d}{2^\lambda}\right). \tag{5.7}$$

Let $t \in G$ and assume that $t/T \le \{(nd+j)2^{-\mu}\} < (t+1)/T$. By the second assumption we obtain

$$\left\lfloor\frac{nd+j}{2^\mu}\right\rfloor+\frac{t}{T}+\frac{\ell d}{2^\mu}\le\frac{(n+\ell)d+j}{2^\mu}<\left\lfloor\frac{nd+j}{2^\mu}\right\rfloor+\frac{t+1}{T}+\frac{\ell d}{2^\mu}$$

and using the first assumption yields

$$\left\lfloor\frac{(n+\ell)d+j}{2^\mu}\right\rfloor=\left\lfloor\frac{nd+j}{2^\mu}\right\rfloor+\left\lfloor\frac{t}{T}+\frac{\ell d}{2^\mu}\right\rfloor$$

for all $\ell \in [0,L)\cup[r2^\tau,r2^\tau+L)$. For $t \in G$ we obtain therefore

$$S_5 = \sum_{k<2^{\lambda-\mu}}\sum_{\substack{n<N\\ \frac{t}{T}\le\left\{\frac{nd+j}{2^\mu}\right\}<\frac{t+1}{T}\\ \left\lfloor\frac{nd+j}{2^\mu}\right\rfloor\equiv k \bmod 2^{\lambda-\mu}}}\mathrm{e}\Bigg(\frac{1}{2}\sum_{\ell\in\mathbb{Z}}b_\ell^r\Bigg(s_{\lambda-\mu}\left(k+\left\lfloor\frac{t}{T}+\frac{\ell d}{2^\mu}\right\rfloor+md\right)$$
$$-s_{\lambda-\mu}\left(k+\left\lfloor\frac{t}{T}+\frac{\ell d}{2^\mu}\right\rfloor\right)\Bigg)\Bigg)$$

Since the summand does not depend on n, we count the number of times the three conditions under the second summation sign are satisfied. To this end, we use again Lemma 3.3 with $K = 2^{\lambda-\mu}$. We obtain for $t \in G$

$$S_5 = \frac{N}{2^{\lambda-\mu}T} \sum_{k<2^{\lambda-\mu}} \mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r\left(s_{\lambda-\mu}\left(k + \left\lfloor \frac{t}{T} + \frac{\ell d}{2^\mu}\right\rfloor + md\right)\right.\right.$$
$$\left.\left. - s_{\lambda-\mu}\left(k + \left\lfloor \frac{t}{T} + \frac{\ell d}{2^\mu}\right\rfloor\right)\right)\right) + O\left(2^{\lambda-\mu} N D_N\left(\frac{d}{2^\lambda}\right)\right). \tag{5.8}$$

Note that this expression is independent of the shift j. Moreover, as we noted earlier, we see that it is necessary that we have $N \geq 2^{\lambda-\mu}$ in order to get a useful result, since the error term would be too large otherwise. Setting

$$i_\ell^{d,t} = \left\lfloor \frac{t}{T} + \ell\left\{\frac{d}{2^\mu}\right\}\right\rfloor,$$

we get the almost trivial identity

$$\left\lfloor \frac{t}{T} + \frac{\ell d}{2^\mu}\right\rfloor = \ell\left\lfloor\frac{d}{2^\mu}\right\rfloor + i_\ell^{d,t}.$$

In Lemma 3.5 we set $t = md$ and

$$f(n) = \mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r s_{\lambda-\mu}\left(n + \ell\left\lfloor\frac{d}{2^\mu}\right\rfloor + i_\ell^{d,t}\right)\right)$$

and obtain for $t \in G$

$$S_5 = \frac{N}{T}\sum_{h<2^{\lambda-\mu}} \left|G_{\lambda-\mu}^{i^{d,t},b^r}\left(h, \left\lfloor\frac{d}{2^\mu}\right\rfloor\right)\right|^2 \mathrm{e}\left(\frac{hmd}{2^{\lambda-\mu}}\right) + O\left(2^{\lambda-\mu} N D_N\left(\frac{d}{2^\lambda}\right)\right), \tag{5.9}$$

where the Fourier coefficients $G(h,d)$ are defined by (2.3). By the definitions of S_4 and S_5 we have

$$S_4 = \frac{1}{M}\sum_{|m|<M}\left(1 - \frac{|m|}{M}\right)\sum_{t<T} S_5. \tag{5.10}$$

Using (5.6), (5.7), (5.9) and (5.10) we obtain

$$S_4 = \frac{N}{T}\sum_{t<T}\frac{1}{M}\sum_{|m|<M}\left(1-\frac{|m|}{M}\right)\sum_{h<2^{\lambda-\mu}}\left|G_{\lambda-\mu}^{i^{d,t},b^r}\left(h,\left\lfloor\frac{d}{2^\mu}\right\rfloor\right)\right|^2 \mathrm{e}\left(\frac{hmd}{2^{\lambda-\mu}}\right)$$
$$+ O\left(N\frac{L}{T} + 2^{\lambda-\mu} T N D_N\left(\frac{d}{2^\lambda}\right)\right), \tag{5.11}$$

where the reinsertion of the indices $t \notin G$ is accounted for by the error term NL/T, which can be seen using Parseval's identity.

Note the important fact that the right hand side gives an estimate for S_4 that is independent of the shift j (that is, independent of the residue class modulo d). Using also the nonnegativity of the main term, which follows from the elementary identity

$$\sum_{|m|<M}(M - |m|)\,\mathrm{e}(mx) = \left|\sum_{m<M}\mathrm{e}(mx)\right|^2,$$

we may remove the maximum together with the absolute value in (5.5) while keeping the important factor $\mathrm{e}\left(hmd/2^{\lambda-\mu}\right)$ in (5.11). We obtain, treating the summand $m = 0$ separately,

$$S_3 \ll \frac{N^2}{T} \sum_{t<T} \frac{1}{M} \sum_{1\le |m|<M} \left(1 - \frac{|m|}{M}\right) S_6 \\ + 2^\nu N^2 O\left(\frac{1}{M} + \frac{2^\mu M}{N} + \frac{L}{T} + 2^{\lambda-\mu} T \frac{1}{2^\nu} \sum_{d<2^{\nu+1}} D_N\left(\frac{d}{2^\lambda}\right)\right), \tag{5.12}$$

where

$$S_6 = S_6(d,r,m,t,\lambda,\mu,\nu) \\ = \sum_{2^\nu \le d < 2^{\nu+1}} \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i^{d,t},b^r}\left(h, \left\lfloor \frac{d}{2^\mu} \right\rfloor\right)\right|^2 \mathrm{e}\left(\frac{hmd}{2^{\lambda-\mu}}\right). \tag{5.13}$$

We want to show that S_6 is substantially smaller than 2^ν (which is a trivial upper bound by Parseval's identity). In order to estimate the right hand side of (5.13), we note that the first factor depends only in a weak way on d. We state this more precisely in the following.

The term $\lfloor d/2^\mu \rfloor$ does not depend on the lowest μ binary digits of d. Moreover, we want to decompose $[0, 2^\mu)$ into few intervals $I_u^{r,t}$ in such a way that the values $i_\ell^{d,t}$, where $\ell \in M$ and $M = [0, L) \cup [r2^\tau, r2^\tau + L)$, are constant for $d \in I_u^{r,t}$. (Note that the indices $\ell \notin M$ are not of interest, since $b_\ell^r = 0$ for these.)

Let $t < T$ be given. We define the (lexicographical) order on $\mathbb{N}^M$ by $(i_\ell)_{\ell\in M} < (j_\ell)_{\ell\in M}$ if and only if $i_\ell \ne j_\ell$ for some $\ell \in M$ and $i_\ell < j_\ell$ for $\ell = \min\{k \in M : i_k \ne j_k\}$. It is easy to check that the assignment

$$d \mapsto i^{d,t} \,|_M$$

is 2^μ-periodic and nondecreasing for $d < 2^\mu$ with respect to this total ordering. It follows that the set $\{0, \ldots, 2^\mu - 1\}$ decomposes into intervals $I_0^{r,t}, \ldots, I_{U-1}^{r,t}$ such that the same sequence $\left(i_\ell^{d,t}\right)_{\ell\in M}$ is defined for each $d \in I_u^{r,t} + 2^\mu\mathbb{Z}$. By the property $0 \le i_{\ell+1}^{r,t} - i_\ell^{r,t} \le 1$, the number U of intervals thus defined satisfies

$$U \le 2^{2L}(r2^\tau - L) \le r2^{2L+\tau}. \tag{5.14}$$

From (5.13), we obtain sums of the form

$$\sum_{d\in I_u^{r,t} + 2^\mu k} \mathrm{e}\left(hmd/2^{\lambda-\mu}\right)$$

for $u < U$ and some $k \in \mathbb{Z}$. Using also the estimate for the Fourier coefficients $G_{\lambda-\mu}$ from Proposition 2.7, we will estimate S_6 nontrivially.

