Normalized Variable and Space Formulation Methodology for High-Resolution Schemes

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ABSTRACT

The Normalized Variable Formulation (NVF) methodology of Leonard [1] provides the proper framework for the development and analysis of High-Resolution convection-diffusion schemes, which combine the accuracy of Higher-Order schemes, with the stability and boundedness of the first-order upwind scheme. However, in its current form the NVF methodology helps in deriving convective schemes for uniformly or nearly uniformly discretized spaces. To remove this shortcoming, a new Normalized Variable and Sapce Formulation methodology is developed (NVSF). In the newly developed technique, spatial parameters are introduced so as to extend the applicability of the NVF methodology to non-uniformly discretized domains. Furthermore, the required conditions for accuracy and boundedness of convective schemes on non-uniform grids are also derived. Several schemes formulated using NVF, are generalized to non-uniform grid using the suggested method. Both formulations are tested on non-uniform grids by solving two problems. Computational results show substantial improvement in accuracy when using the NVSF methodology with third order High-Resolution schemes.

NOMENCLATURE

A	Surface area of control volume face.
a	Coefficients of the discretized equation.
b	Source term in the discretized equation.
F	Convective flux coefficient at control volume face.
f()	Functional relationship.
J	Total scalar flux across cell face.
M	Slope in NVD.
Q	Source term integrated over one control volume in the discretized equation.
RE	Residual error.
S	Source term.
u, v	Velocity components in the x- and y- directions.
ϕ	General dependent variable..
Γ	Diffusion.
ρ	Density.

SUPERSCRIPS

U	Upwind formulation.
D	Diffusion contribution.
C	Convection contribution.
˜	Refers to normalized variable.

SUBSCRIPTS

e,w,n,s	Refers to control volume faces.
E,W,N,S	Refers to neighbours of P grid point.
P	Main grid point
f	Refers to control volume face.
U	Upstream grid point.
D	Downstream grid point.
C	Central grid point.
nb	Refers to neighbours.
dc	Deferred correction

INTRODUCTION

Since the development of the first order upwind scheme [2], used for discretizing the convective flux, workers have tried to devise schemes of higher order of accuracy. However, successful in solving the accuracy problem, they were faced with more complex issues of numerical stability and physical boundedness. This, in turn, has lead to further studies in an attempt to develop more accurate schemes that give physically plausible results (bounded schemes) and which are numerically stable.

First order schemes (e.g. upwind [3]) are numerically stable but highly diffusive in situations when the flow field is oblique to the grid lines in combination with a non-zero gradient of the dependent variable in the direction normal to the flow. The numerical diffusion introduced as a result of first order upwinding is desirable for numerical stability but often leads to highly inaccurate results and causes smearing of sharp gradients. To overcome this shortcoming and to increase the accuracy of the predicted results, researchers have developed a variety of higher-order schemes such as the QUICK scheme of Leonard [4], the third order scheme of Agarwal [5], and the second order upwind scheme of Shyy [6] to cite a few. The difficulties associated with the development of reliable higher-order schemes stem from the conflicting requirements of accuracy, stability, and boundedness. Solutions predicted with the above mentioned schemes are more accurate than the first order upwind scheme and more stable than the second order central difference scheme, but tend to provoke oscillations in the solution when the local Peclet number is high in combination with steep gradients of the flow properties.

To suppress oscillations associated with higher order schemes many techniques have been advertised and may be broadly classified into two groups which will be referred to as the flux blending method and the composite flux limiter method in this paper. Both methods attempt to suppress oscillations in the predicted solutions, without affecting the accuracy, but do so in different ways.

In the flux blending method either an anti-diffusive flux is added to a first order upwind scheme such that the resulting scheme is capable of resolving sharp gradients without oscillations (under/over shoots) or some kind of diffusive agencies are introduced into an unbounded higher order scheme to damp oscillations. The Flux Corrected Transport (FCT) method of Zalesak [7] is an example of the first type, and the Filtering Remedy And Methodology (FRAM) of Chapman [8], the flux blending method of Peric [9], and the method of Zu and Leschziner [10] are examples of the second type. The determination of the blending factor is critical to the successful application of such method. Furthermore, because of their multistep nature, flux-blending techniques tend to be very expensive computationally and are often unable to provide the desired "optimum blend" between accuracy and boundedness. Although flux-blending methods are much more accurate than the first order upwind scheme, they still generate some diffusion when attempting to simulate sharp gradients.

In the composite flux limiter approach, the numerical flux at the interface of the computational cell is modified by employing a flux limiter that enforces a boundedness criterion. The family of schemes based on the Total Variational Diminishing flux limiters (TVD) [11] used in aerodynamic simulations for capturing shock waves are examples of this approach. Leonard [12] has recently generalized the formulation of the high resolution flux-limiter schemes using what is called the Normalized Variable Formulation (NVF). The NVF methodology has provided a good framework for the development of HR schemes that combine simplicity of implementation with high accuracy and boundedness, and since its introduction in 1981 [13] has attracted many workers (Zhu and Rodi [14], Gaskell and Lau [15], Zhu [16], Lin and Chieng [17], and Darwish [18]).

A literature survey reveals that all composite high-resolution schemes based on the NVF methodology have been formulated on the assumption of a uniform grid in each coordinate direction. This has greatly hindered its application for problems involving distorted or non-Cartesian grids. One objective of the paper is to extend this formulation, in the context of the finite volume method, into situations where the grid, in any coordinate direction, is non-uniform. As will be shown later, this amounts to normalizing the space coordinates, and hence the acronym NVSF (Normalized Variable and Space Formulation) has been coined for the extended NVF methodology. A second objective of the paper is to compare, in terms of accuracy, the results of some test problems obtained using both the NVF and NVSF formulations.

