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Notes* I

EFFECTS OF SUBSTITUTING A LINEAR GOAL FOR A FRACTIONAL GOAL IN THE GOAL PROGRAMMING PROBLEM†

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In a recent article in *Management Science*, Awerbuch et al. [1] discuss the occurrence of nonlinearities in the application of goal programming and conclude that direct linearization may not be possible when goal constraints are of a fractional nature.

The purpose of this note is to identify when the surrogate objective function min $M|x - Gt|$ can be used in place of $M|x/t - G|$ directly and when it cannot. In the event it cannot be used in a straightforward manner, some suggestions are given for solving the fractional problem by permuting the order of the goals of the surrogate linear problem.

Awerbuch et al. present the fractional goal programming problem[1]

$$\begin{aligned} &\min|x/t - G| = z \\ &\text{s.t.} \quad \text{(a)} \quad \mathbf{Ay} = \mathbf{b}, \\ &\qquad\quad \text{(b)} \quad \mathbf{y} \geqslant 0, \quad x \geqslant 0, \quad t \geqslant 0, \end{aligned} \tag{1}$$

where $\mathbf{y} = (x_1, x_2, \ldots, x_n)$ and x and t are linear functions of the x_i, and contrast this problem with the linear goal programming problem

$$\begin{aligned} &\min|x - Gt| = w \\ &\text{s.t.} \quad \text{(a)} \quad \mathbf{Ay} = \mathbf{b}, \\ &\qquad\quad \text{(b)} \quad \mathbf{y} \geqslant 0, \quad x \geqslant 0, \quad t \geqslant 0. \end{aligned} \tag{2}$$

In order to demonstrate that (1) is not equivalent to (2) in general, the problems $\min|x/t - 0.5|$ subject to $x \leqslant 0.4t$, $x \geqslant 0$, $t \geqslant 0$ and $\min|x - 0.5t|$ subject to $x \leqslant 0.4t$, $x \geqslant 0$, $t \geqslant 0$ are considered. It is stated that the latter problem has a minimum value of zero (at $x = 0$, $t = 0$), while the former problem has a minimum value of 0.1.

It should first be noted that the former problem should not include the value $t = 0$ in the set of feasible solutions since this solution would necessitate division by zero in the objective function. Instead, the two problems should be stated as $\min|x/t - 0.5|$ subject to $x \leqslant 0.4t$, $x \geqslant 0$, $t \geqslant c > 0$ and $\min|x - 0.5t|$ subject to $x \geqslant 0.4t$, $x \geqslant 0$, $t \geqslant c > 0$ respectively, where c is a positive number very close to zero. In this revised formulation, (2) has the objective function value $(0.1)c$, and the optimal values of x and t are $x = 0.4c$ and $t = c$. In (1) the objective function value is 0.1, and there are infinitely many optimal values for x and t.

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[1] Note: Terms denoting vectors are in boldface type.

Awerbuch et al. also indicate that the problems

$$\min M_1|x/t - 0.5| + M_2|t - 10| \quad \text{subject to} \quad x \leq 0.4t, \quad x \geq 0, \quad t \geq 0, \quad \text{and} \tag{3}$$

$$\min M_1|x - 0.5t| + M_2|t - 10| \quad \text{subject to} \quad x \leq 0.4t, \quad x \geq 0, \quad t \geq 0 \tag{4}$$

have identical solutions in the event that $M_2 >>> M_1$, but have different solutions if $M_1 >>> M_2$.

The following is an attempt to characterize the situations in which $\min M|x - Gt|$ may and may not be used to replace $\min M|x/t - G|$ in goal programming problems.

THEOREM 1. *If there is a unique pair of values* $(x, t) = (x^*, t^*)$ *such that* $|x^*/t^* - G| < |x/t - G|$ *for all* (x, t) *satisfying* $\mathbf{Ay} = \mathbf{b}$, $\mathbf{y} \geq 0$, $x \geq 0$, $t \geq c > 0$ *such that* $(x, t) \neq (x^*, t^*)$, *then* $|x^* - Gt^*| < |x - Gt|$ *for all* (x, t) *satisfying* $\mathbf{Ay} = \mathbf{b}$, $\mathbf{y} \geq 0$, $x \geq 0$, $t \geq c > 0$, *and* $(x, t) \neq (x^*, t^*)$.

