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Normalized weighted geometric Dombi Bonferroni mean operator with interval grey numbers: Application in multicriteria decision making

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ABSTRACT

The main purpose of this paper is to provide a multi-criteria decision-making that combines interval grey numbers and normalized weighted geometric Dombi-Bonferroni mean operator to address the situations where attribute values take the form of interval grey numbers under uncertain information. As in recent decade, evaluation objects are becoming more and more complicated, the interval grey numbers (GNs) are employed to more accurately express uncertainty of the evaluation objects. Firstly, operations and comparison method for interval grey numbers are defined. Subsequently, the interval grey number normalized weighted geometric Bonferroni mean (GNWGBM) operator, is presented accordingly in some desirable characteristics.

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1. Introduction

In second half of 20th century, multi-criteria decision making (MCDM) process has emerged as fastest growing sector in management science domain at finding the best option from all of the feasible alternatives based on inconsistent criteria (Stanujkic te al., 2012). But in imprecise based real life decision problems in various fields as engineering, economics, and management, decision experts have incomplete and inconsistent information about alternatives with respect to conflicting attributes (Wang, 2016). Considering the complexity and uncertainty of MCDM process, decision makers (DMs) may not be able to express their evaluations in precise numbers, but provide some approximate form with their knowledge and perception (Xie & Xin, 2014). Thus to deal with it, Deng (1982) introduced the grey system theory but no satisfactory result is drawn from it. While term “fuzzy” involve uncertain factors in the evaluation information caused by the impreciseness of human thinking, “grey” denote objective uncertainty due to insufficient and incomplete information (Xie & Liu, 2009). Grey system theory is one of the new mathematical approaches born from the concept of the grey sets (Xie & Xin, 2014). As the basic element of grey system, a grey number is a real number in fact while the precise value could not be determined but the potential range of values can be defined. So a grey number is represented as a set of potential discrete, continuous or mixed values (Liu & Wang, 2016).

Generally, aggregation operators are important tools for fusing information in multi-criteria decision making (MCDM) problems. The most widely used operators in fuzzy theory are the min and the max operator.

However, in the case of min-max operators, the main disadvantage is that the result is determined only by one variable and the other has no influence. Also, the min-max operators are not analytic, their second derivative is not continuous (Dombi, 2009). These disadvantages of traditional min-max operators in fuzzy environment are successfully eliminated by generalized Dombi operator class. Also, Dombi T-norms (TN) and T-conorms (TCN) have general parameters of general TN and TCN, and this can make the aggregation process more flexible. Making decisions in real systems requires rational understanding of the relationship between attributes and elimination of the impact of awkward data. For this purpose, Bonferroni (Bonferroni, 1950) introduced the Bonferroni mean (BM) operator, allowing the presentation of interconnections between elements and their fusion into unique score function. Obviously, Bonferroni and Dombi aggregators can successfully achieve this goal. According to the author's knowledge, there has been no research up to date considering the fusion of the Dombi and Bonferroni aggregators in interval grey environment represented by interval grey numbers (GNs). Therefore, logical goal and motivation for this study is to show hybrid Dombi-Bonferroni aggregator for the transformation by GNs.

The remainder of this paper is structured as follows: the second section of the paper shows the basic settings of GN, arithmetic operations with GN and mathematical presentation of Dobmi i Bonferroni aggregators. The third section of the paper presents a novel hybrid GNNWDBM aggregator and its application.

2. Preliminaries

2.1. Operations on Interval Grey Numbers

Definition 6. Grey number operation is a special operation defined on sets of intervals. In analog to the real numbers, arithmetic operations between two grey numbers $\otimes\Upsilon_1=[\underline{\Upsilon}_1,\overline{\Upsilon}_1]$ and $\otimes\Upsilon_2=[\underline{\Upsilon}_2,\overline{\Upsilon}_2]$ are defined by the following expressions (Deng, 1982):

(1) Adding of grey numbers "+"

$$\otimes\Upsilon_1+\otimes\Upsilon_2=[\underline{\Upsilon}_1,\overline{\Upsilon}_1]+[\underline{\Upsilon}_2,\overline{\Upsilon}_2]=[\underline{\Upsilon}_1+\underline{\Upsilon}_2,\overline{\Upsilon}_1+\overline{\Upsilon}_2] \tag{1}$$