DISCRETIZATION OF THE TRANSPORT EQUATION

The equation expressing the conservation principle for a general specific property ϕ for a two-dimensional, steady state situation can be written in the following form,

$$\nabla .(\rho \mathbf{v}\phi) = \nabla .(\Gamma \nabla \phi) + S \quad (1)$$

where Γ is the diffusion coefficient and S is the source term. If $\mathbf{J}$ denotes the total flux i.e.,

$$\mathbf{J} = \rho \mathbf{v}\phi - \Gamma \nabla \phi \qquad (2)$$

then equation (1) is transformed to

$$\nabla .\mathbf{J} = S \, . \qquad (3)$$

Adopting the control volume approach and using Cartesian coordinates, the discretized flux-conservation equation is obtained by integrating equation (3) over the control volume surrounding point P (Fig. 1), and for the total flux term transforming the volume integral to a surface integral using the divergence theorem. Its final form is given by

$$J_e - J_w + J_n - J_s = S_P \, , \qquad (4)$$

where J_f represents the total convective-diffusive flux of ϕ across cell face f (f=e, w, n, or s), and S_P is the average source term over the control volume surrounding point P. By denoting the convective and diffusive contributions to the total flux by J_f^C and J_f^D respectively, the total convective-diffusive flux J_f may be expressed as:

$$J_f = J_f^C + J_f^D \, . \qquad (5)$$

The diffusive flux is discretized using a second order central difference scheme. For a cartesian coordinate system the diffusion flux J_e^D for the east face e, is given by:

$$J_e^D = \;= \Gamma_e \, A_e \left(\frac{\partial \phi}{\partial x}\right)_e = \Gamma_e \, A_e \frac{\phi_E - \phi_P}{\Delta x} \qquad (6)$$

The fluxes along the west, north and south faces are found in a similar manner.

The discretization of the convective flux however, requires a special attention and is the subject of the various schemes developed. The mathematical representation of the convective flux is

$$J_f^C = (\rho \mathbf{v}.\mathbf{A})_f \; \phi_f = F_f \; \phi_f \tag{7}$$

where $\mathbf{A}_f$ is the surface of cell face f, and F_f is the mass flow rate across cell face f. The value of the dependent variable ϕ_f is not known and should be estimated, using an interpolation procedure, from the values at the main neighbouring grid points. Therefore, the accuracy, stability, and boundedness of the solution depends on the procedure used. In general, ϕ_f can be explicitly formulated in terms of its neighbouring node values by a functional relationship of the form:

$$\phi_f = f(\phi_{nb}) \tag{8}$$

where the subscript nb designates neighbouring grid points. Combining equations (3) through (8), the discretized flux-conservation equation becomes:

$$\left\{J_e^D + F_e\left[f(\phi_{nb})\right]_e\right\} - \left\{J_w^D + F_w\left[f(\phi_{nb})\right]_w\right\} + \left\{J_n^D + F_n\left[f(\phi_{nb})\right]_n\right\} - \left\{J_s^D + F_s\left[f(\phi_{nb})\right]_s\right\} = S_P \tag{9}$$

With higher order schemes, the evaluation of ϕ_f may involve a large number of neighbouring grid points. Therefore, in order to simplify the solution of the resulting system of algebraic equations a compacting procedure is usually used. The deferred correction procedure, of Rubin and Khosla [19], adopted in this work is based on replacing the convective flux at the control volume face by an equivalent flux given by:

$$J_f^C = F_f \; \phi_f = F_f \; \phi_f^U - F_f (\phi_f^U - \phi_f) \tag{10}$$

where the superscript U denotes values obtained using the first order upwind scheme, and ϕ_f represents cell face value computed using a high resolution scheme. By combining equations (9) and (10) the conservation equation is transformed to,

$$\begin{aligned}\left\{J_e^D + F_e \phi_e^U\right\} - \left\{J_w^D + F_w \phi_w^U\right\} + \left\{J_n^D + F_n \phi_n^U\right\} - \left\{J_s^D + F_s \phi_s^U\right\} = \\ S_P + \left[F_e (\phi_e^U - \phi_e) - F_w (\phi_w^U - \phi_w) + F_n (\phi_n^U - \phi_n) - F_s (\phi_s^U - \phi_s)\right]\end{aligned} \tag{11}$$

With the above treatment, each discretized equation contains five unknowns (in 2-D) and the matrix of coefficients of the resulting system of equations is pentadiagonal and always diagonally dominant since it is formed using the first order upwind scheme. Upon expanding equation (11) in terms of nodal values, the final form of the discretized equation is given as:

$$a_P \; \phi_P = a_E \; \phi_E + a_W \; \phi_W + a_N \; \phi_N + a_S \; \phi_S + b_P + b_{dc} \tag{12}$$

where a_P , a_E , a_W , a_N , and a_S are the convection-diffusion coefficients obtained from a first order upwind discretization, b_P is the original source term contribution, and b_{dc} is the contribution due to the adopted deferred correction procedure. In calculating source terms, the latest available values of the dependent variable are used.

NORMALIZED VARIABLE FORMULATION ON NON-UNIFORM GRID

Many of the simple and composite high resolution convective schemes are reformulated, in this section, on non-uniform grid using the NVSF methodology. The normalized variables needed in the derivations are first defined followed by the derived expressions of some simple unbounded higher order schemes. Then, the required conditions for accuracy, numerical stability, and physical boundedness are discussed. Finally, the functional relationships of several composite high resolution schemes are presented.

NORMALIZED VARIABLES

The derivations are pertinent to second and third order convective schemes involving the use of three neighbouring grid points (2 upstream and 1 downstream) surrounding the control volume face f. As shown in Fig. 2, the upstream, central, and downstream grid points designated by U, C, and D, are located at distances x_U , x_C , and x_D from the origin, respectively. The values of the dependent variable at these nodes are designated by ϕ_U , ϕ_C and ϕ_D . Moreover, the value of the dependent variable at the control volume face located at a distance x_f from the origin is expressed by ϕ_f . Since a normalized variable and space formulation is sought, the following normalized variables are defined:

$$\tilde{\phi} = \frac{\phi - \phi_U}{\phi_D - \phi_U} \qquad \tilde{x} = \frac{x - x_U}{x_D - x_U} \tag{13}$$

The use of the above normalized parameters simplifies the functional representation of simple and composite high resolution schemes and helps defining the stability and boundedness conditions that they should satisfy. In addition, the normalized functional relationship of any scheme may be plotted on a Normalized Variable Diagram (NVD) which, as will be seen, is an effective tool in assessing the accuracy, boundedness, and relative diffusivity of convective schemes. In general, the value of ϕ_f is represented by the following parametric relation

$$\phi_f = f(\phi_U , \phi_C , \phi_D , x_U , x_C , x_f , x_D) \tag{14}$$

which, upon normalizing, is simplified to

$$\tilde{\phi}_f = f(\tilde{\phi}_C , \tilde{x}_C , \tilde{x}_f) \tag{15}$$

By comparing equations (14) and (15) it is clear that, one of the normalization benefits is a reducion in the number of parameters involved in the functional relationship. This is due to the normalized values of ϕ_U , ϕ_D , x_U and x_D being equal to 0, 1, 0, and 1 respectively.

