Normalized solutions of nonlinear Schrödinger equations

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Abstract

We consider the problem

$$\begin{cases} -\Delta u - g(u) = \lambda u, \\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} u^2 = 1,\ \lambda \in \mathbb{R}, \end{cases}$$

in dimension $N \geq 2$. Here g is a superlinear, subcritical, possibly nonhomogeneous, odd nonlinearity. We deal with the case where the associated functional is not bounded below on the L^2-unit sphere, and we show the existence of infinitely many solutions.

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1 Introduction

In this note we consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta u - g(u) = \lambda u, \\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} u^2 = 1,\ \lambda \in \mathbb{R}, \end{cases} \tag{1.1}$$

in dimension $N \geq 2$. The nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is superlinear, subcritical, and possibly nonhomogeneous. A model nonlinearity is

$$g(u) = \left(\sum_{i=1}^{k} |u|^{p_i-2} \right) u, \quad 2 < p_1 < \ldots < p_k < 2^*, \tag{1.2}$$

where $2^* = 2N/(N-2)$ if $N \geq 3$ and ∞ if $N = 2$, the critical Sobolev exponent.

This problem possesses many physical motivations, e. g. it appears in models for Bose-Einstein condensation (see [9]). Looking for standing wave solutions $\Psi(t,x) = e^{imt}u(x)$ of the dimensionless nonlinear Schrödinger equation

$$i\Psi_t - \Delta_x \Psi = f(|\Psi|)\Psi$$

one is lead to problem (1.1) with $g(u) = f(|u|)u$. As in these physical frameworks Ψ is a wave function, it seems natural to search for *normalized* solutions, i. e. solutions of the equation satisfying $\int_{\mathbb{R}^N} u^2 = 1$.

If g is homogeneous ($k = 1$ in (1.2)) then one can use the classical results from [3,4], for instance, to solve $-\Delta u + u = g(u)$, and then rescale u in order to obtain normalized solutions of (1.1). This does not work for a general nonlinearity, it fails already in the case $k \geq 2$ in (1.2). If g is not homogeneous and does not grow too fast (for g as in (1.2) this means all $p_i < 2 + \frac{4}{N}$) then one can minimize the associated functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u), \quad \text{with } G(t) = \int_0^t g(s)\,ds, \tag{1.3}$$

on the L^2-unit sphere $S = \{u \in H^1_{\text{rad}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = 1\}$ to obtain a solution. Here $H^1_{\text{rad}}(\mathbb{R}^N)$ denotes the space of radial H^1-functions. The parameter λ appears as Lagrange multiplier. Rather general conditions on g which allow minimization, even in a nonradial setting, can be found in [7] and the references therein. If g is odd, as in the case $g(u) = f(|u|)u$ appearing in applications, and if g does not grow too fast then one can obtain infinitely many solutions using classical min-max arguments based on the Krasnoselski genus.

However for fast growing g, J is not bounded below on S, hence minimization doesn't work. Moreover, the genus of the sublevel sets $J^c = \{u \in S : J(u) \leq c\}$ is always infinite, so the Krasnoselski genus arguments do not apply. In [8], Jeanjean was able to treat nonhomogeneous, fast growing nonlinearities and showed the existence of one solution of (1.1) using a mountain pass structure for J on S. The object of this short note is to prove that for the same class of nonlinearities considered in [8], (1.1) actually has infinitely many solutions.

In order to state our result we recall the assumptions on the function g made in [8]:

(H_1) $g : \mathbb{R} \to \mathbb{R}$ is continuous and odd,

(H_2) there exists $\alpha, \beta \in \mathbb{R}$ satisfying

$$2 + \frac{4}{N} < \alpha \leq \beta < 2^*$$

such that

$$0 < \alpha G(s) \leq g(s)s \leq \beta G(s).$$

The condition $G > 0$ in (H_2) is not stated in [8] but used implicitely.

Theorem 1.1. *If assumptions* (H_1) *and* (H_2) *hold, then problem* (1.1) *possesses an unbounded sequence of pairs of radial solutions* $(\lambda_n, \pm u_n)$.