(2) Subtraction of grey numbers "$-$"

$$\otimes\Upsilon_1-\otimes\Upsilon_2=[\underline{\Upsilon}_1,\overline{\Upsilon}_1]-[\underline{\Upsilon}_2,\overline{\Upsilon}_2]=[\underline{\Upsilon}_1-\overline{\Upsilon}_2,\overline{\Upsilon}_1-\underline{\Upsilon}_2] \tag{2}$$

(3) Multiplication of grey numbers "$\times$"

$$\otimes\Upsilon_1\times\otimes\Upsilon_2=[\underline{\Upsilon}_1,\overline{\Upsilon}_1]\times[\underline{\Upsilon}_2,\overline{\Upsilon}_2]=\begin{bmatrix}\min\left\{\underline{\Upsilon}_1\underline{\Upsilon}_2,\underline{\Upsilon}_1\overline{\Upsilon}_2,\overline{\Upsilon}_1\underline{\Upsilon}_2,\overline{\Upsilon}_1\overline{\Upsilon}_2\right\},\\ \max\left\{\underline{\Upsilon}_1\underline{\Upsilon}_2,\underline{\Upsilon}_1\overline{\Upsilon}_2,\overline{\Upsilon}_1\underline{\Upsilon}_2,\overline{\Upsilon}_1\overline{\Upsilon}_2\right\}\end{bmatrix} \tag{3}$$

(4) Division of grey numbers "/"

$$\otimes\Upsilon_1\div\otimes\Upsilon_2=[\underline{\Upsilon}_1,\overline{\Upsilon}_1]\times\left[\frac{1}{\overline{\Upsilon}_2},\frac{1}{\underline{\Upsilon}_2}\right],\ \otimes\Upsilon_2\notin 0 \tag{4}$$

(5) The n[th] root of grey $\otimes\Upsilon$

$$\left(\otimes\Upsilon\right)^{1/n}=\left[(\underline{\Upsilon})^{1/n},(\overline{\Upsilon})^{1/n}\right] \tag{5}$$

Example 2. If we assume that $\otimes\Upsilon_1\in\left[3,5\right]$ and $\otimes\Upsilon_2\in\left[2,6\right]$ are two interval grey numbers and n=2, then we can show that:

$$\otimes\Upsilon_1+\otimes\Upsilon_2=[3,5]+[2,6]=[5,11]$$

$$\otimes\Upsilon_1-\otimes\Upsilon_2=[3,5]-[2,6]=[-3,3]$$

$$\otimes\Upsilon_1\times\otimes\Upsilon_2=[3,5]\times[2,6]=\begin{bmatrix}\min\left\{6,18,10,30\right\},\\ \max\left\{6,18,10,30\right\}\end{bmatrix}=[6,30]$$

$$\otimes\Upsilon_1\div\otimes\Upsilon_2=[3,5]\times\left[\frac{1}{6},\frac{1}{2}\right]=\begin{bmatrix}\min\left\{\frac{3}{6},\frac{3}{2},\frac{5}{6},\frac{5}{2}\right\},\\ \max\left\{\frac{3}{6},\frac{3}{2},\frac{5}{6},\frac{5}{2}\right\}\end{bmatrix}=[0.5,2.5]$$

$$\left(\otimes\Upsilon_1\right)^{1/2}=\left[(3)^{1/2},(5)^{1/2}\right]=[1.73,2.24]$$

2.2. Bonferroni mean operator

Definition 1 (Bonferroni, 950): Let (a_1, a_2, ..., a_n) be a set of non-negative numbers and $p,q \geq 0$. If

$$BM^{p,q}(a_1,a_2,...,a_n)=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \tag{6}$$

then $BM^{p,q}$ is called a Bonferroni mean (BM) operator.
Definition 4 (Sun & Liu, 2013): Let (a_1,a_2,...,a_n) be a set of non-negative numbers and $p,q \geq 0$. If

$$NWGBM^{p,q}(a_1,a_2,...,a_n)=\frac{1}{p+q}\prod_{i,j=1}^{n}\left(pa_i+qa_j\right)^{\frac{w_i w_j}{1-w_i}} \tag{7}$$

Then $NWGBM^{p,q}$ is called a normalized weighted geometric BM (NWGBM) operator.