Let $\mathcal{I} = \mathcal{I}_{L+r2^\tau-1}$ be the set of sequences $i_0, \ldots, i_{L+r2^\tau-1}$ such that $i_0 = 0$ and $0 \le i_{\ell+1} - i_\ell \le 1$ for $0 \le \ell < L + r2^\tau - 1$. Assume that $m \ne 0$ and $\lambda \ge \nu \ge \mu$. Writing $d = d_1 + 2^\mu d_2$, and choosing i_u such that $i_u = i^{d_1,t}$ for all $d_1 \in I_u^{r,t}$, we obtain

$$\begin{aligned}|S_6| &= \left| \sum_{u<U} \sum_{d_1\in I_u^{r,t}} \sum_{d_2<2^{\nu-\mu}} \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i^{d_1,t},b^r}(h,d_2)\right|^2 \mathrm{e}\left(\frac{hm(d_1 + d_2 2^\mu)}{2^{\lambda-\mu}}\right)\right| \\ &= \left| \sum_{u<U} \sum_{d_2<2^{\nu-\mu}} \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i_u,b^r}(h,d_2)\right|^2 \sum_{d_1\in I_u^{r,t}} \mathrm{e}\left(\frac{hm(d_1 + d_2 2^\mu)}{2^{\lambda-\mu}}\right)\right| \\ &\le \sum_{u<U} \max_{i\in\mathcal{I}} \sum_{d_2,h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i,b^r}(h,d_2)\right|^2 \left| \sum_{d_1\in I_u^{r,t}} \mathrm{e}\left(\frac{hmd_1}{2^{\lambda-\mu}}\right)\right|.\end{aligned}$$

We apply the Cauchy–Schwarz inequality to the sum over h and d_2 and obtain with the help of Parseval's identity

$$\begin{aligned}
\left|S_6\right| &\leq \sum_{u<U} \max_{i\in\mathcal{I}} \left(\sum_{d_2,h<2^{\lambda-\mu}} \left| G^{i,b^r}_{\lambda-\mu}(h,d_2) \right|^4 \right)^{1/2} \\
&\qquad\qquad \times \left(\sum_{d_2,h<2^{\lambda-\mu}} \left| \sum_{d_1\in I_u^{r,t}} \mathrm{e}\left(\frac{hmd_1}{2^{\lambda-\mu}}\right) \right|^2 \right)^{1/2} \\
&\leq 2^{(\lambda-\mu)/2} \max_{i\in\mathcal{I}} \left(\sum_{d_2<2^{\lambda-\mu}} \max_{h<2^{\lambda-\mu}} \left| G^{i,b^r}_{\lambda-\mu}(h,d_2) \right|^2 \right)^{1/2} \qquad (5.15) \\
&\qquad\qquad \times \sum_{u<U} \left(\sum_{h<2^{\lambda-\mu}} \left| \sum_{k<2^{\lambda-\mu}} a_k^u \, \mathrm{e}\left(\frac{hk}{2^{\lambda-\mu}}\right) \right|^2 \right)^{1/2},
\end{aligned}$$

where

$$a_k^u = a_k^{u,r,t,m} = \left| \{ d_1 \in I_u^{r,t} : d_1 m \equiv k \bmod 2^{\lambda-\mu} \} \right|.$$

In order to estimate the sum over d_2 in (5.15), we use Proposition 2.7: there exist $\eta > 0$ and C, which only depend on L, such that for all $\lambda \geq \mu \geq 0$ and all $r \geq 1$ satisfying $\nu_2(r)+\tau \leq (\lambda-\mu)/4$ we have

$$\max_{i\in\mathcal{I}} \left(\sum_{d_2<2^{\lambda-\mu}} \max_{h<2^{\lambda-\mu}} \left| G^{i,b^r}_{\lambda-\mu}(h,d_2) \right|^2 \right)^{1/2} \leq C 2^{(1-\eta)(\lambda-\mu)/2}. \qquad (5.16)$$

The sum over h in (5.15) can be estimated by Lemma 3.9 (the large sieve inequality) and the estimate

$$a_k^{u,r,t,m} \leq \begin{cases} 2^{\nu_2(m)} & \mu \leq \lambda-\mu \\ 2^{\nu_2(m)} 2^{\mu}/2^{\lambda-\mu} & \mu > \lambda-\mu \end{cases}$$

(which is clear for odd m and follows easily by the decomposition $m = m_0 2^{\nu_2(m)}$ otherwise). This gives

$$\begin{aligned}
\left(\sum_{h<2^{\lambda-\mu}} \left| \sum_{k<2^{\lambda-\mu}} a_k^{u,r,t,m} \, \mathrm{e}\left(\frac{hk}{2^{\lambda-\mu}}\right) \right|^2 \right)^{1/2} &\ll \left(2^{\lambda-\mu} \sum_{k<2^{\lambda-\mu}} \left| a_k^{u,r,t,m} \right|^2 \right)^{1/2} \qquad (5.17) \\
&\ll 2^{\lambda-\mu} 2^{\nu_2(m)} \max\{1, 2^{2\mu-\lambda}\}.
\end{aligned}$$

Combining (5.16), (5.17) and (5.14), we get

$$\begin{aligned}
|S_6| &\ll r 2^{2L+\tau} 2^{\nu_2(m)} \max\{1, 2^{2\mu-\lambda}\} 2^{(2-\eta/2)(\lambda-\mu)} \\
&\ll r 2^{\nu_2(m)} \max\{1, 2^{2\mu-\lambda}\} 2^{(2-\eta/2)(\lambda-\mu)}.
\end{aligned}$$

with an implied constant depending only on L. We translate this estimate back to an estimate for S_1. By the estimate

$$\sum_{1\leq m<M} 2^{\nu_2(m)} \ll M \log M$$

valid for $M \geq 1$, which is easy to show by splitting the summation according to the value of $\nu_2(m)$, we obtain

$$\frac{1}{RM} \sum_{1\leq r<R} \sum_{1\leq |m|<M} \left(1 - \frac{|m|}{M}\right) S_6 \ll R \log M \max\{1, 2^{2\mu-\lambda}\} 2^{(2-\eta/2)(\lambda-\mu)}. \tag{5.18}$$

We collect the error terms, using (5.2), (5.12) and (5.18) and use the discrepancy estimate (3.2), obtaining

$$\begin{aligned}\left|\frac{S_1(N, 2^\nu, \xi)}{N 2^\nu}\right|^4 \leq C\Bigg(&\frac{1}{R^2} + \left(\frac{R2^\nu}{2^\lambda}\right)^2 + \left(\frac{R}{N}\right)^2 + \frac{1}{M} + \frac{2^\mu M}{N} + \frac{1}{T} \\ &+ T\frac{2^{\lambda-\mu}}{N}\frac{N+2^\lambda}{2^\nu}\left(\log^+ N\right)^2 + R 2^{(2-\eta/2)(\lambda-\mu)-\nu} \log M \max\{1, 2^{2\mu-\lambda}\}\Bigg)\end{aligned} \tag{5.19}$$

with a constant C depending only on L. This estimate is valid for all integers $M, N, R, T \geq 1$ and $\lambda, \mu, \nu \geq 0$ such that $\mu \leq \nu < \nu + 1 \leq \lambda$ and $R2^\tau \leq 2^{(\lambda-\mu)/4}$, and for all real numbers ξ. Moreover, this estimate also holds for real-valued parameters M, R, T, λ, μ satisfying these restrictions (with a possibly different constant C). In order to finish the proof of Proposition 2.5, we have to choose the parameters M, R, T, λ and μ, depending on N and D. Let $0 < \rho_1 \leq \rho_2 < 2$ be given and choose θ and ε in such a way that

$$\max\left\{1, \rho_2, \frac{3\rho_2}{1+\rho_2}, 2 - \eta/2\right\} < \theta < 2$$

and

$$0 < \varepsilon < \min\left\{2 - \theta, \theta/\rho_2 - 1, \theta - 1, \theta\frac{1+\rho_2}{\rho_2} - 3, \theta - (2 - \eta/2), 1/4\right\}.$$

Assume that $D \geq 1$ is a real number and that $N, \nu \geq 1$ are integers such that $N^{\rho_1} \leq D \leq N^{\rho_2}$ and $D < 2^\nu \leq 2D$. Set $\mu = \nu/\theta$, $\lambda = 2\mu$ and $R = M = T = 2^{\varepsilon\mu}$. Using these choices it is not difficult to check, proceeding term by term, that

$$\frac{S_1(N, 2^\nu, \xi)}{N 2^\nu} \leq C 2^{-\nu\eta_1}\left(\log^+ N\right)^2$$

for some C and $\eta_1 > 0$ depending only on ρ_2 and L. Finally, we insert the lower bound $N^{\rho_1} \leq 2^\nu$ (so far we did not use ρ_1), which completes the proof of Proposition 2.5.