The proof is based on variational methods applied to the functional J constrained to S. We shall present a new linking geometry for constrained functionals which is motivated by the fountain theorem [2, Theorem 2.5]; see also [10, Section 3]. The classical symmetric mountain pass theorem applies to functionals on Banach spaces, not on spheres. Another difficulty due to the constraint is that $J|_S$ does not satisfy the Palais-Smale condition although the embedding $H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ of the space of radial H^1-functions into the L^p-spaces is compact for $2 < p < 2^*$. In fact, there exist bounded Palais-Smale sequences for $J|_S$ converging weakly to 0, and there may exist unbounded Palais-Smale sequences.

2 Proof of Theorem 1.1

In order to recover some compacity, we will work in $E = H^1_{\text{rad}}(\mathbb{R}^N)$, provided with the standard scalar product and norm: $\|u\|^2 = |\nabla u|_2^2 + |u|_2^2$. Here and in the sequel we write $|u|_p$ to denote the L^p-norm. As we look for normalized solutions, we consider the functional J constrained to the L^2-unit sphere in E:

$$J_S : S = \{u \in E : |u|_2 = 1\} \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).$$

Observe that $\nabla J_S(u) = \nabla J(u) - \lambda_u u$ for some $\lambda_u \in \mathbb{R}$.

The main theorem's proof will follow from several lemmas. We fix a strictly increasing sequence of finite-dimensional linear subspaces $V_n \subset E$ such that $\bigcup_n V_n$ is dense in E.

Lemma 2.1. *For* $2 < p < 2^*$ *there holds:*

$$\mu_n(p) = \inf_{u \in V_{n-1}^\perp} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)}{\left(\int_{\mathbb{R}^N} |u|^p\right)^{2/p}} = \inf_{u \in V_{n-1}^\perp} \frac{\|u\|^2}{|u|_p^2} \to \infty \quad \text{as } n \to \infty.$$

Proof. Arguing by contradiction, suppose there exists a sequence $(u_n) \subset E$ such that $u_n \in V_{n-1}^\perp$, $|u_n|_p = 1$ and $\|u_n\| \to c < \infty$. Then there exists $u \in E$ with $u_n \rightharpoonup u$ in E and $u_n \to u$ in L^p up to a subsequence. Let $v \in E$ and $(v_n) \subset E$ such that $v_n \in V_{n-1}$ and $v_n \to v$ in V. We have, in E,

$$|\langle u_n, v\rangle| \le |\langle u_n, v - v_n\rangle| + |\langle u_n, v_n\rangle| \le \|u_n\|\|v - v_n\| \to 0$$

so that $u_n \rightharpoonup 0 = u$, while $|u|_p = 1$, a contradiction. □

We introduce now the constant

$$K = \max_{x>0} \frac{|G(x)|}{|x|^\alpha + |x|^\beta},$$

which is well defined thanks to assumption (H_2). For $n \in \mathbb{N}$ we define

$$\rho_n = \frac{M_n^{\beta/(2(\beta-2))}}{L^{1/(\beta-2)}},$$

where

$$M_n = \left(\mu_n(\alpha)^{-\alpha/2} + \mu_n(\beta)^{-\beta/2}\right)^{-2/\beta} \quad \text{and} \quad L = 3K \max_{x>0} \frac{(1+x^2)^{\beta/2}}{1+x^\beta}.$$

We also define

$$B_n = \left\{u \in V_{n-1}^\perp \cap S : |\nabla u|_2 = \rho_n\right\}.$$

Then we have:

Lemma 2.2. $\inf_{u \in B_n} J(u) \to \infty$ *as* $n \to \infty$.

Proof. For any $u \in B_n$, we deduce, using the preceding lemma with $p = \alpha$ and $p = \beta$,

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u) \ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - K\int_{\mathbb{R}^N} |u|^\alpha - K\int_{\mathbb{R}^N} |u|^\beta \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{K}{\mu_n(\alpha)^{\alpha/2}}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + 1\right)^{\alpha/2} \\
&\qquad - \frac{K}{\mu_n(\beta)^{\beta/2}}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + 1\right)^{\beta/2} \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{K}{M_n^{\beta/2}}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + 1\right)^{\beta/2} \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{L}{3M_n^{\beta/2}}\left(\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\beta/2} + 1\right) \\
&= \frac{1}{2}\rho_n^2 - \frac{L}{3M_n^{\beta/2}}\rho_n^\beta + o(1) = \left(\frac{1}{2} - \frac{1}{3}\right)\rho_n^2 + o(1) \to \infty.
\end{aligned}$$