2.3. Dombi operations of GN

Definition 2. Let p and q be any two real numbers. Then, the Dombi T-norm and T-conorm between p and q are defined as follows (Dombi, 2009):

$$O_D(p,q)=\frac{1}{1+\left\{\left(\frac{1-p}{p}\right)^{\rho}+\left(\frac{1-q}{q}\right)^{\rho}\right\}^{1/\rho}} \tag{8}$$

$$O_D^c(p,q)=1-\frac{1}{1+\left\{\left(\frac{p}{1-p}\right)^{\rho}+\left(\frac{q}{1-q}\right)^{\rho}\right\}^{1/\rho}} \tag{9}$$

where $\rho>0$ and $(p,q)\in[0,1]$.

According to the Dombi T-norm i T-conorm, we define the Dombi operations of grey numbers (GN).

Definition 3. Suppose $\otimes\varphi_1=[\underline{\varphi}_1,\overline{\varphi}_1]$ and $\otimes\varphi_2=[\underline{\varphi}_2,\overline{\varphi}_2]$ are two GN, $\rho,\gamma>0$ and let it be $f\left(\otimes\varphi_i\right)=\left[f\left(\underline{\varphi}_i\right),f\left(\overline{\varphi}_i\right)\right]=\left[\frac{\underline{\varphi}_i}{\sum_{i=1}^{n}\underline{\varphi}},\frac{\overline{\varphi}_i}{\sum_{i=1}^{n}\overline{\varphi}_i}\right]$ grey function, then some operational lows of grey numbers based on the Dombi T-norm and T-conorm can be defined as follows

(1) Addition "+"

$$\otimes\varphi_1+\otimes\varphi_2=\left[\begin{array}{c}\sum_{j=1}^{2}\underline{\varphi}_j-\frac{\sum_{j=1}^{2}\underline{\varphi}_j}{1+\left\{\left(\frac{f\left(\underline{\varphi}_1\right)}{1-f\left(\underline{\varphi}_1\right)}\right)^{\rho}+\left(\frac{f\left(\underline{\varphi}_2\right)}{1-f\left(\underline{\varphi}_2\right)}\right)^{\rho}\right\}^{1/\rho}},\\ \sum_{j=1}^{2}\overline{\varphi}_j-\frac{\sum_{j=1}^{2}\overline{\varphi}_j}{1+\left\{\left(\frac{f\left(\overline{\varphi}_1\right)}{1-f\left(\overline{\varphi}_1\right)}\right)^{\rho}+\left(\frac{f\left(\overline{\varphi}_2\right)}{1-f\left(\overline{\varphi}_2\right)}\right)^{\rho}\right\}^{1/\rho}}\end{array}\right] \tag{10}$$

(2) Multiplication "×"

$$\otimes\varphi_1\times\otimes\varphi_2=\left[\frac{\sum_{j=1}^{2}\underline{\varphi}_j}{1+\left\{\left(\frac{1-f\left(\underline{\varphi}_1\right)}{f\left(\underline{\varphi}_1\right)}\right)^{\rho}+\left(\frac{1-f\left(\underline{\varphi}_2\right)}{f\left(\underline{\varphi}_2\right)}\right)^{\rho}\right\}^{1/\rho}},\frac{\sum_{j=1}^{2}\overline{\varphi}_j}{1+\left\{\left(\frac{1-f\left(\overline{\varphi}_1\right)}{f\left(\overline{\varphi}_1\right)}\right)^{\rho}+\left(\frac{1-f\left(\overline{\varphi}_2\right)}{f\left(\overline{\varphi}_2\right)}\right)^{\rho}\right\}^{1/\rho}}\right] \tag{11}$$

(3) Scalar multiplication, where $\gamma > 0$

$$\gamma \otimes \varphi_1 = \left[\underline{\varphi}_1 - \frac{\underline{\varphi}_1}{1+\left\{\gamma\left(\frac{\underline{\varphi}_1}{1-\underline{\varphi}_1}\right)^{\rho}\right\}^{1/\rho}}, \underline{\varphi}_1 - \frac{\overline{\varphi}_1}{1+\left\{\gamma\left(\frac{\overline{\varphi}_1}{1-\overline{\varphi}_1}\right)^{\rho}\right\}^{1/\rho}} \right] \tag{12}$$

