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Note—Competitive Bidding with Asymmetric Information Reanalyzed

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COMPETITIVE BIDDING WITH ASYMMETRIC INFORMATION REANALYZED†

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This note reanalyzes the following problem, formerly treated by Wilson [3]: two parties have to submit bids for an object. One of them knows the value with certainty, the other does not. The equilibrium derived differs from Wilson's solution and yields a simple explanation for the case cited by Wilson: the value of the game is essentially zero for the party with incomplete information.
(GAMES/GROUP DECISIONS–BIDDING; GAMES/GROUP DECISIONS–GAMBLING)

I. Introduction

In his paper [3], Wilson analyzed a bidding situation where one party has complete knowledge of the situation and the opposing party has only information in the form of a subjective distribution of the value of the object bid for. The motivation for treating this problem goes back to a case cited by Wood: two oil companies submit sealed tenders for an offshore parcel. For one (the informed player) the value is known, because he has a contiguous parcel. The uninformed party has only imperfect information. Wilson provides two examples, one of which yielded clearly unacceptable results. We reanalyze this game, showing first that Wilson's solution of the first example does not yield a pair of Nash-equilibrium strategies. The solution in [3] assumes that the uninformed player decides first, taking the reaction of player two into account. The informed player acts as a follower. This does not necessary yield a Nash-equilibrium. If it does not, Wilson's solution goes wrong because the optimal strategy of the uninformed player is no longer randomized, as it should be. Next we derive the Nash-equilibrium strategies and illustrate, for purposes of comparison, with the examples in Wilson's paper.

II. Wilson's "Easy Example"

For ease of reference we first state the problem. Two players have to submit bids for an object with value v, known by player 2, but unknown by player 1. The player with incomplete information assesses a density $f(v)$ over the value. This density is assumed known by the other player. Wilson derives a solution for this game and gives two examples.

In the first example $f(v)$ is taken as a uniform density over [0, 1]. The solution proposed is the following: player 1 uses in equilibrium a randomized strategy, with distribution function:

$$G(p) = (1+\alpha)^{1/\alpha} p^{1/\alpha} \quad \text{for } p \in \left[0, (1+\alpha)^{-1}\right]. \tag{1}$$

Player 2 uses a pure strategy (we depart somewhat from Wilson's notation for

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typographical reasons):

$$s(v) = (1 + \alpha)^{-1}v \quad \text{where } \alpha = (1 + \sqrt{5})/2. \tag{2}$$

It is trivial to show that this is not a solution of the game: the expected payoffs with this pair of strategies are $\alpha^{-6}/2$ for player 1 and v^{α}/α for player 2.

Player 1's strategy is not in equilibrium however; because if player 2 sticks to the strategy given in equation (2) the expected profit for player 1, using the pure strategy:

$$p = (1 + \alpha)^{-1} \tag{3}$$

which is a corner solution, becomes $\alpha^{-3}/2$. This is clearly larger than $\alpha^{-6}/2$.

The second example treated by Wilson remained unsolved.

III. The Equilibrium Strategies

In order to derive the equilibrium strategies we can start from the following observations:

—first, there is no equilibrium in pure strategies for player 1, nor in mixed strategies for player 2.

These assertions are easily proved by contradiction.

—Second, for a randomized strategy to be optimal for player 1, an equalizer strategy has to be used by player 2. (See Karlin [1, p. 172].) This means a strategy such that any pure strategy in the support of a randomized one should yield equal expected profit to player 1 in equilibrium.