NVSF OF SIMPLE UNBOUNDED HIGHER ORDER SCHEMES ON NON-UNIFORM GRID

In the following, the QUICK scheme [4], the first and second order upwind schemes [2,6], the Central Difference scheme, and Fromm's method [20] (in its steady state limit) are reformulated using NVSF methodology on non-uniform grid.

QUICK scheme

In the third order QUICK scheme, a parabolic profile is used to describe the variation of the dependent variable over the interval $[x_U , x_D]$ (Fig. 3). Mathematically,

$$\phi = a\, x^2 + bx + c \tag{16}$$

subject to:

$$\phi = \phi_U \qquad \text{for } x = x_U \tag{17}$$

$$\phi = \phi_C \qquad \text{for } x = x_C \tag{18}$$

$$\phi = \phi_D \qquad \text{for } x = x_D \tag{19}$$

Upon applying the conditions given by equations (17)-(19) and normalizing, the NVSF form of QUICK is

$$\tilde{\phi}_f = \frac{\tilde{x}_f (\tilde{x}_f - \tilde{x}_C)}{1- \tilde{x}_C} + \frac{\tilde{x}_f (\tilde{x}_f -1)}{\tilde{x}_C (\tilde{x}_C - 1)} \tilde{\phi}_C \tag{20}$$

For later use, equation (20) is rewritten in the following form

$$\tilde{\phi}_f = \tilde{x}_f + \frac{\tilde{x}_f (\tilde{x}_f -1)}{\tilde{x}_C (\tilde{x}_C - 1)} (\tilde{\phi}_C - \tilde{x}_C) \tag{21}$$

The derivations of other simple unbounded schemes follow the same procedure. However, for compactness of presentation, only their final forms are given next.

First order upwinding

$$\tilde{\phi}_f = \tilde{\phi}_C \tag{22}$$

Second order upwinding

$$\tilde{\phi}_f = \frac{\tilde{x}_f}{\tilde{x}_C} \tilde{\phi}_C \tag{23}$$

central difference

$$\tilde{\phi}_f = \frac{\tilde{x}_f - \tilde{x}_C}{1 - \tilde{x}_C} + \frac{\tilde{x}_f - 1}{\tilde{x}_C - 1} \tilde{\phi}_C \tag{24}$$

Fromm's method

$$\tilde{\phi}_f = \tilde{\phi}_C + (\tilde{x}_f - \tilde{x}_C) \tag{25}$$

ACCURACY AND BOUNDEDNESS REQUIREMENTS

The convection boundedness criterion for implicit steady state flow calculation as formulated by Gaskell and Lau [15] is applicable here. This criterion, based on the normalized variable analysis, states that for a scheme to have the boundedness property its functional relationship should be continous and bounded from below by $\tilde{\phi}_f$ =

$\tilde{\phi}_C$ and from above by unity, should pass through the point (0,0) and (1,1) in the monotonic range $0< \tilde{\phi}_C <1$, and for $\tilde{\phi}_C < 0$ or $\tilde{\phi}_C >1$ the functional relationship $f(\tilde{\phi}_C)$ should equal $\tilde{\phi}_C$. Mathematically these conditions are

$$\begin{cases} f(\tilde{\phi}_C) & \text{is continuous} \\ f(\tilde{\phi}_C) = 0 & \text{for } \tilde{\phi}_C = 0 \\ f(\tilde{\phi}_C) = 1 & \text{for } \tilde{\phi}_C = 1 \\ f(\tilde{\phi}_C)<1 \text{ and } f(\tilde{\phi}_C)>\tilde{\phi}_C & \text{for } 0<\tilde{\phi}_C<1 \\ f(\tilde{\phi}_C) = \tilde{\phi}_C & \text{for } \tilde{\phi}_C<0 \text{ and } \tilde{\phi}_C>1 \end{cases} \tag{26}$$

The above conditions may also be described geometrically on a normalized variable diagram as shown in Fig. 4. In Fig. 5, the linear relations given by equations (21)-(26) are also plotted on a normalized variable diagram (NVD). From this plot, it may easily be seen that, the only scheme fully satisfying the boundedness criterion (Eq. 26) is the first order upwind scheme [2]. Therefore, with the exception of this scheme, other schemes may in general give physically unrealistic results. Furthermore, schemes that have an NVD plot close to the first order upwind NVD plot tend to be highly diffusive, while schemes whose NVD plots are near the first order downwind NVD plot (the line $\tilde{\phi}_f =1$) tend to be highly compressive.

Concerning accuracy, Leonard [21] has developed the conditions that a scheme, derived on a uniforn grid using the NVF methodology, should satisfy in order to be second or third order accurate. According to his formulation, the necessary and sufficient condition for a scheme to be second order is for its functional relationship to pass through the point Q (0.5,0.75). If in addition it passes through Q with a slope of 0.75 then the scheme is third order. For the case of non-uniform grid (NVSF) these conditions can be derived by noting that all second and third order schemes described above may be represented using the following functional relationship:

$$\tilde{\phi}_f = \tilde{x}_f + M\,(\tilde{\phi}_C - \tilde{x}_C) \tag{27}$$

where M is the slope of the linear function. Knowing that the QUICK scheme is third order accurate and comparing equations (20) and (26), the necessary and sufficient condition for a scheme formulated on non-uniform grid to be second order accurate is for its normalized function to pass through the point $Q(\tilde{x}_C , \tilde{x}_f)$ (Fig. 5), and to be third order accurate its slope at Q must be equal to

$$M = \frac{\tilde{x}_f\,(\tilde{x}_f - 1)}{\tilde{x}_C\,(\tilde{x}_C - 1)} \tag{28}$$

The same results may be obtained from a Taylor series expansion around the control volume face f.

Having developed the required conditions for accuracy and boundedness, the shortcomings of the simple unbounded higher order schemes are eliminated through the use of composite schemes satisfying all above requirements. These schemes, developed for uniform grid, are extended next to non-uniform grid using the NVSF methodology.

NVSF OF COMPOSITE HIGH-RESOLUTION SCHEMES ON NON-UNIFORM GRID

In this section, the MINMOD [11] (or SOUCOUP [14]), OSHER [22], MUSCL [23], CLAM [24], SMART [15] and STOIC [18] schemes are reformulated on non-uniform grid. Since these schemes are extensively discussed in the literature, it is deemed unnecessary to elaborate on them here. Furthermore, due to the lengthy algebraic manipulations needed, only the final form of their functional relationships are presented.