(4) Power, where $\gamma > 0$

$$\{\otimes\varphi_1\}^{\gamma} = \left[\frac{\underline{\varphi}_1}{1+\left\{\gamma\left(\frac{1-\underline{\varphi}_1}{\underline{\varphi}_1}\right)^{\rho}\right\}^{1/\rho}}, \frac{\overline{\varphi}_1}{1+\left\{\gamma\left(\frac{1-\overline{\varphi}_1}{\overline{\varphi}_1}\right)^{\rho}\right\}^{1/\rho}} \right] \tag{13}$$

3. Normalized weighted geometric Dombi Bonferroni mean operator with grey numbers

Based on the GN operators (10)-(13) we propose the GN Normalized Weighted Dombi Bonferroni mean (GNNWDBM) operator.

Theorem 1. Let it be $\otimes a_j = [\underline{a}_j, \overline{a}_j]$; $(j = 1, 2, ..., n)$, collection of GNs in R, then GNNWDBM operator is defined as follows

$$GNNWDBM^{p,q,\rho}(\otimes a_1, \otimes a_2, ..., \otimes a_n) =$$

$$= \frac{1}{p+q} \prod_{i,j=1}^{n} \left(p \otimes a_i + q \otimes a_j \right)^{\frac{w_i w_j}{1-w_i}}$$

$$= \left[\begin{array}{l} \sum_{i=1}^{n} \underline{a}_i - \dfrac{\sum_{i=1}^{n} \underline{a}_i}{\left\{ 1 + \dfrac{1}{(p+q) w_i w_j} \dfrac{1-w_i}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \dfrac{1}{p\left(\frac{f(\underline{a}_i)}{1-f(\underline{a}_i)}\right)^{\rho} + q\left(\frac{f(\underline{a}_j)}{1-f(\underline{a}_j)}\right)^{\rho}} \right)} \right\}^{1/\rho}}, \\ \sum_{i=1}^{n} \overline{a}_i - \dfrac{\sum_{i=1}^{n} \overline{a}_i}{\left\{ 1 + \dfrac{1}{(p+q) w_i w_j} \dfrac{1-w_i}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \dfrac{1}{p\left(\frac{f(\overline{a}_i)}{1-f(\overline{a}_i)}\right)^{\rho} + q\left(\frac{f(\overline{a}_j)}{1-f(\overline{a}_j)}\right)^{\rho}} \right)} \right\}^{1/\rho}} \end{array} \right] \tag{14}$$

where $f(\otimes a_i) = \left(f(\underline{a}_i), f(\overline{a}_i) \right) = \left(\frac{\underline{a}_i}{\sum_{i=1}^{n} \underline{a}_i}, \frac{\overline{a}_i}{\sum_{i=1}^{n} \overline{a}_i} \right)$ represents grey function.

Proof: We need to prove the Eq. (14) is kept. According to the operational laws of GNs, we get

$$pa_i = \left[\underline{a}_i - \frac{\underline{a}_i}{p\left(\frac{f(\underline{a}_i)}{1-f(\underline{a}_i)}\right)^{1/\rho}}, \overline{a}_i - \frac{\overline{a}_i}{p\left(\frac{f(\overline{a}_i)}{1-f(\overline{a}_i)}\right)^{1/\rho}} \right],$$

$$qa_i = \left[\underline{a}_i - \frac{\underline{a}_i}{q\left(\frac{f(\underline{a}_i)}{1-f(\underline{a}_i)}\right)^{1/\rho}}, \overline{a}_i - \frac{\overline{a}_i}{q\left(\frac{f(\overline{a}_i)}{1-f(\overline{a}_i)}\right)^{1/\rho}} \right] \text{ and}$$

$$p \otimes a_i + \otimes a_j = \left(\begin{array}{c} \underline{a}_i + \underline{a}_j - \dfrac{\underline{a}_i + \underline{a}_j}{1 + \left\{ p\left(\frac{f(\underline{a}_i)}{1-f(\underline{a}_i)}\right)^{\rho} + q\left(\frac{f(\underline{a}_j)}{1-f(\underline{a}_j)}\right)^{\rho} \right\}^{1/\rho}}, \\ \overline{a}_i + \overline{a}_j - \dfrac{\overline{a}_i + \overline{a}_j}{1 + \left\{ p\left(\frac{f(\overline{a}_i)}{1-f(\overline{a}_i)}\right)^{\rho} + q\left(\frac{f(\overline{a}_j)}{1-f(\overline{a}_j)}\right)^{\rho} \right\}^{1/\rho}} \end{array} \right)$$