Denoting player 2's strategy by $s(v)$ and its inverse by $h(p)$ (assuming as usual that $s(v)$ is strictly increasing), the expected payoff for a pure strategy by player 1 becomes:

$$E(\pi_1) = \int_{-\infty}^{h(p)} (v - p)f(v)\, dv. \tag{4}$$

Because the maximal $E(\pi_1)$ has to be constant over an interval of values for p, the following must hold:

$$\frac{dE(\pi_1)}{dp} = -\int_{-\infty}^{h(p)} f(v)dv + h'(p)[h(p) - p]f[h(p)] = 0 \tag{5}$$

or

$$-F[h(p)] + h'(p)h(p)f[h(p)] - h'(p)pf[h(p)] = 0. \tag{6}$$

Substituting $p = s(v)$ and $h'(p) = 1/s'(v)$ and multiplying through by $s'(v)$ gives:

$$-F(v)s'(v) + vf(v) - s(v)f(v) = 0, \tag{7}$$

$$F(v)s'(v) + s(v)f(v) = vf(v), \tag{8}$$

$$d\{s(v)F(v)\}/dv = vf(v) \tag{9}$$

or

$$s(v) = \frac{\int_0^v uf(u)\, du}{F(v)} - \frac{K}{F(v)}. \tag{10}$$

This result, incidentally, is identical to the equilibrium condition in symmetrical games (see Ortega-Reichert [2]). It is easily checked, at least for the examples considered, that K is the profit level which player 2 allows to player 1. A positive K

implies that negative bids have positive probability, and even worse, that bids in the interval $(-\infty, M]$ where M is an arbitrary negative constant have positive probability. This feature makes a solution with K positive unrealistic, as the bidtaker will not accept such bids. Conversely, a negative K implies that player 1 can only incur an expected loss. Thus K can be taken to be zero in most cases.

In order to derive an equilibrium we have to look for a strategy of player 1 that makes (10) optimal for player 2.

$$E(\pi_2) = (v - p)G(p) \tag{11}$$

where $G(p)$ is the distribution of bids resulting from player 1's randomized strategy. This yields:

$$\frac{\partial E(\pi_2)}{\partial p} = (v - p)g(p) - G(p) = 0. \tag{12}$$

Substituting player 2's strategy:

$$(v - s(v))g(s(v)) - G(s(v)) = 0 \tag{13}$$

yields a differential equation, the solution of which corresponds to player 1's equilibrium strategy. Equation (13) can be written with $v = h(p)$:

$$[h(p) - p]g(p) - G(p) = 0. \tag{14}$$

IV. Examples

For the example based on a uniform density of v, one obtains with K equal to zero, from (7):

$$s(v) = v/2 \quad \text{and} \quad h(p) = 2p.$$

Substituting in (14):

$$pg(p) = G(p). \tag{15}$$

The solution of (15) is:

$$G(p) = Cp. \tag{16}$$

The two intervals of serious bids have to be identical, thus:

$$g(p) = 2, \qquad p \in [0, 1/2]. \tag{17}$$

Wilson's second example with $f(v) = \theta e^{\theta v}$, $v \in (-\infty, 0]$, yields again for $K = 0$

$$s(v) = \frac{\theta v - 1}{\theta}, \tag{18}$$

$$h(p) = \frac{\theta p + 1}{\theta}. \tag{19}$$

Substituting:

$$\frac{1}{\theta} g(p) - G(p) = 0 \tag{20}$$

which results in:

$$G(p) = \theta e^{\theta p}. \tag{21}$$

V. Conclusion

The solution obtained corresponds closely to the situation described by Wilson: player 1 can as well abstain from bidding, as it is most likely that player 2 will bid such that the object has no expected value for him. This is especially clear if the value of the object lies in an interval closed from below. It is somewhat less evident if this is not the case, as in the second example, because the large negative bids could result from large negative values.

References

1. KARLIN, S., *Mathematical Methods and Theory in Games, Programming and Economics*, Vol. II, Addison-Wesley, Reading, Mass., 1959.
2. ORTEGA-REICHERT, A., "Models for Competitive Bidding under Uncertainty," Unpublished Ph.D. dissertation, Stanford University, 1963.
3. WILSON, R. B., "Competitive Bidding with Asymmetric Information," *Management Sci.*, Vol. 13 (July 1967), pp. 816–820.