6. Proof of Proposition 2.6

We follow the proof of Proposition 2.5 and start, without loss of generality, with the same assumptions. Assume that $L \geq 1$ is an integer, $a_0 = 1$ and $a_1, \ldots, a_{L-1} \in \{0, 1\}$. Choose $m \geq 5$ such that $2^{m-5} \leq L < 2^{m-4}$ and set $\tau = 2m$. Assume that $D, N, \nu \geq 1$, where N and ν are integers satisfying $N^{\rho_1} \leq N \leq N^{\rho_2}$ and $D < 2^\nu \leq 2D$.

We apply van der Corput's inequality for $K = 2^\tau$ and obtain, in analogy to (5.2),

$$\begin{aligned}\left|\tilde{S}_1\right|^2 \ll (2^\nu N)^2 O\left(\frac{1}{R} + \frac{R2^\tau 2^\nu}{2^\lambda} + \frac{R2^\tau}{N}\right)& \\ + 2^{3\nu/2} N \left(\frac{1}{R}\sum_{1\leq r<R} \tilde{S}_3(N, r, \lambda, \nu)\right)^{1/2}&,\end{aligned} \tag{6.1}$$

where

$$\tilde{S}_3(N, r, \lambda, \nu) = \int_{2^\nu}^{2^{\nu+1}} \max_{\beta\geq 0} \left|\sum_{n<N} \mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r s_\lambda\left(\lfloor (n+\ell)\alpha + \beta\rfloor\right)\right)\right|^2 \mathrm{d}\alpha \tag{6.2}$$

and $b_\ell^r = a_{\ell-r2^\tau} - a_\ell$, and a_ℓ is assumed to be zero for $\ell \notin [0,L)$. The estimate (6.1) is valid for $N, R \geq 1$ and $\lambda, \nu \geq 0$.

Next we want to "cut off" the μ lowest digits, that is, replace s_λ by $s_{\mu,\lambda}$. This was accomplished by a simple application of Lemma 3.2, setting $K = 2^\mu$, in the proof of Proposition 2.5. Since we are now considering Beatty sequences (having in general non-integer step length α), we need to modify our strategy. To do so, we use Diophantine approximation, more precisely, Farey series. Let $\alpha \in \mathbb{R}$ be given. We assign a fraction $p(\alpha)/q(\alpha)$ to α according to the Farey dissection of the circle: consider reduced fractions $a/b < c/d$ that are neighbours in the Farey series $\mathcal{F}_{2^{\mu+\sigma}}$, where $\sigma \geq 1$ is chosen later, such that $a/b \leq \alpha/2^\mu < c/d$. If $\alpha/2^\mu < (a+c)/(b+d)$, then set $p(\alpha) = a$ and $q(\alpha) = b$, otherwise set $p(\alpha) = c$ and $q(\alpha) = d$. Lemma 3.8 implies

$$\left| q(\alpha)\alpha - p(\alpha)2^\mu \right| < 2^{-\sigma}. \tag{6.3}$$

Applying Lemma 3.2 with $K = q(\alpha)$ and noting that $q(\alpha) \leq 2^{\mu+\sigma}$, we obtain for all integers $M \geq 1$ and $\mu \geq 0$

$$\begin{aligned}
&\left| \sum_{n<N} \mathrm{e}\left(\frac{1}{2} \sum_{\ell\in\mathbb{Z}} b_\ell^r s_\lambda \left(\lfloor (n+\ell)\alpha + \beta \rfloor \right) \right) \right|^2 \\
&\qquad \ll O(2^{\mu+\sigma} MN) + \frac{N}{M} \sum_{|m|<M} \left(1 - \frac{|m|}{M} \right) \\
&\qquad\quad \times \sum_{n<N} \mathrm{e}\left(\frac{1}{2} \sum_{\ell\in\mathbb{Z}} b_\ell^r \big(s_\lambda \left(\lfloor (n+\ell+m\,q(\alpha))\alpha + \beta \rfloor \right) - s_\lambda \left(\lfloor (n+\ell)\alpha + \beta \rfloor \right) \big) \right).
\end{aligned} \tag{6.4}$$

In order to reduce this expression to a sum analogous to S_4, we want to shift the expression $m\,q(\alpha)\alpha$ out of the floor function. To this end, we use (6.3) and the argument that $m\,q(\alpha)\alpha$ is close to an integer, while $(n+\ell)\alpha+\beta$ usually is not. This can be made precise as follows. Assume that

$$\|(n+\ell)\alpha + \beta\| \geq M/2^\sigma \tag{6.5}$$

and that $2M < 2^\sigma$. Using part (iii) of Lemma 3.1 with $\varepsilon = 1/2^\sigma$ and (6.3), moreover noting that $\sigma \geq 1$, we obtain

$$\langle m\,q(\alpha)\alpha \rangle = m \langle q(\alpha)\alpha \rangle = m\,p(\alpha)2^\mu.$$

Applying part (i) of Lemma 3.1, setting $\varepsilon = M/2^\sigma$, we see that (6.5) implies

$$\lfloor (n+\ell+m\,q(\alpha))\alpha + \beta \rfloor = m\,p(\alpha)2^\mu + \lfloor (n+\ell)\alpha + \beta \rfloor. \tag{6.6}$$

The number of n where hypothesis (6.5) fails for some ℓ can be estimated by discrepancy estimates for $\{n\alpha\}$-sequences: for all positive integers N and $2M < 2^\sigma$ we have

$$\begin{aligned}
&\left| \left\{ n < N : \|(n+\ell)\alpha + \beta\| \leq M/2^\sigma \right\} \right| \\
&\qquad = \left| \left\{ n < N : (n+\ell)\alpha + \beta \in [-M/2^\sigma, M/2^\sigma] + \mathbb{Z} \right\} \right| \\
&\qquad = \left| \left\{ n < N : n\alpha \in [0, 2M/2^\sigma] - \beta - \ell\alpha - M/2^\sigma + \mathbb{Z} \right\} \right| \\
&\qquad \leq N D_N(\alpha) + 2MN/2^\sigma.
\end{aligned} \tag{6.7}$$

Multiplying this error by $2L$ (which is $O(1)$ according to our conventions stated in the introduction), we obtain an upper bound for the number of $n < N$ such that $\|(n+\ell)\alpha + \beta\| \leq M/2^\sigma$ for some $\ell \in [0,L) \cup [r2^\tau, r2^\tau + L)$. Treating these integers separately and using (6.4) through (6.7), we obtain

$$\left| \sum_{n<N} \mathrm{e}\left(\frac{1}{2} \sum_{\ell\in\mathbb{Z}} b_\ell^r s_\lambda \left(\lfloor (n+\ell)\alpha + \beta \rfloor \right) \right) \right|^2 \tag{6.8}$$

$$\ll N\bigl|\tilde{S}_4\bigr| + O\bigl(2^{\mu+\sigma}NM + N^2D_N(\alpha) + N^2M/2^\sigma\bigr),$$

where

$$\tilde{S}_4 = \tilde{S}_4(N,M,\alpha,\beta,r,\mu,\lambda) = \frac{1}{M}\sum_{|m|<M}\left(1-\frac{|m|}{M}\right)$$
$$\times \sum_{n<N} \mathrm{e}\Biggl(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r\biggl(s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)\alpha+\beta}{2^\mu}\right\rfloor + m\,p(\alpha)\right)$$
$$-s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)\alpha+\beta}{2^\mu}\right\rfloor\right)\biggr)\Biggr).$$

Note that (6.8) is, except for the error terms, completely analogous to equation (5.4) in the proof of Proposition 2.5. From (6.2) and (6.8) we get

$$\tilde{S}_3(N,r,\lambda,\nu) \ll N\int_{2^\nu}^{2^{\nu+1}} \max_{\beta\geq 0}\bigl|\tilde{S}_4\bigr|\,\mathrm{d}\alpha + N^2\int_{2^\nu}^{2^{\nu+1}} D_N(\alpha)\,\mathrm{d}\alpha + 2^{\mu+\sigma+\nu}MN + N^2 2^\nu M/2^\sigma. \tag{6.9}$$

Let t,T be integers such that $0\leq t<T$ and define

$$\tilde{S}_5 = \sum_{\substack{n<N\\ \frac{t}{T}\leq\left\{\frac{n\alpha+\beta}{2^\mu}\right\}<\frac{t+1}{T}}} \mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r\biggl(s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)\alpha+\beta}{2^\mu}\right\rfloor + m\,p(\alpha)\right)\right.$$
$$\left. - s_{\lambda-\mu}\left(\left\lfloor\frac{(n+\ell)\alpha+\beta}{2^\mu}\right\rfloor\right)\biggr)\right)$$

and

$$G = G(T,\alpha,r,\mu) = \Biggl\{t<T : \left[\frac{t}{T}+\frac{\ell\alpha}{2^\mu},\frac{t+1}{T}+\frac{\ell\alpha}{2^\mu}\right)\cap\mathbb{Z}=\emptyset$$
$$\text{for all } \ell\in[0,L)\cup[r2^\tau,r2^\tau+L)\Biggr\}.$$