MINMOD or SOUCOUP

$$\begin{aligned}
&\tilde{\phi}_f = \frac{\tilde{x}_f}{\tilde{x}_C}\tilde{\phi}_C && 0 < \tilde{\phi}_C < \tilde{x}_C \\
&\tilde{\phi}_f = \frac{\tilde{x}_C - \tilde{x}_f}{\tilde{x}_C - 1} + \frac{\tilde{x}_f - 1}{\tilde{x}_C - 1}\tilde{\phi}_C && \tilde{x}_C < \tilde{\phi}_C < 1 \\
&\tilde{\phi}_f = \tilde{\phi}_C && \text{elsewhere}
\end{aligned} \tag{29}$$

OSHER

$$\begin{aligned}
&\tilde{\phi}_f = \frac{\tilde{x}_f}{\tilde{x}_C}\tilde{\phi}_C && 0 < \tilde{\phi}_C < \frac{\tilde{x}_C}{\tilde{x}_f} \\
&\tilde{\phi}_f = 1 && \frac{\tilde{x}_C}{\tilde{x}_f} < \tilde{\phi}_C < 1 \\
&\tilde{\phi}_f = \tilde{\phi}_C && \text{elsewhere}
\end{aligned} \tag{30}$$

MUSCL

$$\begin{aligned}
&\tilde{\phi}_f = \frac{2\tilde{x}_f - \tilde{x}_C}{\tilde{x}_C}\tilde{\phi}_C && 0 < \tilde{\phi}_C < \frac{\tilde{x}_C}{2\tilde{x}_f} \\
&\tilde{\phi}_f = \tilde{x}_f - \tilde{x}_C + \tilde{\phi}_C && \frac{\tilde{x}_C}{2\tilde{x}_f} < \tilde{\phi}_C < 1 + \tilde{x}_C - \tilde{x}_f \\
&\tilde{\phi}_f = 1 && 1 + \tilde{x}_C - \tilde{x}_f < \tilde{\phi}_C < 1 \\
&\tilde{\phi}_f = \tilde{\phi}_C && \text{elsewhere}
\end{aligned} \tag{31}$$

CLAM

$$\begin{aligned}
&\tilde{\phi}_f = \frac{\tilde{x}_C^2 - \tilde{x}_f}{\tilde{x}_C(\tilde{x}_C - 1)}\tilde{\phi}_C + \frac{\tilde{x}_f - \tilde{x}_C}{\tilde{x}_C(\tilde{x}_C - 1)}\tilde{\phi}_C^2 && 0 < \tilde{\phi}_C < 1 \\
&\tilde{\phi}_f = \tilde{\phi}_C && \text{elsewhere}
\end{aligned} \tag{32}$$

SMART

$$\begin{aligned}
&\tilde{\phi}_f = -\frac{\tilde{x}_f(1 - 3\tilde{x}_C + 2\tilde{x}_f)}{\tilde{x}_C(\tilde{x}_C - 1)}\tilde{\phi}_C && 0 < \tilde{\phi}_C < \frac{\tilde{x}_C}{3} \\
&\tilde{\phi}_f = \frac{\tilde{x}_f(\tilde{x}_f - \tilde{x}_C)}{1 - \tilde{x}_C} + \frac{\tilde{x}_f(\tilde{x}_f - 1)}{\tilde{x}_C(\tilde{x}_C - 1)}\tilde{\phi}_C && \frac{\tilde{x}_C}{3} < \tilde{\phi}_C < \frac{\tilde{x}_C}{\tilde{x}_f}(1 + \tilde{x}_f - \tilde{x}_C)
\end{aligned}$$

$$\tilde{\phi}_f = 1 \qquad \frac{\tilde{x}_C}{\tilde{x}_f}(1 + \tilde{x}_f - \tilde{x}_C) < \tilde{\phi}_C < 1 \tag{33}$$

$$\tilde{\phi}_f = \tilde{\phi}_C \qquad \text{elsewhere}$$

STOIC

$$\tilde{\phi}_f = -\frac{\tilde{x}_f(1 - 3\tilde{x}_C + 2\tilde{x}_f)}{\tilde{x}_C(\tilde{x}_C - 1)}\tilde{\phi}_C \qquad 0 < \tilde{\phi}_C < \frac{\left(\tilde{x}_C - \tilde{x}_f\right)\tilde{x}_C}{\tilde{x}_C + \tilde{x}_f + 2\tilde{x}_f^2 - 4\tilde{x}_f\tilde{x}_C}$$

$$\tilde{\phi}_f = \frac{\tilde{x}_C - \tilde{x}_f}{\tilde{x}_C - 1} + \frac{\tilde{x}_f - 1}{\tilde{x}_C - 1}\tilde{\phi}_C \qquad \frac{\left(\tilde{x}_C - \tilde{x}_f\right)\tilde{x}_C}{\tilde{x}_C + \tilde{x}_f + 2\tilde{x}_f^2 - 4\tilde{x}_f\tilde{x}_C} < \tilde{\phi}_C < \tilde{x}_C$$

$$\tilde{\phi}_f = \frac{\tilde{x}_f(\tilde{x}_f - \tilde{x}_C)}{1 - \tilde{x}_C} + \frac{\tilde{x}_f(\tilde{x}_f - 1)}{\tilde{x}_C(\tilde{x}_C - 1)}\tilde{\phi}_C \qquad \tilde{x}_C < \tilde{\phi}_C < \frac{\tilde{x}_C}{\tilde{x}_f}(1 + \tilde{x}_f - \tilde{x}_C) \tag{34}$$

$$\tilde{\phi}_f = 1 \qquad \frac{\tilde{x}_C}{\tilde{x}_f}(1 + \tilde{x}_f - \tilde{x}_C) < \tilde{\phi}_C < 1$$

$$\tilde{\phi}_f = \tilde{\phi}_C\,, \qquad \text{elsewhere}$$

RESULTS AND DISCUSSION

The performance of the various composite HR convective schemes formulated using both NVF and NVSF methodologies is examined in this section by solving two typical problems.
Calculations are performed on an 18x18 highly non-uniform grid in order to demonstrate the virtues of the NVSF methodology. In both tests, computational results were considered converged when the residual error (RE) defined as:

$$RE = \sum \left| a_P\,\phi_P - \left(\sum_{nb} a_{nb}\phi_{nb} + b_P + b_{dc}\right) \right| \tag{35}$$

became smaller than 0.075%.