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ERRATUM TO "A MINIMAX ORDERING POLICY FOR THE INFINITE STAGE DYNAMIC INVENTORY PROBLEM"

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(INVENTORY/PRODUCTION–STOCHASTIC MODELS; DYNAMIC PROGRAMMING)

In my recent paper "A Minimax Ordering Policy for the Infinite Stage Dynamic Inventory Problem," *Management Sci.*, Vol. 24, No. 11 (July 1978), pp. 1138–1145, I noticed some errors in the proof of Theorem A.5 in the Appendix which made the proof somewhat incomplete. I have given below a correct version of the proof, mentioning only its main points.

PROOF OF THEOREM A.5. Denote by $F_0(p, t)$ a two-point distribution that attributes probability masses p and q to points $y = \sigma\sqrt{q/p}$ and $w = -\sigma\sqrt{p/q}$, respectively. Then $F_0(p, t) \in \Gamma(0, \sigma)$. For a given $F \in \Gamma(0, \sigma)$, define $\int_x^\infty t\, dF(t) = p_0$. By Schwarz inequality, we then have

$$\left[\int_x^\infty dF(t)\right] < \left[\int_x^\infty t^2\, dF(t)\right]^{1/2} \sqrt{p_0} \quad \text{and}$$

$$\left[\int_{-\infty}^x t\, dF(t)\right] < \left[\int_{-\infty}^x t^2\, dF(t)\right]^{1/2} \sqrt{q_0}\,,$$

which together imply

$$\left[\int_x^\infty t\, dF(t)\right] < \sigma\sqrt{q_0 p_0} = \int_x^\infty t\, dF_0(p_0, t).$$

Let $g_1(t) = \alpha_1 t^+ + \alpha_2(t-a)^+$, where $\alpha_1 > 0$, $\alpha_1 + \alpha_2 > 0$ and $a > 0$. Thus $g_1(t)$ is a piecewise linear nondecreasing continuous function over R.

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Let $p' = \int_{x+a}^{\infty} dF(t)$. Then

$$\int_x^{\infty} g_1(t-x)\,dF(t) = \alpha_1 \int_x^{\infty} (t-x)\,dF(t) + \alpha_2 \int_{x+a}^{\infty} (t-x-a)\,dF(t)$$
$$\leq \alpha_1\left[\sigma\sqrt{q_0 p_0} - p_0 x\right] + \alpha_2\left[\sigma\sqrt{q' p'} - p'(x+a)\right]$$
$$\leq \int_x^{\infty} g_1(t-x)\,dF_0(p_1, t),$$

for some p_1 such that $0 \leq p \leq 1$.

By induction, we can then similarly show that if $g_k(t)$ is a piecewise linear nondecreasing continuous function we can find a constant p_k such that

$$\int_x^{\infty} g_k(t-x)\,dF(t) \leq \int_x^{\infty} g_k(t-x)\,dF_0(p_k, x).$$

Let $g(t)$ be a uniform pointwise limit of $g_k(t)$ such that $g_k(t) \downarrow g(t)$. Then by monotone convergence theorem,

$$\int_x^{\infty} g(t-x)\,dF(t) \leq \int_x^{\infty} g(t-x)\,dF_0(p_{\infty}, x),$$

where $p_{\infty} = \lim p_k$. This completes the proof.

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Communications

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Comment on Designing Scientific Journals

In a recent article in this journal, King, Kilmann and Sochats [1] report the results of a survey of authors and readers of *Management Science*, and discuss some interesting implications on the Designing of Scientific Journals. Seven-point Likert scales were used to measure the respondents' interest in 87 management science topics (where 1 was "not at all interested" and 7 "extremely interested"). The overall response means (across the 87 scales) were found to be 3.8, 2.9, and 3.2 for readers, authors, and editors, respectively (p. 780). There are potential reasons for the differences between these means which King et al. explain (pp. 780 and 781). But then, they use the same (raw) data "to analyze the relative congruence of interest between (1) authors and readers, and (2) academic readers and practitioner readers." To this end, they compute the t-values for mean differences for each of the 87 items (Table 2, pp. 779–780). Since the difference between the overall means is 0.9 between

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