□

Let $P_{n-1} : E \to V_{n-1}$ be the orthogonal projection, and set

$$h_n : S \to V_{n-1} \times \mathbb{R}^+, \quad u \mapsto \left(P_{n-1}u, |\nabla u|_2\right).$$

Then clearly $B_n = h_n^{-1}(0, \rho_n)$. With $\pi : V_{n-1} \times \mathbb{R}^+ \to \mathbb{R}^+$ denoting the projection we define

$$\Gamma_n = \Big\{\gamma : [0,1] \times (S \cap V_n) \to S \mid \gamma \text{ is continuous, odd in } u \text{ and such that} \\ \forall u:\ \pi \circ h_n \circ \gamma(0,u) < \rho_n/2,\ \pi \circ h_n \circ \gamma(1,u) > 2\rho_n\Big\}.$$

It is easy to see that $\Gamma_n \neq \emptyset$. To describe a particular element $\gamma \in \Gamma_n$, let

$$m : \mathbb{R} \times E \to E, \quad m(s,u) = s * u,$$

be the action of the group $\mathbb{R}$ on E defined by

$$(s * u)(x) = e^{sN/2} u(e^s x) \quad \forall s \in \mathbb{R},\ u \in E,\ x \in \mathbb{R}^N.$$

Observe that $s * u \in S$ if $u \in S$. The map $\gamma(t,u) = (2s_n t - s_n) * u$ lies in Γ_n for $s_n > 0$ large.

We now need the following linking property.

Lemma 2.3. *For every $\gamma \in \Gamma_n$, there exists $(t,u) \in [0,1] \times (S \cap V_n)$ such that $\gamma(t,u) \in B_n$.*

For the proof of this lemma we need to recall some properties of the cohomological index for spaces with an action of the group $G = \{-1, 1\}$. This index goes back to [5] and has been used in a variational setting in [6]. It associates to a G-space X an element $i(X) \in \mathbb{N}_0 \cup \{\infty\}$. We only need the following properties.

(I_1) If G acts on $\mathbb{S}^{n-1}$ via multiplication then $i(\mathbb{S}^{n-1}) = n$.

(I_2) If there exists an equivariant map $X \to Y$ then $i(X) \leq i(Y)$.

(I_3) Let $X = X_0 \cup X_1$ be metrisable and $X_0, X_1 \subset X$ be closed G-invariant subspaces. Let Y be a G-space and consider a continuous map $\phi : [0,1] \times Y \to X$ such that each $\phi_t = \phi(t, \cdot) : Y \to X$ is equivariant. If $\phi_0(Y) \subset X_0$ and $\phi_1(Y) \subset X_1$ then

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \geq i(Y).$$

Properties (I_1) and (I_2) are standard and hold also for the Krasnoselskii genus. Property (I_3) has been proven in [1, Corollary 4.11, Remark 4.12]. We can now prove Lemma 2.3.

Proof. We fix $\gamma \in \Gamma_n$, and consider the map

$$\phi = h_n \circ \gamma : [0,1] \times (S \cap V_n) \to V_{n-1} \times \mathbb{R}^+ =: X.$$

Since

$$\phi_0(S \cap V_n) \subset V_{n-1} \times (0, \rho_n] =: X_0$$

and

$$\phi_1(S \cap V_n) \subset V_{n-1} \times [\rho_n, \infty) =: X_1,$$

it follows from (I_1) – (I_3) that

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \geq i(S \cap V_n) = \dim V_n.$$

If there would not exist $(t,u) \in [0,1] \times (S \cap V_n)$ with $\gamma(t,u) \in B_n$, then

$$\mathrm{Im}(\phi) \cap X_0 \cap X_1 \subset (V_{n-1} \setminus \{0\}) \times \{\rho_0\}.$$

Now (I_1), (I_2) imply that

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \leq i((V_{n-1} \setminus \{0\}) \times \{\rho_0\}) = \dim V_{n-1},$$

contradicting $\dim V_{n-1} < \dim V_n$. □

It follows from Lemma 2.3 that

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{\substack{t \in [0,1] \\ u \in S \cap V_n}} J(\gamma(t,u)) \geq \inf_{u \in B_n} J(u) \to \infty. \tag{2.1}$$

We will show that c_n is a critical value of J, which finishes the proof of Theorem 1.1. We fix n from now on.