Again, we have

$$G\geq T-2L \tag{6.10}$$

and we distinguish between the cases $t\in G$ and $t\notin G$. For $t\notin G$ we estimate $\tilde{S}_5$ trivially, applying Lemma 3.3 with $K=2^{\lambda-\mu}$. We obtain

$$\tilde{S}_5 \ll \frac{N}{T} + 2^{\lambda-\mu}ND_N\left(\frac{\alpha}{2^\lambda}\right). \tag{6.11}$$

Let $t\in G$ and assume that $t/T\leq\{(n\alpha+\beta)2^{-\mu}\}<(t+1)/T$. Then, as before,

$$\left\lfloor\frac{(n+\ell)\alpha+\beta}{2^\mu}\right\rfloor = \left\lfloor\frac{n\alpha+\beta}{2^\mu}\right\rfloor + \left\lfloor\frac{t}{T}+\frac{\ell\alpha}{2^\mu}\right\rfloor$$

for $\ell\in[0,L)\cup[r2^\tau,r2^\tau+L)$. For $t\in G$ we obtain

$$\tilde{S}_5 = \sum_{k<2^{\lambda-\mu}}\sum_{\substack{n<N\\ \frac{t}{T}\leq\left\{\frac{n\alpha+\beta}{2^\mu}\right\}<\frac{t+1}{T}\\ \left\lfloor\frac{n\alpha+\beta}{2^\mu}\right\rfloor\equiv k \bmod 2^{\lambda-\mu}}} \mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r\biggl(s_{\lambda-\mu}\left(k+\left\lfloor\frac{t}{T}+\frac{\ell\alpha}{2^\mu}\right\rfloor + m\,p(\alpha)\right)\right.$$

$$- s_{\lambda-\mu}\left(k + \left\lfloor \frac{t}{T} + \frac{\ell\alpha}{2^\mu}\right\rfloor\right)\Bigg)\Bigg).$$

We apply Lemma 3.3, setting $K = 2^{\lambda-\mu}$, and obtain for $t \in G$

$$\tilde{S}_5 = \frac{N}{2^{\lambda-\mu}T} \sum_{k<2^{\lambda-\mu}} \mathrm{e}\Bigg(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r \Bigg(s_{\lambda-\mu}\left(k + \left\lfloor \frac{t}{T} + \frac{\ell\alpha}{2^\mu}\right\rfloor + m\,p(\alpha)\right) - s_{\lambda-\mu}\left(k + \left\lfloor \frac{t}{T} + \frac{\ell\alpha}{2^\mu}\right\rfloor\right)\Bigg)\Bigg) + O\left(2^{\lambda-\mu} N D_N\left(\frac{\alpha}{2^\lambda}\right)\right). \tag{6.12}$$

Setting

$$i_\ell^{\alpha,t} = \left\lfloor \frac{t}{T} + \ell\left\{\frac{\alpha}{2^\mu}\right\}\right\rfloor,$$

we have

$$\left\lfloor \frac{t}{T} + \frac{\ell\alpha}{2^\mu}\right\rfloor = \ell\left\lfloor\frac{\alpha}{2^\mu}\right\rfloor + i_\ell^{\alpha,t}.$$

In Lemma 3.5 we set $t = m\,p(\alpha)$ and

$$f(n) = \mathrm{e}\left(\frac{1}{2}\sum_{\ell\in\mathbb{Z}} b_\ell^r s_{\lambda-\mu}\left(n + \ell\left\lfloor\frac{\alpha}{2^\mu}\right\rfloor + i_\ell^{\alpha,t}\right)\right)$$

and obtain for $t \in G$

$$\tilde{S}_5 = \frac{N}{T}\sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i^{\alpha,t},b^r}\left(h, \left\lfloor\frac{\alpha}{2^\mu}\right\rfloor\right)\right|^2 \mathrm{e}\left(\frac{hm\,p(\alpha)}{2^{\lambda-\mu}}\right) + O\left(2^{\lambda-\mu}ND_N\left(\frac{\alpha}{2^\lambda}\right)\right). \tag{6.13}$$

Using the identity

$$\tilde{S}_4 = \frac{1}{M}\sum_{|m|<M}\left(1 - \frac{|m|}{M}\right)\sum_{t<T}\tilde{S}_5$$

as well as (6.10), (6.11) and (6.13), we obtain, in analogy to (5.11),

$$\begin{aligned}\tilde{S}_4 = {}& \frac{N}{T}\sum_{t<T}\frac{1}{M}\sum_{|m|<M}\left(1 - \frac{|m|}{M}\right)\\ &\times \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i^{\alpha,t},b^r}\left(h, \left\lfloor\frac{\alpha}{2^\mu}\right\rfloor\right)\right|^2 \mathrm{e}\left(\frac{hm\,p(\alpha)}{2^{\lambda-\mu}}\right)\\ &+ O\left(\frac{N}{T} + 2^{\lambda-\mu}TND_N\left(\frac{\alpha}{2^\lambda}\right)\right).\end{aligned} \tag{6.14}$$

From (6.9) and (6.14), treating the summand for $m = 0$ separately, we get

$$\tilde{S}_3 \ll \frac{N^2}{T}\sum_{t<T}\frac{1}{M}\sum_{1\le|m|<M}\left(1 - \frac{|m|}{M}\right)\tilde{S}_6 + 2^\nu N^2\, O\Bigg(\frac{1}{M} + \frac{2^{\mu+\sigma}M}{N} + \frac{1}{T} + \frac{2^{\lambda-\mu}T}{2^\nu}\int_{2^\nu}^{2^{\nu+1}} D_N\left(\frac{\alpha}{2^\lambda}\right)\mathrm{d}\alpha + \frac{1}{2^\nu}\int_{2^\nu}^{2^{\nu+1}} D_N(\alpha)\,\mathrm{d}\alpha + \frac{M}{2^\sigma}\Bigg), \tag{6.15}$$

where

$$\tilde{S}_6 = \tilde{S}_6(\alpha, r, m, t, \lambda, \mu, \nu)$$

$$= \int_{2^\nu}^{2^{\nu+1}} \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i^{\alpha,t},b^r}\left(h, \left\lfloor \frac{\alpha}{2^\mu} \right\rfloor\right) \right|^2 \mathrm{e}\left(\frac{hm\,p(\alpha)}{2^{\lambda-\mu}}\right) \mathrm{d}\alpha.$$

As in the proof of Proposition 2.5 we obtain intervals $I_0^{r,t}, \ldots, I_{U-1}^{r,t} \subseteq \mathbb{R}$, where $U \le r2^{2L+\tau}$, such that $\alpha \mapsto i^{\alpha,t}|_M$ is constant on each $I_u^{r,t}$. Define $\mathcal{I}$ as before, that is, as the set of sequences $(i_\ell)_{\ell<L+r2^\tau-1}$ such that $0 \le i_{\ell+1} - i_\ell \le 1$ for $0 \le i < L + r2^\tau - 1$. Moreover, we assume from now on that $m \ne 0$ and $\lambda \ge \nu \ge \mu$. We obtain, applying the Cauchy–Schwarz inequality to the sum over (h, d_2),

$$\begin{aligned}
\left|\tilde{S}_6\right| &= \left| \sum_{u<U} \int_{I_u^{r,t}} \sum_{d_2<2^{\nu-\mu}} \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i^{\alpha,t},b^r}(h,d_2) \right|^2 \mathrm{e}\left(\frac{hm\,p(\alpha+d_2 2^\mu)}{2^{\lambda-\mu}}\right) \mathrm{d}\alpha \right| \\
&\le \sum_{u<U} \max_{i\in\mathcal{I}} \sum_{d_2<2^{\lambda-\mu}} \sum_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i,b^r}(h,d_2) \right|^2 \left| \int_{I_u^{r,t}} \mathrm{e}\left(\frac{hm\,p(\alpha+d_2 2^\mu)}{2^{\lambda-\mu}}\right) \mathrm{d}\alpha \right| \\
&\le \sum_{u<U} \max_{i\in\mathcal{I}} \left(\sum_{d_2,h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i,b^r}(h,d_2) \right|^4 \right)^{1/2} \\
&\qquad\qquad \times \left(\sum_{d_2,h<2^{\lambda-\mu}} \left| \int_{I_u^{r,t}} \mathrm{e}\left(\frac{hm\,p(\alpha+d_2\, 2^\mu)}{2^{\lambda-\mu}}\right) \mathrm{d}\alpha \right|^2 \right)^{1/2} \\
&\le \max_{i\in\mathcal{I}} \left(\sum_{d_2<2^{\lambda-\mu}} \max_{h<2^{\lambda-\mu}} \left| G_{\lambda-\mu}^{i,b^r}(h,d_2) \right|^2 \right)^{1/2} \\
&\qquad\qquad \times \sum_{u<U} \left(\sum_{d,h<2^{\lambda-\mu}} \left| \sum_{k<2^{\lambda-\mu}} a_k^{u,d}\, \mathrm{e}\left(\frac{hk}{2^{\lambda-\mu}}\right) \right|^2 \right)^{1/2},
\end{aligned}$$

where

$$a_k^{u,d} = a_k^{r,m,t,u,d,\mu,\lambda} = \boldsymbol{\lambda}\left(\{\alpha \in I_u^{r,t} : m\,p(\alpha + d\,2^\mu) \equiv k \bmod 2^{\lambda-\mu}\}\right),$$

and $\boldsymbol{\lambda}$ denotes the Lebesgue measure. Assume that $r2^\tau \le 2^{(\lambda-\mu)/4}$. Then, using Proposition 2.7, the first factor can be estimated by $C2^{(1-\eta)(\lambda-\mu)/2}$. The second factor can be estimated by the large sieve inequality (Lemma 3.9) and $|U| \ll r$, which yields

$$\left|\tilde{S}_6\right| \ll r\, 2^{(1-\eta)(\lambda-\mu)/2} \max_{u<U} \left(\sum_{d<2^{\lambda-\mu}} 2^{\lambda-\mu} \sum_{k<2^{\lambda-\mu}} \left| a_k^{u,d} \right|^2 \right)^{1/2}.$$