TRANSPORT OF A STEP PROFILE IN AN OBLIQUE UNIFORM VELOCITY FIELD

The physical situation under consideration along with the mesh network generated are shown in Fig. 6. This problem has been used by many researchers in studying artificial diffusion which inherently plagues many numerical schemes and particularly so at high Peclet numbers and for velocity fields that are oblique to the grid system. The governing conservation equation of the problem is

$$\frac{\partial(u\phi)}{\partial x} + \frac{\partial(v\phi)}{\partial y} = 0 \tag{36}$$

where ϕ is the depedent variable and u and v are the Cartesian components of the uniform velocity vector V, which, in this problem is taken to be at an angle of 45 degrees with respect to the horizontal. From Eq. (36), it is clear that the diffusion coefficient is set to zero. Thus, ϕ is transported purely by convection and the exact solution to the problem is $\phi = 1$ above the 45 degrees line and $\phi = 0$ below it (Fig. 6).

Before elaborating on the computational results, it should be pointed out that the results generated by the NVSF methodology are the exact ones in terms of the scheme used. The NVF results obtained on non-uniform grids, contain approximations due to the formulation of the scheme with the assumption of a uniform grid. However , in some instances, this approximation may cause *false compression* and be in favour of the overall solution. This occurs whenever the functional relationship of the HR scheme using NVSF is closer to the line $\tilde{\phi}_f = \tilde{\phi}_C$ (which is the functional relationship of the first order upwind scheme) than the one obtained using NVF resulting in a more diffusive profile for NVSF. But, it should be remembered that, mathematically, NVSF gives the exact formulation.

With this in mind, the computational ϕ values along the vertical centerline of the domain are shown for several HR schemes in Figs. 7(a-e). Their functional relationships are respectively displayed in the NVD's depicted in Figs. 8(a-e). Specific attention to each profile is not required and rather attention may be focused on results for a typical composite second order scheme (e.g. OSHER [22], Figs. 7(c) and 8(c)) and a typical composite third order scheme (e.g. STOIC [18], Figs. 7(e) and 8(e)). It is easily seen that the second order schemes have not improved in performance and, depending on the scheme, there may be slight deterioration in accuracy. This may be explained by referring to the NVD plot of the OSHER scheme(Fig. 8(c)). As shown in Fig. 8(c), Q($\tilde{x}_C$,$\tilde{x}_f$) is closer to the line $\tilde{\phi}_f = \tilde{\phi}_C$ than the point (0.5,0.75). Therefore, the NVSF results of this scheme are expected to be more diffusive, as obtained. If attention is directed to Fig. 7(a) (MINMOD [11]), the ϕ profile is seen to have deteriorated slightly in the lower portion and ameliorated slightly in the upper part. Again this is easily explained by inspection of the NVD plot (Fig. 8(a)). In the lower part, the NVSF profile is more diffusive and in the upper part is more compressive than the NVF profile. In general, the formulation of the second order composite schemes gives very diffusive profiles whether using NVF or NVSF methodologies and this is why improvement is not very pronounced. The third order methods (STOIC [18], SMART [15]) show excellent improvements (Figs. 7(d) and 7(e)). As seen in their NVD plots (Figs. 8(d) and 8(e)), their profiles are becoming less diffusive due to the aforesaid reasons. This will always be the case independent of the grid network used. The virtues of the NVSF methodology, as shown by the above example, are substantial for third order schemes. Furthermore, as depicted in Fig. 7(f) where the profiles obtained by the various schemes using NVSF are plotted, second order composite schemes do not provide the required accuracy and are still relatively very diffusive.

TRANSPORT OF A SEMI-ELLIPTIC PROFILE IN AN OBLIQUE UNIFORM VELOCITY FIELD

A schematic of the physical situation under consideration along with the grid system used are depicted in Fig. 8. The governing conservation equation and the variables of the problem are the same as in the previous one. The velocity field is again set at 45 degrees to the horizontal in order to simulate a flow for which false diffusion is maximum. The numerical results using both NVF and NVSF formulations are shown in Fig. 9. The trend of results is similar to that of the previous problem with substantial improvements obtained with the third order STOIC and SMART schemes and nearly no improvement obtained with the second order composite schemes.

The various profiles obtained using NVSF are displayed in Fig. 9(f). From this Fig. and Fig. 7(f) it can be inferred once more that the second order composite schemes are very diffusive as compared to the third order ones. Since, the use of the NVSF methodology along with the deferred correction procedure permits easy implementation of these third order composite schemes in existing CFD codes, it is recommended that these schemes be used in CFD applications. Furthermore, the increase in computational cost with the employment of third order NVSF schemes is outweighed by the improvement in accuracy obtained and the need to use a smaller number of grid points for a desired level of accuracy.

CONCLUSION

A new Normalized Variable and Space Formulation methodology (NVSF) for the development and analysis of High-Resolution convection-diffusion schemes is presented. The method is an extension of the NVF methodology of Leonard [1] into non-uniformly discretized domains. The required conditions for boundedness and accuracy of HR schemes when using the newly developed technique were discussed. The method can easily be implemented in existing CFD computer codes. Several second and third order HR schemes formulated using NVF were generalized using NVSF. The technique was applied to two test problems using several generalized second and third order composite convection schemes. While improvements with second order schemes were mild, the accuracy of third order schemes (SMART, STOIC) improved substantially with NVSF.

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FIGURE CAPTIONS

Fig. 1 Typical grid point cluster and control volume.

Fig. 2 Interpolation points used in calculating ϕ_f.

Fig. 3 Original and normalized variables and profiles.

Fig. 4 Convective Boundedness Criterion (CBC).

Fig. 5 Normalized Variable Diagram (NVD) for several linear schemes formulated using NVSF.

Fig. 6 Physical domain and grid network used for the transport of a step profile problem.

Fig. 7 ϕ profiles for the transport of a step profile problem along the vertical centerline of the domain for various HR schemes using NVF and NVSF methodologies.

Fig. 8 NVD plots for various HR schemes using NVF and NVSF methodologies.

Fig. 9 Physical domain and grid network used for the transport of an elliptic profile problem.

Fig. 10 ϕ profiles for the transport of a semi-elliptic profile problem along a vertical line ($x = 0.25$) of the domain for various HR schemes using NVF and NVSF methodologies.