further we get

$$\left(p \otimes a_i + \otimes a_j\right)^{\frac{w_i w_j}{1-w_i}} = \left(\begin{array}{c} \dfrac{\underline{a}_i + \underline{a}_j}{1 + \left\{ \frac{1}{1-w_i} \frac{w_i w_j}{p\left(\frac{1-f(\underline{a}_i)}{f(\underline{a}_i)}\right)^{\rho} + q\left(\frac{1-f(\underline{a}_j)}{f(\underline{a}_j)}\right)^{\rho}} \right\}^{1/\rho}}, \\ \dfrac{\overline{a}_i + \overline{a}_j}{1 + \left\{ \frac{1}{1-w_i} \frac{w_i w_j}{p\left(\frac{1-f(\overline{a}_i)}{f(\overline{a}_i)}\right)^{\rho} + q\left(\frac{1-f(\overline{a}_j)}{f(\overline{a}_j)}\right)^{\rho}} \right\}^{1/\rho}} \end{array} \right)$$

Thereafter,

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(p \otimes a_i + \otimes a_j\right)^{\frac{w_i w_j}{1-w_i}} = \left(\begin{array}{c} \dfrac{\sum_{i=1}^{n} \underline{a}_i}{1 + \left\{ \frac{w_i w_j}{1-w_i} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{1}{p\left(\frac{1-f\left(\underline{a}_i\right)}{f\left(\underline{a}_i\right)}\right)^{\rho} + q\left(\frac{1-f\left(\underline{a}_j\right)}{f\left(\underline{a}_j\right)}\right)^{\rho}} \right) \right\}^{1/\rho}}, \\ \dfrac{\sum_{i=1}^{n} \overline{a}_i}{1 + \left\{ \frac{w_i w_j}{1-w_i} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{1}{p\left(\frac{1-f\left(\overline{a}_i\right)}{f\left(\overline{a}_i\right)}\right)^{\rho} + q\left(\frac{1-f\left(\overline{a}_j\right)}{f\left(\overline{a}_j\right)}\right)^{\rho}} \right) \right\}^{1/\rho}} \end{array} \right)$$

Therefore,

$$GNNWDBM^{p,q,\rho}(\otimes a_1,\otimes a_2,...,\otimes a_n)=\frac{1}{p+q}\prod_{i,j=1}^{n}\left(p\otimes a_i+q\otimes a_j\right)^{\frac{w_iw_j}{1-w_i}}$$

$$=\left[\begin{array}{l}\sum_{i=1}^{n}\underline{a}_i-\dfrac{\sum_{i=1}^{n}\underline{a}_i}{1+\left\{\dfrac{1}{(p+q)w_iw_j}\dfrac{1-w_i}{\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\dfrac{1}{p\left(\dfrac{f(\underline{a}_i)}{1-f(\underline{a}_i)}\right)^{\rho}+q\left(\dfrac{f(\underline{a}_j)}{1-f(\underline{a}_j)}\right)^{\rho}}\right)}\right\}^{1/\rho}},\\ \sum_{i=1}^{n}\overline{a}_i-\dfrac{\sum_{i=1}^{n}\overline{a}_i}{1+\left\{\dfrac{1}{(p+q)w_iw_j}\dfrac{1-w_i}{\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\dfrac{1}{p\left(\dfrac{f(\overline{a}_i)}{1-f(\overline{a}_i)}\right)^{\rho}+q\left(\dfrac{f(\overline{a}_j)}{1-f(\overline{a}_j)}\right)^{\rho}}\right)}\right\}^{1/\rho}}\end{array}\right]$$

Theorem 2 (Idempotency): Let it be $\otimes a_j=[\underline{a}_j,\overline{a}_j]$; $\left(j=1,2,...,n\right)$, collection of GNs in R, then, if $\otimes a_i=\otimes a$, then $GNNWDBM^{p,q,\rho}(\otimes a_1,\otimes a_2,..,\otimes a_n)=GNNWDBM^{p,q,\rho}(\otimes a,\otimes a,..,\otimes a)$.