It remains to find estimates for $a_k^{u,d}$. In order to do so, we first note that

$$\begin{aligned} q(\alpha + d\,2^\mu) &= q(\alpha) \\ p(\alpha + d\,2^\mu) &= p(\alpha) + q(\alpha)d \end{aligned} \tag{6.16}$$

for all $d \in \mathbb{N}$. Lemma 3.8 implies

$$\boldsymbol{\lambda}\left(\{\alpha \in [0, 2^\mu) : p(\alpha) = p, q(\alpha) = q\}\right) \le 2\frac{2^\mu}{2^{\mu+\sigma}q}. \tag{6.17}$$

Formulas (6.17) and (6.16) together with the estimate

$$\left|\{p < 2^{\mu+\sigma} : mp \equiv k \bmod 2^{\lambda-\mu}\}\right| \le 2^{\nu_2(m)} \max\{1, 2^{2\mu-\lambda+\sigma}\}$$

imply

$$\begin{aligned}\boldsymbol{\lambda}\left(\{\alpha\in[0,2^\mu): m\,p(\alpha+d\,2^\mu)\equiv k \bmod 2^{\lambda-\mu}, q(\alpha+d\,2^\mu)=q\}\right)\\ =\boldsymbol{\lambda}\left(\{\alpha\in[0,2^\mu): m\,p(\alpha)\equiv k-d\,q \bmod 2^{\lambda-\mu}, q(\alpha)=q\}\right)\\ \le 2^{\nu_2(m)}\max\{1,2^{2\mu-\lambda+\sigma}\}\frac{2}{2^\sigma q}\end{aligned}$$

By a summation over $q\le 2^{\mu+\sigma}$ we obtain

$$a_k^{u,d}\ll 2^{\nu_2(m)}\max\{1,2^{2\mu-\lambda+\sigma}\}(\mu+\sigma)/2^\sigma.$$

Therefore

$$\left|\tilde S_6\right|\ll R2^{\nu_2(m)}(\mu+\sigma)\,2^{(2-\eta')(\lambda-\mu)}\max\{2^{-\sigma},2^{2\mu-\lambda}\},$$

which leads to

$$\begin{aligned}&\frac{1}{RM}\sum_{1\le r<R}\sum_{1\le|m|<M}\left(1-\frac{|m|}{M}\right)\tilde S_6\\ &\qquad\ll R\log M(\mu+\sigma)\,2^{(2-\eta')(\lambda-\mu)}\max\{2^{-\sigma},2^{2\mu-\lambda}\}.\end{aligned}\tag{6.18}$$

By analogous reasoning as in the proof of Proposition 2.6, leading to (5.19), and using (6.1), (6.15) and (6.18) and the estimate for the integral mean of the discrepancy (Lemma 3.4), we obtain

$$\begin{aligned}\left|\frac{\tilde S_1(N,\nu,\xi)}{N2^\nu}\right|^4\ll \frac{1}{R^2}+\left(\frac{R2^{\nu+\tau}}{2^\lambda}\right)^2+\left(\frac{R2^\tau}{N}\right)^2+\frac{1}{M}+\frac{2^{\mu+\sigma}M}{N}+\frac{1}{T}+\frac{M}{2^\sigma}\\ +T\frac{2^{\lambda-\mu}}{N}\left(\log^+N\right)^2+R\log M(\mu+\sigma)2^{(2-\eta')(\lambda-\mu)+2\mu-\lambda-\nu}\end{aligned}$$

for all integers $T,R,M,N,\sigma\ge 1$ and $\lambda,\mu,\nu\ge 0$ such that $\mu\le\nu\le\lambda\le 2\nu$, $2M<2^\sigma$, $R2^\tau\le 2^{(\lambda-\mu)/4}$ and for all $\xi\in\mathbb{R}$. An analogous argument as in the proof of Proposition 2.5 finishes the proof.

7. Proof of Proposition 2.7

7.1. **A recurrence for Fourier coefficients.** In order to get started with the proof of Proposition 2.7, we recall the definition of the discrete Fourier coefficients $G_\lambda^{i,b}(h,d)$, which is equation (2.3). For nonnegative integers d and λ, for sequences $i:\mathbb{N}\to\mathbb{N}$ (for notational reasons we use, in this section, the function notation $i(\ell)$ to denote the ℓ-th element) and $b:\mathbb{N}\to\mathbb{Z}$, where b has finite support, and for $h\in\mathbb{Z}$ we have

$$G_\lambda^{i,b}(h,d)=\frac{1}{2^\lambda}\sum_{u<2^\lambda}\mathrm{e}\left(\frac12\sum_{\ell\ge0}b_\ell s_\lambda(u+\ell d+i(\ell))-\frac{hu}{2^\lambda}\right).$$

For any $K\ge 0$, we denote by $\mathcal{I}_K$ the set of sequences $i:\mathbb{N}\to\mathbb{N}$ with support in $[0,K)$ such that $i(\ell+1)-i(\ell)\in\{0,1\}$ for $0\le\ell<K-1$. For $(\delta,\varepsilon)\in\{0,1\}^2$, we define a transformation $T_{\delta,\varepsilon}:\mathcal{I}_K\to\mathcal{I}_K$ by

$$T_{\delta,\varepsilon}(i)(\ell)=\left\lfloor\frac{i(\ell)+\ell\delta+\varepsilon}{2}\right\rfloor$$

for $0\le\ell<K$. Note that this transformation is well-defined. We also define *weights* by

$$f_{\delta,\varepsilon}^{i,b}=\mathrm{e}\left(\frac12\sum_{\ell<K}b_\ell(i(\ell)+\ell\delta+\varepsilon)\right).$$

These quantities appear in the following recurrence for the discrete Fourier coefficients, compare [7, Lemma 13].

Lemma 7.1. *Assume that $b:\mathbb{N}\rightarrow\mathbb{Z}$ and $i\in\mathcal{I}_K$. Then for all integers $d,\lambda\geq 0$ and h, and $\varepsilon\in\{0,1\}$ we have*

$$G_\lambda^{i,b}(h,2d+\delta)=\frac{1}{2}\sum_{\varepsilon=0}^{1}\mathrm{e}\left(-\frac{h\varepsilon}{2^\lambda}\right)f_{\delta,\varepsilon}^{i,b}\,G_{\lambda-1}^{T_{\delta,\varepsilon}(i),b}(h,d). \tag{7.1}$$

Proof. By splitting the sum in the definition of $G_\lambda^{i,b}(h,d)$ according to the parity of u, we obtain (writing $\varepsilon_0(n)$ for the lowest binary digit of n)

$$\begin{aligned}
&G_\lambda^{i,b}(h,2d+\delta)\\
&\quad=\frac{1}{2^\lambda}\sum_{\varepsilon=0}^{1}\sum_{u<2^{\lambda-1}}\mathrm{e}\left(\frac{1}{2}\sum_{\ell<K}b_\ell s_\lambda(2u+\varepsilon+\ell(2d+\delta)+i(\ell))-h(2u+\varepsilon)2^{-\lambda}\right)\\
&\quad=\frac{1}{2^\lambda}\sum_{\varepsilon=0}^{1}\mathrm{e}\left(-\frac{h\varepsilon}{2^\lambda}\right)\sum_{u<2^{\lambda-1}}\mathrm{e}\left(\frac{1}{2}\sum_{\ell<K}b_\ell\left(s_{\lambda-1}\left(u+\ell d+\left\lfloor\frac{i(\ell)+\ell\delta+\varepsilon}{2}\right\rfloor\right)\right.\right.\\
&\qquad\qquad\left.\left.+\varepsilon_0(i(\ell)+\ell\delta+\varepsilon)\right)-hu2^{\lambda-1}\right)\\
&\quad=\frac{1}{2}\sum_{\varepsilon=0}^{1}\mathrm{e}\left(-\frac{h\varepsilon}{2^\lambda}\right)f_{\delta,\varepsilon}^{i,b}\,G_{\lambda-1}^{T_{\delta,\varepsilon}(i),b}(h,d).
\end{aligned}$$