FIGURES

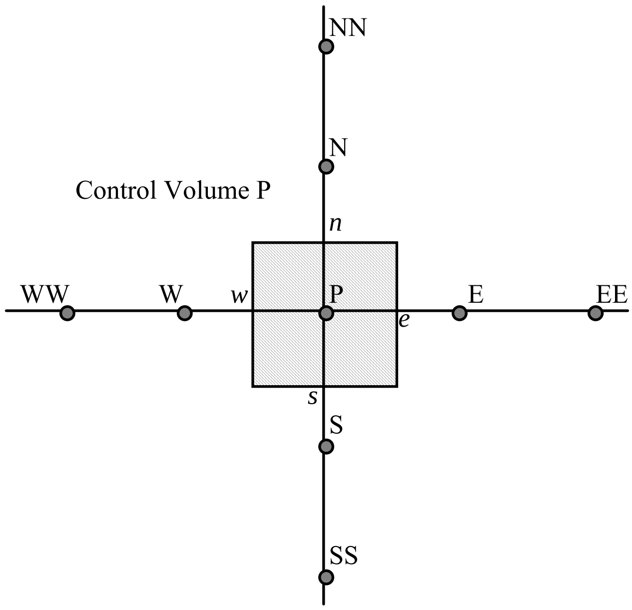

Fig. 1: Typical grid point cluster and control volume.

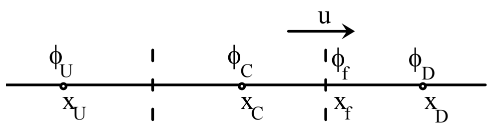

Fig. 2: Interpolation points used in calculating ϕ_f .

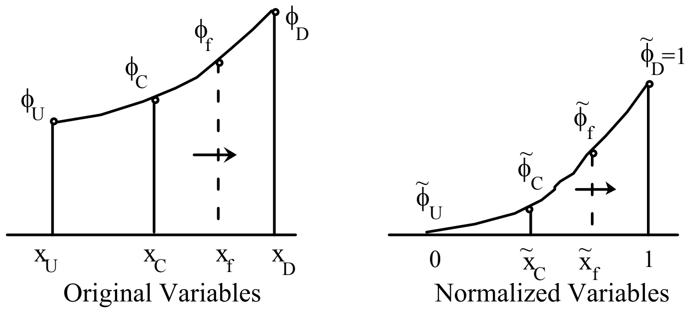

Fig. 3: Original and normalized variables and profiles.

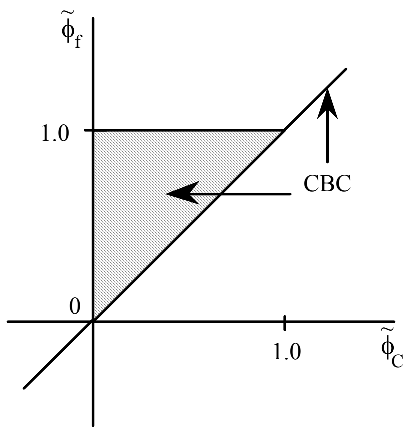

Fig. 4: Convective Boundedness Criterion (CBC).

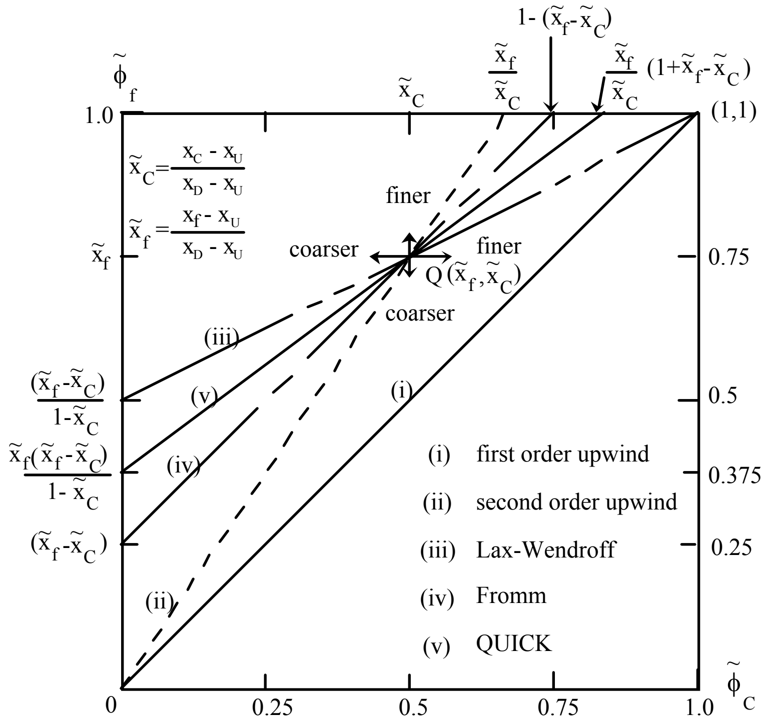

Fig. 5: Normalized Variable Diagram (NVD) for several linear schemes formulated using NVSF.

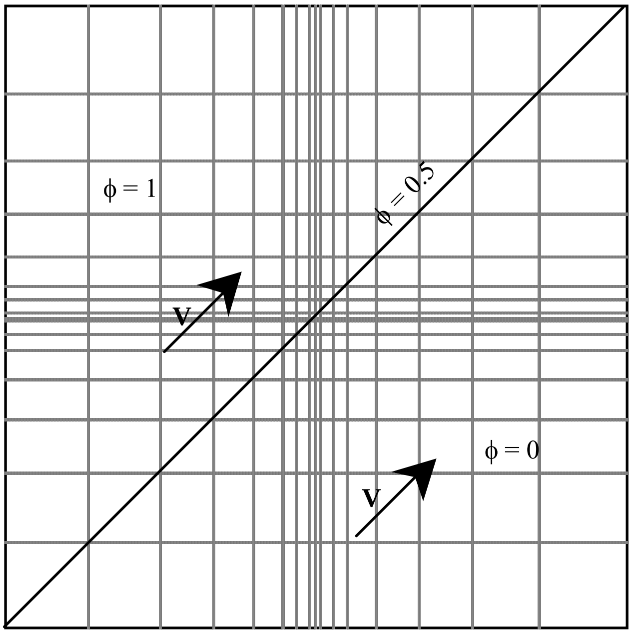

Fig. 6: Physical domain and grid network used for the transport of a step profile problem.