Proof: Since $\otimes a_i=\otimes a$, i.e. $\underline{a}_i=\underline{a}$, $\overline{a}_i=\overline{a}$ then

$GNNWDBM^{p=1,q=1,\rho=1}(\otimes a_1,\otimes a_2,...,\otimes a_n)=(\otimes a,\otimes a,...,\otimes a)$

$$
=\left(\frac{\sum_{i=1}^{n}\underline{a}_i}{1+\left\{\frac{1}{p+q}\frac{n(n-1)}{\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{\left(p\left(\frac{1-f\left(\underline{a}_i\right)}{f\left(\underline{a}_i\right)}\right)^{\rho}+q\left(\frac{1-f\left(\underline{a}_j\right)}{f\left(\underline{a}_j\right)}\right)^{\rho}\right)}}\right\}^{1/\rho}},\frac{\sum_{i=1}^{n}\overline{a}_i}{1+\left\{\frac{1}{p+q}\frac{n(n-1)}{\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{\left(p\left(\frac{1-f\left(\overline{a}_i\right)}{f\left(\overline{a}_i\right)}\right)^{\rho}+q\left(\frac{1-f\left(\overline{a}_j\right)}{f\left(\overline{a}_j\right)}\right)^{\rho}\right)}}\right\}^{1/\rho}}\right)
$$

$$
=\left(\frac{2\underline{a}_i}{1+\left\{\frac{1}{1+1}\frac{2}{\sum_{i=1}^{n}\frac{1}{\left(\left(\frac{1-f(\underline{a}_i)}{f(\underline{a}_i)}\right)^{\rho}+\left(\frac{1-f(\underline{a}_j)}{f(\underline{a}_j)}\right)^{\rho}\right)}}\right\}^{1/\rho}},\frac{2\overline{a}_i}{1+\left\{\frac{1}{1+1}\frac{2}{\sum_{i=1}^{n}\frac{1}{\left(\left(\frac{1-f(\overline{a}_i)}{f(\overline{a}_i)}\right)^{\rho}+\left(\frac{1-f(\overline{a}_j)}{f(\overline{a}_J)}\right)^{\rho}\right)}}\right\}^{1/\rho}}\right)
$$

$$=\left(\frac{2\underline{a}_i}{1+\left\{\dfrac{1}{\sum_{i=1}^{n}\dfrac{1}{(1+1)\left(\dfrac{1-f(\underline{a}_i)}{f(\underline{a}_i)}\right)^{\rho}}}\right\}^{1/\rho}},\frac{2\overline{a}_i}{1+\left\{\dfrac{1}{\sum_{i=1}^{n}\dfrac{1}{(1+1)\left(\dfrac{1-f(\overline{a}_i)}{f(\overline{a}_i)}\right)^{\rho}}}\right\}^{1/\rho}}\right)=\left(\underline{a}_i,\overline{a}_i\right)=\otimes a$$

The proof of Theorem 2 is completed.

Theorem 3 (Boundedness): Let it be $\otimes a_j=[\underline{a}_j,\overline{a}_j]$; $\left(j=1,2,...,n\right)$, collection of GNs in R, let $\otimes a^-=\left(\min\underline{a}_j,\min\overline{a}_j\right)$ and $\otimes a^+=\left(\max\underline{a}_j,\max\overline{a}_j\right)$ then

$\otimes a^-\le GNNWDBM^{p,q,\rho}(\otimes a_1,\otimes a_2,...,\otimes a_n)\le\otimes a^+$.

Proof: Let $\otimes a^-=\min(\otimes a_1,\otimes a_2,...,\otimes a_n)=\left(\min\underline{a}_j,\min\overline{a}_j\right)$ and

$\otimes a^+=\max(\otimes a_1,\otimes a_2,...,\otimes a_n)=\left(\max\underline{a}_j,\max\overline{a}_j\right)$. Then, it can be stated that $\underline{a}^-=\min_j(\underline{a}_j)$, $\overline{a}^-=\min_j(\overline{a}_j)$, $\underline{a}^+=\max_j(\underline{a}_j)$ and $\overline{a}^+=\max_j(\overline{a}_j)$. Based on that, the following inequalities can be formulated:

$\otimes a^-\le\otimes a_j\le\otimes a^+;\min_j(\underline{a}_j)\le\underline{a}_j\le\max_j(\underline{a}_j);$

$\min_j(\overline{a}_j)\le\overline{a}_j\le\max_j(\overline{a}_j);$

According to the inequalities shown above, it can be concluded that $\otimes a^-\le GNNWDBM^{p,q,\rho}(\otimes a_1,\otimes a_2,...,\otimes a_n)\le\otimes a^+$ holds.