□

We want to study compositions of elementary transformations $T_{\delta,\varepsilon}$. We therefore extend this notation as follows. Assume that $d,e,m\geq 0$ are integers. For $d,e<2^m$, we define $T_{d,e}^{(m)}:\mathcal{I}_K\rightarrow\mathcal{I}_K$ by setting, for $i\in\mathcal{I}_K$ and $\ell<K$,

$$T_{d,e}^{(m)}(i)(\ell)=\left\lfloor\frac{i(\ell)+\ell d+e}{2^m}\right\rfloor. \tag{7.2}$$

For general integers $d,e\geq 0$ we set

$$T_{d,e}^{(m)}=T_{d\bmod 2^m,\,e\bmod 2^m}^{(m)}.$$

Note that we have

$$T_{d,0}^{(m)}(i)(\ell)\leq T_{d,e}^{(m)}(i)(\ell)\leq T_{d,0}^{(m)}(i)(\ell)+1 \tag{7.3}$$

for all $e\geq 0$. By a straightforward induction it follows that

$$T_{d,e}^{(m)}=T_{\delta_{m-1},\varepsilon_{m-1}}\circ\cdots\circ T_{\delta_0,\varepsilon_0}, \tag{7.4}$$

for $m\geq 1$, where $\sum_{i\geq 0}\delta_i2^i$ and $\sum_{i\geq 0}\varepsilon_i2^i$ are the binary expansions of d and e respectively. Moreover, $T_{d,e}^{(0)}$ is the identity on $\mathcal{I}_K$. We also define, generalizing the notation $f_{\delta,\varepsilon}^{i,b}$,

$$f_{d,e}^{(m),i,b}=f_{\delta_{m-1},\varepsilon_{m-1}}^{T_{d,e}^{(m-1)}(i),b}\cdots f_{\delta_1,\varepsilon_1}^{T_{d,e}^{(1)}(i),b}\cdot f_{\delta_0,\varepsilon_0}^{i,b}. \tag{7.5}$$

In order to obtain a recurrence relation for $|G_\lambda^{i,b}(h,d)|^2$, we define

$$\Phi_\lambda^{i_1,i_2,b}(h,d)=G_\lambda^{i_1,b}(h,d)\overline{G_\lambda^{i_2,b}(h,d)}.$$

Using (7.1), this immediately yields

$$\begin{aligned}(7.6)\quad &\Phi_\lambda^{i_1,i_2,b}(h,2d+\delta)\\ &= \frac{1}{4}\sum_{\varepsilon_1<2}\sum_{\varepsilon_2<2} \mathrm{e}\left(-\frac{(\varepsilon_1-\varepsilon_2)h}{2^\lambda}\right) f_{\delta,\varepsilon_1}^{i_1,b}\overline{f_{\delta,\varepsilon_2}^{i_2,b}}\,\Phi_{\lambda-1}^{T_{\delta,\varepsilon_1}(i_1),T_{\delta,\varepsilon_2}(i_2),b}(h,d)\end{aligned}$$

for $\delta \in \{0,1\}$, and applying this identity iteratively one gets, for $m \in \mathbb{N}$ and $d' < 2^m$,

$$\begin{aligned}\Phi_\lambda^{i_1,i_2,b}(h,2^m d+d') = \frac{1}{4^m}&\sum_{e_1<2^m}\sum_{e_2<2^m} \mathrm{e}\left(-\frac{(e_1-e_2)h}{2^\lambda}\right)\\ &\times f_{d',e_1}^{(m),i_1,b}\overline{f_{d',e_2}^{(m),i_2,b}}\,\Phi_{\lambda-m}^{T_{d',e_1}^{(m)}(i_1),T_{d',e_2}^{(m)}(i_2),b}(h,d).\end{aligned}\tag{7.7}$$

Obviously this implies for all $d' < 2^m$

$$\begin{aligned}\left|G_\lambda^{i,b}(h,2^m d+d')\right|^2 &= \left|\Phi_\lambda^{i,i,b}(h,2^m d+d')\right|\\ &\le \max_{e_1,e_2<2^m}\left|\Phi_{\lambda-m}^{T_{d',e_1}^{(m)}(i),T_{d',e_2}^{(m)}(i),b}(h,d)\right|\\ &= \max_{e<2^m}\left|G_{\lambda-m}^{T_{d',e}^{(m)}(i),b}(h,d)\right|^2,\end{aligned}\tag{7.8}$$

an estimate that is also valid for $m = 0$.

7.2. **An estimate for Fourier coefficients.** In this section, we are concerned with such sequences b originating from a sequence $a = (a_0,\ldots,a_{L-1}) \in \{0,1\}^L$, where $1 \le L \le r$, via the assignment

$$b_\ell = \begin{cases} a_\ell, & 0 \le \ell < L,\\ -a_{\ell-r}, & r \le \ell < L+r-1,\\ 0, & \text{otherwise.}\end{cases}\tag{7.9}$$

That is, the sequence b consists of two blocks, identical modulo 2. From now on, we assume that b is such a sequence and that $K = L + r$ (such that b_ℓ, for $\ell < K$, captures all nonzero values). For brevity, and since b is constant in what follows, we omit b as an upper index of G, Φ and f. Moreover, we assume throughout this section that $\lambda \ge 0, r \ge 1$ and $m \ge 5$ are integers such that

$$2^{m-5} \le L < 2^{m-4},\tag{7.10}$$

$$2m \le \nu_2(r) \le \lambda/4.\tag{7.11}$$

For brevity, we write $x = \nu_2(r)$.

Lemma 7.2. *Assume that the sequence $i \in \mathcal{I}_K$ satisfies*

$$i(r) \bmod 2^m \in \{1,2\}.\tag{7.12}$$

Let $z \ge 0$ and h be integers and $0 \le d < 2^z$. Then

$$\left|G_{z+m}^i(h,d\,2^m+1)\right|^2 \le (1-\eta)\max_{e<2^m}\left|G_z^{T_{1,e}^{(m)}(i)}(h,d)\right|^2$$

for $\eta = \frac{2}{4^m}$.

Proof. We rewrite the left hand side via the identity (7.7), setting $i_1 = i_2 = i$, and want to find pairs of indices $(e_1',e_2'),(e_1'',e_2'')$ such that the corresponding two summands on the right hand side of (7.7) cancel. This will give the announced saving. That is, we want

$$T_{1,e_1'}^{(m)}(i) = T_{1,e_1''}^{(m)}(i),\tag{7.13}$$

$$T^{(m)}_{1,e'_2}(i) = T^{(m)}_{1,e''_2}(i), \tag{7.14}$$

$$f^{(m),i}_{1,e'_1}\overline{f^{(m),i}_{1,e'_2}} = -f^{(m),i}_{1,e''_1}\overline{f^{(m),i}_{1,e''_2}} \quad \text{and} \tag{7.15}$$

$$e'_1 - e'_2 = e''_1 - e''_2. \tag{7.16}$$

We show that these conditions are satisfied for the choice

$$e'_1 = (0101^{m-3})_2, \quad e'_2 = (1001^{m-3})_2,$$
$$e''_1 = (0111^{m-3})_2, \quad e''_2 = (1011^{m-3})_2,$$

where $(\varepsilon_\nu \ldots \varepsilon_0)_2 = \sum_{i\le\nu} \varepsilon_i 2^i$ and 1^k means k-fold repetition of the digit 1. Condition (7.16) is clearly true. In order to verify (7.13) and (7.14), we note that the binary representations of e'_1, e'_2, e''_1, e''_2 all start with $m-3$ ones. We therefore define

$$j = T^{(m-3)}_{1,(1^{m-3})_2}(i).$$

By (7.10), which implies $i(\ell) + \ell < 2^{m-3}$ for $0 \le \ell < L$, we have

$$j(\ell) = \left\lfloor \frac{i(\ell) + \ell + 2^{m-3} - 1}{2^{m-3}} \right\rfloor = \begin{cases} 0, & \ell = 0, \\ 1, & 1 \le \ell < L. \end{cases} \tag{7.17}$$

Moreover, by (7.12) we obtain $i(r+\ell) + \ell - 1 \bmod 2^m \in \{0, \ldots, 2L-1\}$. Using also (7.10) and (7.11), we get

$$j(r+\ell) = \left\lfloor \frac{i(r+\ell) + r + \ell + 2^{m-3} - 1}{2^{m-3}} \right\rfloor \equiv 1 \bmod 8 \tag{7.18}$$

for $0 \le \ell < L$. Since $j \in \mathcal{I}_K$, this equation implies that the value $j(\ell)$ is constant for $\ell \in [r, r+L)$. By (7.4) we obtain (7.13) and (7.14) as soon as we show that $T^{(3)}_{0,2}(j) = T^{(3)}_{0,3}(j) = T^{(3)}_{0,4}(j) = T^{(3)}_{0,5}(j)$. Using (7.17) and (7.18) we have for $0 \le e \le 6$ and $0 \le \ell < L$

$$T^{(3)}_{0,e}(j)(\ell) = \left\lfloor \frac{j(\ell) + e}{8} \right\rfloor \le \left\lfloor \frac{7}{8} \right\rfloor = 0,$$
$$T^{(3)}_{0,e}(j)(r+\ell) = \left\lfloor \frac{j(r+\ell) + e}{8} \right\rfloor = \left\lfloor \frac{j(r) - 1}{8} + \frac{e+1}{8} \right\rfloor = \frac{j(r) - 1}{8}.$$