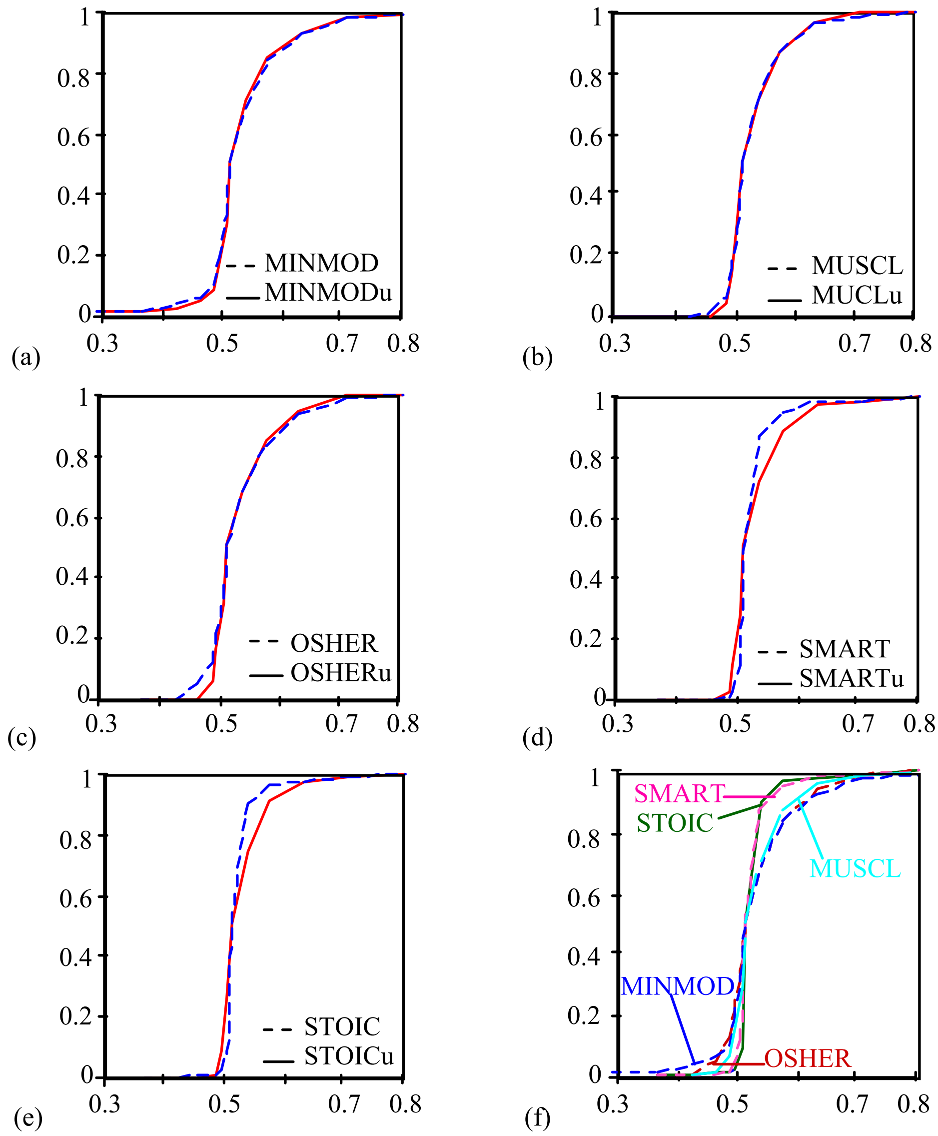

Fig. 7: ϕ profiles for the transport of a step profile problem along the vertical centerline of the domain for various HR schemes using NVF and NVSF methodologies.