Theorem 4 (Commutativity): Let the grey set $(\otimes a_1^{'},\otimes a_2^{'},...,\otimes a_n^{'})$ be any permutation of $(\otimes a_1,\otimes a_2,..,\otimes a_n)$.

Then $GNNWDBM^{p,q,\rho}(\otimes a_1,\otimes a_2,..,\otimes a_n)=GNNWDBM^{p,q,\rho}(\otimes a_1^{'},\otimes a_2^{'},...,\otimes a_n^{'})$.

Proof: This property is obvious.

Example. While buying a car, the buyer applies four criteria for the evaluation of the alternative (the car): Quality (C1), Price (C2), Safety (C3) and Comfort (C4). The criteria for alternative A1 are presented with interval grey numbers $\otimes a_1=\left[2,4\right]$, $\otimes a_2=\left[3,5\right]$, $\otimes a_3=\left[3,5\right]$ and $\otimes a_4=\left[2,4\right]$, *p=q=ρ=1* and $w_j=\left(0.18,0.32,0.33,0.17\right)$. Then, by application of GNNWDBM we can obtain utility function for alternative A1 as follows:

$$GNNWDBM_w^{1,1,1}\left\{[2,4];[3,5];[3,5];[2,4]\right\}=$$

$$=\left[\begin{array}{l}10-\dfrac{2+3+3+4}{1+\left\{\dfrac{1}{1+1}\dfrac{1}{\frac{0.18\cdot0.32}{1-0.18}\frac{1}{0.25^1+0.43^1}+\frac{0.18\cdot0.33}{1-0.18}\frac{1}{0.25^1+0.43^1}+\frac{0.18\cdot0.17}{1-0.18}\frac{1}{0.25^1+0.25^1}+\frac{0.32\cdot0.18}{1-0.32}\frac{1}{0.43^1+0.25^1}+\frac{0.32\cdot0.33}{1-0.32}\frac{1}{0.43^1+0.43^1}+...+\frac{0.17\cdot0.33}{1-0.17}\frac{1}{0.25^1+0.43^1}}\right\}^{1/1}},\\18-\dfrac{4+5+5+4}{1+\left\{\dfrac{1}{1+1}\dfrac{1}{\frac{0.18\cdot0.32}{1-0.18}\frac{1}{0.22^1+0.278^1}+\frac{0.18\cdot0.33}{1-0.18}\frac{1}{0.22^1+0.278^1}+\frac{0.18\cdot0.17}{1-0.18}\frac{1}{0.22^1+0.22^1}+\frac{0.32\cdot0.18}{1-0.32}\frac{1}{0.278^1+0.22^1}+\frac{0.32\cdot0.33}{1-0.32}\frac{1}{0.278^1+0.278^1}+...+\frac{0.17\cdot0.33}{1-0.17}\frac{1}{0.22^1+0.278^1}}\right\}^{1/1}}\end{array}\right]$$

$$=\left[2.61,4.61\right]$$

4. Conclusion

Many real-world decision making problems are related with uncertainties and/or some form of predictions and therefore their solutions can be much better determined using an extended MCDM method which is adapted for the use of fuzzy or interval grey numbers. For practical problems, interval grey numbers can better express the diversified and utilized information that affect the rationality of decision results. GNs can flexibly express uncertain, imprecise and inconsistent information that widely exist in real-life situations. The operation and ranking method for interval grey numbers plays a vital role in development of grey system theory. Using the kernel and the degree of greyness of interval grey number, we propose aggregation operator namely GNNWDBM operator for solving MCDM problems. The usability and effectiveness of the proposed approach are checked by the numerical illustration and final results of the methodology is observable in result discussion section.

In future, we expect to investigate the generalized distance measures and aggregation operators of IGNs. Further integration of the interval grey number approach into existing MCDM models (Pamučar et al., 2018; Roy et al., 2018; Yazdani et al., 2019) would make a significant contribution to deal with both uncertainty and subjectivity in the decision making process.

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