It remains to verify (7.15), which is clearly equivalent to

$$f^{(m),i}_{1,e'_1}\overline{f^{(m),i}_{1,e''_1}} = -f^{(m),i}_{1,e'_2}\overline{f^{(m),i}_{1,e''_2}},$$

since the weights have absolute value 1. By (7.4), (7.5) and the definition of e'_1, e'_2, e''_1, e''_2 this equation is equivalent to

$$f^{(3),j}_{0,2}\overline{f^{(3),j}_{0,3}} = -f^{(3),j}_{0,4}\overline{f^{(3),j}_{0,5}}. \tag{7.19}$$

By (7.9) we have, for any sequence $i \in \mathcal{I}_K$ and all $\delta, \varepsilon \in \{0, 1\}$,

$$f^i_{\delta,\varepsilon} = \mathrm{e}\left(\frac{1}{2} \sum_{\ell<L} a_\ell (i(\ell) - i(r+\ell)) \right).$$

Using this identity, (7.17), (7.18) and the assumption $a_0 = 1$ the verification of (7.19) is straightforward, which completes the proof. □

Let $d = (d_{\lambda-1} \cdots d_0)_2 < 2^\lambda$. We call a position μ *good* (a notion that depends on λ, d, i, r and m), if the following properties are satisfied.

(a) $0 \le \mu \le \lambda - m$.
(b) $(d_{\mu+m-1}, \ldots, d_{\mu+1}, d_\mu) = (0, \ldots, 0, 1)$.

(c) $T_{d,0}^{(\mu)}(i)(r) \equiv 1 \bmod 2^m$.

The point of this notion is that at each good position, using Lemma 7.2, we win a factor $1-\eta$ in the estimate of $\Phi(h,d)$. This argument is carried out in the following lemma.

Lemma 7.3. *Let $i \in \mathcal{I}_K$ and $k \ge 0$ and assume that $d < 2^\lambda$ is such that the number of good positions μ is at least k. Then*

$$\left|G_\lambda^i(h,d)\right|^2 \le (1-\eta)^k$$

holds for all $h \in \mathbb{Z}$ and $\eta = 2/4^m$.

Proof. Let $0 \le \mu_0 < \ldots < \mu_{k-1}$ be good positions. We set $\mu_k = \lambda$. The estimate (7.8) implies that

$$\left|G_\lambda^i(h,d)\right|^2 \le \max_{e<2^{\mu_0}} \left|G_{\lambda-\mu_0}^{T_{d,e}^{(\mu_0)}(i)}(h, \lfloor d/2^{\mu_0} \rfloor)\right|^2.$$

Let $0 \le j < k$. Since the position μ_j is good, we know by (c) and (7.3) that $T_{d,e}^{(\mu_j)}(r) \bmod 2^m \in \{1,2\}$ for all $e < 2^{\mu_j}$. Thus we can apply Lemma 7.2, the identity (7.4) and the estimate (7.8) and obtain

$$\begin{aligned}
&\max_{e<2^{\mu_j}} \left|G_{\lambda-\mu_j}^{T_{d,e}^{(\mu_j)}(i)}(h, \lfloor d/2^{\mu_j} \rfloor)\right|^2 \\
&\qquad \le \max_{e<2^{\mu_j}} (1-\eta) \max_{e'<2^m} \left|G_{\lambda-\mu_j-m}^{T_{1,e'}^{(m)} \circ T_{d,e}^{(\mu_j)}(i)}\left(h, \lfloor d/2^{\mu_j+m} \rfloor\right)\right|^2 \\
&\qquad = (1-\eta) \max_{e<2^{\mu_j+m}} \left|G_{\lambda-\mu_j-m}^{T_{d,e}^{(\mu_j+m)}(i)}\left(h, \lfloor d/2^{\mu_j+m} \rfloor\right)\right|^2 \\
&\qquad \le (1-\eta) \max_{e<2^{\mu_{j+1}}} \left|G_{\lambda-\mu_{j+1}}^{T_{d,e}^{(\mu_{j+1})}(i)}\left(h, \lfloor d/2^{\mu_{j+1}} \rfloor\right)\right|^2.
\end{aligned}$$

This proves the desired upper bound. □

In order to show that for most d there are many good positions, we have a closer look at condition (7.12).

Lemma 7.4. *Write $r = 2^x r_0$ with r_0 odd and assume that $y \ge 0$ and $0 \le d_0 < 2^y$. Let $i \in \mathcal{I}_K$. There exists a unique $d_1 \in \{0,\ldots,2^m-1\}$ such that for all $d_2 \in \{0,\ldots,2^{x-m}-1\}$ we have*

$$T_{d,0}^{(x+y)}(i)(r) \equiv 1 \bmod 2^m,$$

where $d = 2^{y+m}d_2 + 2^y d_1 + d_0$.

If $d_1' < 2^y$ is different from d_1, we have $T_{d,0}^{(x+y)}(i)(r) \not\equiv 1 \bmod 2^m$ for all $d_2 \in \{0,\ldots,2^{x-m}-1\}$.

Proof. Since r_0 is odd, the statements follow from

$$T_{d,0}^{(x+y)}(i)(r) = \left\lfloor \frac{i(r)}{2^{x+y}} + \frac{r_0 d_0}{2^y} \right\rfloor + r_0 d_1 + r_0 2^m d_2 \equiv \left\lfloor \frac{i(r)}{2^{x+y}} + \frac{r_0 d_0}{2^y} \right\rfloor + r_0 d_1 \bmod 2^m.$$

□

Note that the good-ness of a position μ does not depend on the digits of d with indices $\mu - x + m, \ldots, \mu - 1$. Let $\lambda \ge 0$. We decompose the set $\{0,\ldots,\lambda-1\}$ into intervals as follows. Consider the mutually disjoint sets of indices

$$\begin{aligned}
A_1 &= \{2\ell_1 x + \ell_0 m : 0 \le \ell_1 < \lfloor \lambda/(2x) \rfloor \text{ and } 0 \le \ell_0 < \lfloor x/m \rfloor\}, \\
A_2 &= \{(2\ell_1+1) x + \ell_0 m : 0 \le \ell_1 < \lfloor \lambda/(2x) \rfloor \text{ and } 0 \le \ell_0 < \lfloor x/m \rfloor\},
\end{aligned}$$

which form the starting points of intervals of length m. We call these intervals to be *of type 1* and *2* respectively. The integers in $[0,\lambda)$ not contained in an interval of type 1 or 2 form intervals *of type 3*, having total length $\lambda - 2m\lfloor\lambda/(2x)\rfloor\lfloor x/m\rfloor$. Assume that $\lambda \geq 2x$, which will be guaranteed by the hypotheses of Proposition 2.7. Then, beginning at 0, the resulting partition starts with $\lfloor x/m\rfloor$ intervals of type 1, followed possibly (if and only if $m \nmid x$) by an interval of type 3 , which fills up the gap up to position x. This is followed by $\lfloor x/m\rfloor$ intervals of type 2 and possibly an interval of type 3, reaching $2x$. This pattern continues up to $2x\lfloor\lambda/(2x)\rfloor$, the last interval of type 3 however extends up to λ.

Lemma 7.5. *Let M be a k-element subset of A_2. The number of $d < 2^\lambda$ such that M is the set of good positions in A_2 equals $2^{\lambda-2m\lambda_0}(2^{2m}-1)^{\lambda_0-k}$, where $\lambda_0 = |A_1| = |A_2| = \lfloor\lambda/(2x)\rfloor\lfloor x/m\rfloor$.*

Proof. We construct recursively the set of admissible $d = (d_{\lambda-1}\cdots d_0)_2 < 2^\lambda$. In order to do so, we let μ run through the set $A_1 \cup A_2 \cup A_3$ in ascending order and choose digits of d in such a way that all digits up to position μ have already been chosen when we reach μ. If we encounter an index $\mu \in A_1$, we set the $2m$ digits $d_\mu,\ldots,d_{\mu+m-1},d_{\mu+x},\ldots,d_{\mu+x+m-1}$ according to two cases: if $\mu + x \in M$, we have to guarantee good-ness of the position $\mu + x$, we therefore set $(d_{\mu+m-1},\ldots,d_\mu)$ according to Lemma 7.4 and $(d_{\mu+x+m-1},\ldots,d_{\mu+x}) = (0,\ldots,0,1)$. If $\mu + x \notin M$, we may choose any of the remaining $2^{2m}-1$ possibilities for these $2m$ digits, such that the position $\mu + x$ will not be good. If we encounter $\mu \in A_2$, we do nothing, since the corresponding block of length m has already been filled. If we find an index $\mu \in A_3$, it is the starting point of an interval I of type 3. We may set the digits of d with indices in I freely. Therefore we can choose $\lambda - 2m|A_1|$ digits arbitrarily, and $|A_1| - k$ times we may choose out of $2^{2m}-1$ possibilities. This implies the statement of the lemma. □

The proof of Proposition 2.7 is now easy.