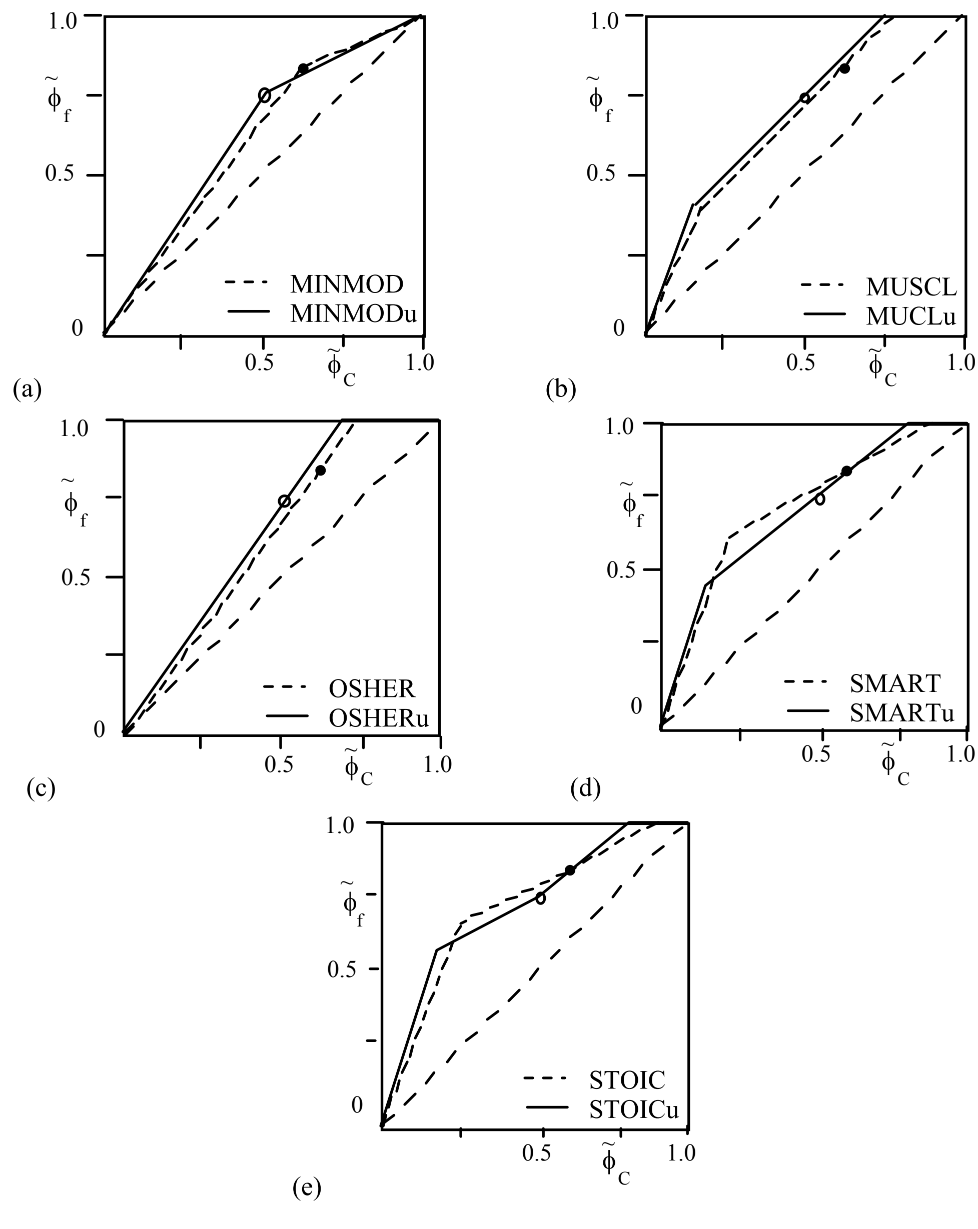

Fig. 8: NVD plots for various HR schemes using NVF and NVSF methodologies.

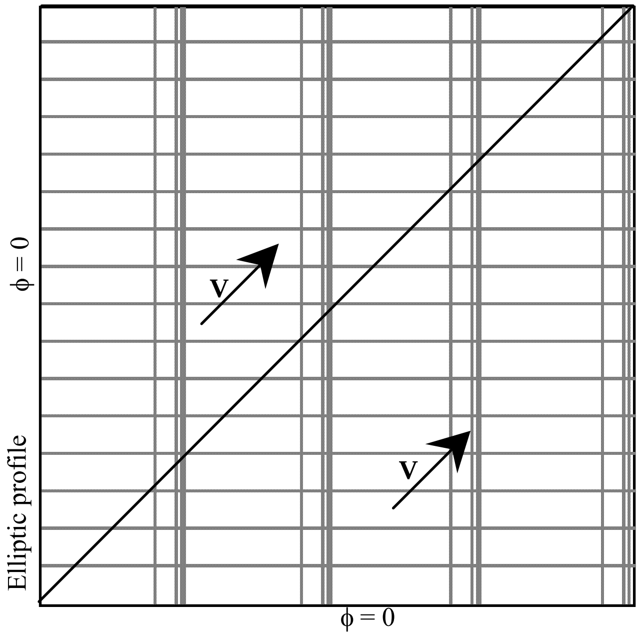

Fig. 9: Physical domain and grid network used for the transport of an elliptic profile problem.

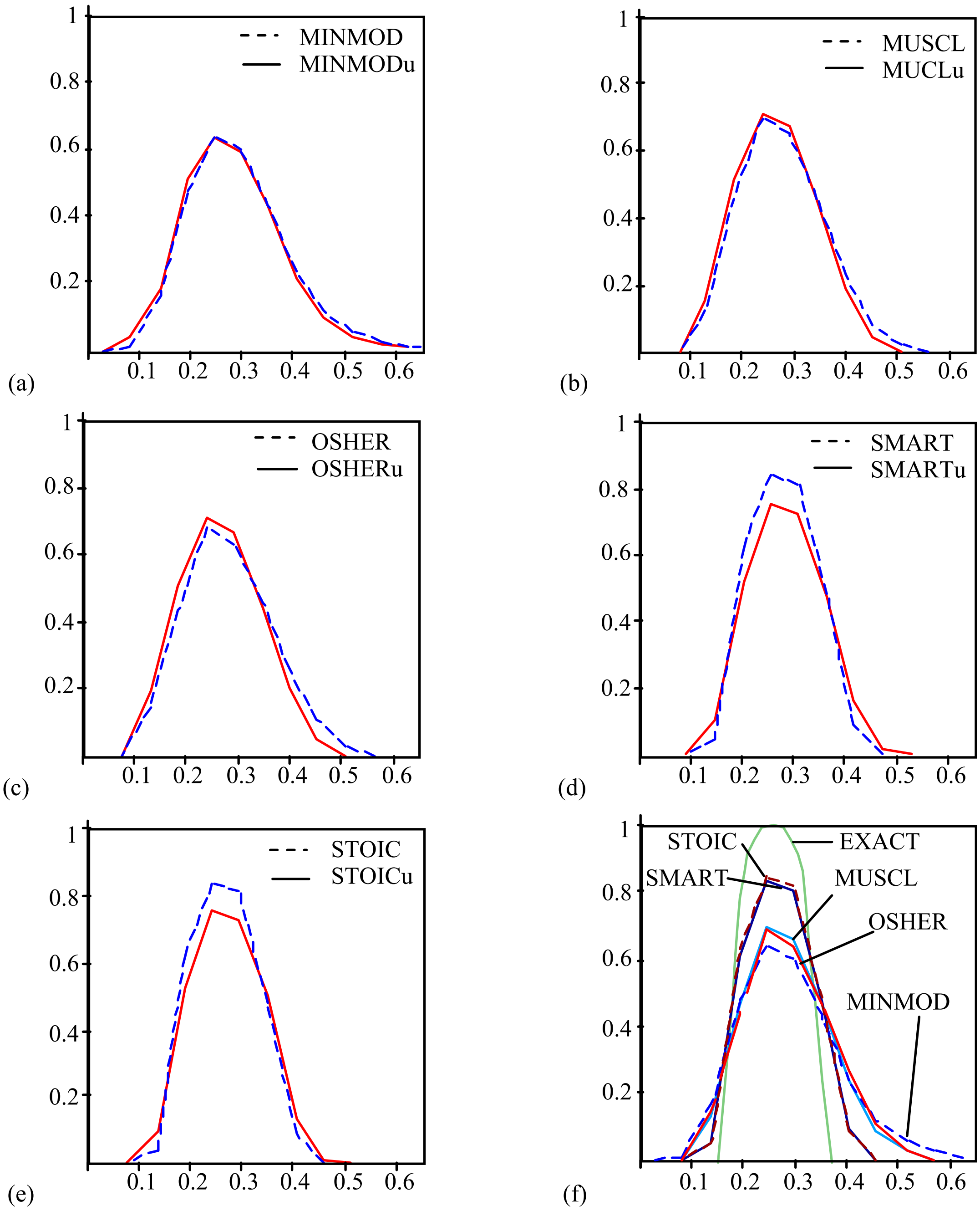

Fig. 10: ϕ profiles for the transport of a semi-elliptic profile problem along a vertical line (x = 0.25) of the domain for various HR schemes using NVF and NVSF methodologies.

TABLES

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