THEOREM 2. *If there is more than one pair of values* x, t *which minimize* $|x/t - G|$ s.t. $\mathbf{Ay} = \mathbf{b}$, $\mathbf{y} \geq 0$, $x \geq 0$, $t \geq c > 0$, *and* $\min|x/t - G| > 0$, *then the pair of these values with the smallest* t *value is the one which minimizes* $|x - Gt|$ s.t $\mathbf{Ay} = \mathbf{b}$, $y \geq 0$, $x \geq 0$, $t \geq c > 0$.

The following is a discussion of the implications of the previous theorems regarding the use of the formulation $\min M|x - Gt|$ as a surrogate for $\min M|x/t - G|$ in the goal programming problem:

(1) In the event that $w = \min M|x - Gt|$ is substituted for $z = \min M|x/t - G|$ as one of the preemptive priority goals in a goal programming problem, if w^*, the optimum value of w, is zero, then z^*, the optimum value of z, is also zero and the optimum values of the decision variables x and t are identical. This is obviously true if there is a unique solution (x^*, t^*) to the fractional problem and is also true in the event of alternative optima since in both problems, the choice of unique optimal x and t values will be postponed until lower level goals are considered.

(2) Suppose $w = \min M|x - Gt|$ is substituted for $z = \min M|x/t - G|$ as one of the preemptive priority goals, where $z^* \neq 0$ and the resulting optimal values for x, t and w are x^*, t^* and w^* respectively. If there is a unique optimal solution for the corresponding goal of the fractional problem subject to the constraints and the higher order goals, then this solution is $x = x^*$, $t = t^*$, $z = w^*/t^*$.

(3) Suppose $w = \min M|x - Gt|$ is substituted for $z = \min M|x/t - G|$ as one of the preemptive priority goals where $z^* \neq 0$. If there is not a unique optimal solution for the corresponding goal of the fractional problem subject to the constraints and the higher order goals, the fractional problem and the surrogate linear problem are likely to have different optimal values of x and t. An example of this occurs in (3) and (4).

In light of comments (2) and (3), it is imperative to be able to detect whether or not there is a unique optimal solution for $z = \min M|x/t - G|$ subject to the constraints and higher order goals in order to confidently substitute the surrogate linear problem for the fractional problem in the event that the optimal value for w is not equal to zero (see comment (1)).

In the event that there is not a unique solution to the fractional problem, the user may still be able to obtain the solution to the fractional problem by judiciously permuting the order of the preemptive priority goals in the surrogate linear problem. For example, although the problem $\min M_1|x - 0.5t| + M_2|t - 10|$ s.t. $x \leq 0.4t$, $x \geq 0$, $t \geq c > 0$, $M_1 >>> M_2$ does not have the same optimal values of x and t as $\min M_1|x/t - 0.5| + M_2|t - 10|$ s.t. $x \leq 0.4t$, $x \geq 0$, $t \geq c > 0$, $M_1 >>> M_2$, if the former problem is altered by letting $M_2 >>> M_1$, the optimal x and t values are identical to those of the fractional problem with $M_1 >>> M_2$.

In general, whenever there is not a unique solution in the fractional problem for a given preemptive priority goal subject to the constraints and higher order goals, the "tie" is broken only after considering lower order goals. If the user can recognize which lower order goal breaks this "tie," the surrogate linear problem can be used to obtain the optimal solution by lowering the priority of the goal which is a surrogate for the fractional goal. The priority should be lowered just below the goal which breaks the tie. This is equivalent to what was done in the problem just mentioned.

In many instances, this can be done quite easily because one of the lower order goals clearly fixes x or t at a specific value (e.g., $M_2|t-10|$ or $M_4|x-5|$). In other cases, the goal which fixes x or t may not be obvious and the user may be better off using the methods suggested by Charnes and Copper [2], Kornbluth [3], or a nonlinear programming code.[1]

[1] The author would like to thank Chun Dar Chen for helpful discussions relating to the topics in this paper.

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Notes* II

PLANT LAYOUT: COMPUTERS VERSUS HUMANS†

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This work offers a critique of the methodology used by Scriabin and Vergin [7] in their study of computer algorithms versus humans in designing plant layouts. It attempts to show that Scriabin's and Vergin's experiments do not provide a useful comparison of computers and humans, and to point out several experimental procedures that would make comparisons of heuristic computer algorithms and humans more valid.

1. Introduction

In a recent issue of *Management Science*, Scriabin and Vergin [7] (hereafter called S&V) compared the performance of human subjects and computer algorithms in solving plant layout problems. They say [7, p. 173], "This project tests the hypothesis that the three computer algorithms . . . do not perform better than certain selected people." Although their experiment was restricted to three computer algorithms, they

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