Proof of Proposition 2.7. Let $\lambda_0 = \lfloor\lambda/(2x)\rfloor\lfloor x/m\rfloor$. It follows from Lemma 7.5 that there are precisely $\binom{\lambda_0}{k}2^{\lambda-2m\lambda_0}\bigl(2^{2m}-1\bigr)^{\lambda_0-k}$ integers $d \in \{0,\ldots,2^\lambda-1\}$ such that exactly k positions from A_2 are good. Since each d occurs for some $k \leq \lambda_0$, Lemma 7.3 implies

$$\begin{aligned}\sum_{d<2^\lambda}\max_{h<2^\lambda}\left|G_\lambda^i(h,d)\right|^2 &\leq 2^{\lambda-2m\lambda_0}\sum_{k=0}^{\lambda_0}\binom{\lambda_0}{k}(2^{2m}-1)^{\lambda_0-k}(1-\eta)^k\\ &= 2^\lambda(1-2/16^m)^{\lambda_0}.\end{aligned}$$

By (7.11) we obtain

$$\lambda_0 = \lfloor\lambda/(2x)\rfloor\lfloor x/m\rfloor \geq \frac{\lambda-2x}{2x}\frac{x-m}{m} \geq \frac{\lambda/2}{2x}\frac{x/2}{m} = \frac{\lambda}{8m}.$$

This finishes the proof. □

8. Proof of Proposition 2.8

The following elementary lemma summarizes (and extends) Lemmas 9 through 11 from [22].

Lemma 8.1. *Let a, b be real numbers such that $a \leq b$ and set $K = b - a$. Assume that $f : [a,b] \to \mathbb{R}$ is twice differentiable and $|f''| \leq B$. Then for all $\alpha \in f'([a,b])$ the following statements hold.*

(i) For $a \leq x \leq b$ we have

$$|x\alpha + f(a) - a\alpha - f(x)| \leq BK^2.$$

(ii) If $x \in [a,b]$ is such that $\|x\alpha + f(a) - a\alpha\| > BK^2$, then

$$\lfloor f(x)\rfloor = \lfloor x\alpha + f(a) - a\alpha\rfloor.$$

(iii) If a,b are integers, $L \ge 1$ is an integer, $f : [a, b+L-1] \to \mathbb{R}$ is twice differentiable, $\alpha \in f'([a,b])$ and $|f''| \le B$, then

$$\begin{aligned}|\{n \in (a,b] : \lfloor f(n+\ell)\rfloor \ne \lfloor (n+\ell)\alpha + f(a) - a\alpha\rfloor \text{ for some } \ell < L\}|\\ \le 2B(K+L-1)^3L + KLD_K(\alpha).\end{aligned}$$

We prove Proposition 2.8. For convenience, let $\varphi(n) = 0$ for $n \le 0$ and set $f(n) = n^c$. We follow the proof of [22, Proposition 1]. By analogous considerations as given there, we may assume that K is an integer and that $2 \le K \le N$.

Define integral partition points $a_i = \lceil N\rceil + iK$ for $i \ge 0$ and set $M = \max\{i : a_i + L - 1 \le 2N\}$. The integer M satisfies the estimate $KM \le N$. We have the decomposition

$$(N, 2N] = (N, \lceil N\rceil] \cup \bigcup_{0\le i<M} (a_i, a_{i+1}] \cup (a_M, 2N]. \tag{8.1}$$

Let $\alpha \in \mathbb{R}$. Then by the triangle inequality and the relation $a_{i+1} - a_i = K$ we have for $i < M$

$$\begin{aligned}\big||\{n \in (a_i, a_{i+1}] : \varphi\left(\lfloor (n+\ell)^c\rfloor\right) = \omega_\ell \text{ for } 0 \le \ell < L\}| - K\delta\big|\\ \le T_1(\alpha, i) + T_2(\alpha, i),\end{aligned} \tag{8.2}$$

where

$$\begin{aligned}T_1(\alpha, i) = &\big||\{n \in (a_i, a_{i+1}] : \varphi\left(\lfloor (n+\ell)^c\rfloor\right) = \omega_\ell \text{ for } 0 \le \ell < L\}|\\ &- |\{n \in (a_i, a_{i+1}] : \varphi\left(\lfloor (n+\ell)\alpha + f(a_i) - a_i\alpha\rfloor\right) = \omega_\ell \text{ for } 0 \le \ell < L\}|\big|\\ T_2(\alpha, i) = &\big||\{n \in (a_i, a_{i+1}] : \varphi\left(\lfloor (n+\ell)\alpha + f(a_i) - a_i\alpha\rfloor\right) = \omega_\ell \text{ for } 0 \le \ell < L\}|\\ &- K\delta\big|\end{aligned}$$

We integrate both sides of (8.2) in the variable α from $f'(a_i)$ to $f'(a_{i+1})$, divide by the length of the integration range, and take the sum over i from 0 to $M-1$, which yields

$$\begin{aligned}\big||\{n \in (a_0, a_M] : \varphi\left(\lfloor (n+\ell)^c\rfloor\right) = \omega_\ell \text{ for } 0 \le \ell < L\}| - MK\delta\big|\\ \le \sum_{0\le i<M} \frac{1}{f'(a_{i+1}) - f'(a_i)} \int_{f'(a_i)}^{f'(a_{i+1})} \Big(T_1(\alpha, i) + T_2(\alpha, i)\Big)\,\mathrm{d}\alpha.\end{aligned} \tag{8.3}$$

We estimate the first summand. If $0 \le i < M$ and $\alpha \in f'([a_i, a_{i+1}])$, Lemma 8.1 gives

$$T_1(\alpha, i) \le 2f''(N)(K+L-1)^3L + LKD_K(\alpha). \tag{8.4}$$

By the Mean Value Theorem we have

$$\frac{1}{f'(a_{i+1}) - f'(a_i)} \ll \frac{N}{K}\frac{1}{f'(2N) - f'(N)} \tag{8.5}$$

for $0 \le i < M$. Using this and the integral mean discrepancy estimate (3.3) we obtain

$$\begin{aligned}&\sum_{0\le i<M} \frac{1}{f'(a_{i+1}) - f'(a_i)} \int_{f'(a_i)}^{f'(a_{i+1})} D_K(\alpha)\,\mathrm{d}\alpha\\ &\le \frac{N}{K}\frac{1}{f'(2N) - f'(N)} \sum_{0\le i<M} \int_{f'(a_i)}^{f'(a_{i+1})} D_K(\alpha)\,\mathrm{d}\alpha\\ &\le \frac{N}{K}\frac{f'(2N) - f'(N) + 1}{f'(2N) - f'(N)} \int_0^1 D_K(\alpha)\,\mathrm{d}\alpha\\ &\ll \Big(N + \frac{1}{f''(N)}\Big)\frac{\left(\log^+ K\right)^2}{K^2}.\end{aligned}$$

By the estimates $KM \le N$ and $N \ge C/f''(N)$ this implies

$$\sum_{0\le i<M} \frac{1}{f'(a_{i+1})-f'(a_i)} \int_{f'(a_i)}^{f'(a_{i+1})} T_1(\alpha,i)\,\mathrm{d}\alpha \le C_1 N L\biggl(f''(N)K^2 + \frac{\bigl(\log^+ N\bigr)^2}{K}\biggr)$$

for some constant C_1 depending on c and L. We turn our attention to the second summand in (8.3). Inserting (8.5) and the definition of $T_2(\alpha,i)$, we easily obtain

$$\sum_{0\le i<M} \frac{1}{f'(a_{i+1})-f'(a_i)} \int_{f'(a_i)}^{f'(a_{i+1})} T_2(\alpha,i)\,\mathrm{d}\alpha \ll N\,J(N,K). \tag{8.6}$$

Estimating also the contributions of the first and the last interval in (8.1) trivially and collecting the error terms, we obtain

$$\left|\frac{1}{N}\bigl|\bigl\{n\in(N,2N] : \varphi\bigl(\lfloor (n+\ell)^c\rfloor\bigr)=\omega_\ell \text{ for } 0\le \ell<L\bigr\}\bigr| - \delta\right| \le C_2\biggl(f''(N)K^2 + \frac{(\log N)^2}{K} + J(N,K) + \frac{K}{N}\biggr),$$

for some C_2 depending on c and L. By the estimate $f''(N)\ge C/N$ the last term is dominated by the first, which finishes the proof of Proposition 2.8.

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