Lemma 2.4. *There exists a Palais-Smale sequence $(u_k)_k$ for J_S at the level c_n satisfying*

$$|\nabla u_k|_2^2 + N \int_{\mathbb{R}^N} G(u_k) - \frac{N}{2} \int_{\mathbb{R}^N} g(u_k) u_k \to 0. \tag{2.2}$$

For the proof we recall the stretched functional from [8]:

$$\tilde{J} : \mathbb{R} \times E \to \mathbb{R}, \quad (s,u) \mapsto J(s * u).$$

Now we define

$$\tilde{\Gamma}_n = \Big\{ \tilde{\gamma} : [0,1] \times (S \cap V_n) \to \mathbb{R} \times S \mid \tilde{\gamma} \text{ is continuous, odd in } u, \text{ and such that } m \circ \tilde{\gamma} \in \Gamma_n \Big\},$$

where $m(s,u) = s * u$, and

$$\tilde{c}_n = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{\substack{t \in [0,1] \\ u \in S \cap V_n}} \tilde{J}(\tilde{\gamma}(t,u)).$$

Lemma 2.5. *We have $\tilde{c}_n = c_n$.*

Proof. The maps

$$\Phi : \Gamma_n \to \tilde{\Gamma}_n, \quad \gamma \mapsto [(0,\gamma) : \ (t,u) \mapsto (0,\gamma(t,u))],$$

and

$$\Psi : \tilde{\Gamma}_n \to \Gamma_n, \quad \tilde{\gamma} \mapsto [m \circ \gamma : \ (t,u) \mapsto m(\tilde{\gamma}(t,u))],$$

satisfy

$$\tilde{J}(\Phi(\gamma)(t,u)) = J(\gamma(t,u)), \quad \text{and} \quad J(\Psi(\tilde{\gamma})(t,u)) = \tilde{J}(\tilde{\gamma}(t,u)).$$

The lemma is an immediate consequence. □

Proof of Lemma 2.4. By Ekeland's variational principle there exists a Palais-Smale sequence $(s_k, u_k)_k$ for $\tilde{J}|_{\mathbb{R}\times S}$ at the level c_n. From $\tilde{J}(s,u) = \tilde{J}(0, s * u)$ we deduce that $(0, s_k * u_k)_k$ is also a Palais-Smale sequence for $\tilde{J}|_{\mathbb{R}\times S}$ at the level c_n. Thus we may assume that $s_k = 0$. This implies, firstly, that $(u_k)_k$ is a Palais-Smale sequence for J_S at the level c_n, and secondly, using $\partial_s \tilde{J}(0, u_k) \to 0$, that (2.2) holds. □

Lemma 2.6. *If the sequence $(u_k)_k$ in S satisfies $J'_S(u_k) \to 0$, $J_S(u_k) \to c > 0$, and* (2.2), *then it is bounded and has a convergent subsequence.*

Proof. That $(u_k)_k$ is bounded in E, hence $u_k \rightharpoonup \bar{u}$ along a subsequence, can be proved as in [8, pp. 1644-1644]. The compactness of the embedding $H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ yields $g(u_k) \to g(\bar{u})$ in E^*. From $J'_S(u_k) \to 0$ it follows that

$$-\Delta u_k - \lambda_k u_k - g(u_k) \to 0 \quad \text{in } E^* \tag{2.3}$$

for some sequence $\lambda_k \in \mathbb{R}$. Using $J_S(u_k) \to c > 0$ and (2.2), we deduce as in [8, Lemma 2.5] that $\lambda_k \to \bar{\lambda} < 0$ along a subsequence. Then $-\Delta - \bar{\lambda}$ is invertible and (2.3) implies $u_k \to (-\Delta - \bar{\lambda})^{-1}(g(\bar{u}))$ in E. □

Theorem 1.1 follows from (2.1), Lemma 2.4 and Lemma 2.6.

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