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Note*

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NOTE ON "OPTIMAL ORDERING QUANTITY TO REALIZE A PRE-DETERMINED LEVEL OF PROFIT"†

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In this note we consider the one-period inventory model in which it is required to determine the order quantity which maximizes the probability of realizing a predetermined level of profit R. We give a condition for determining the optimal order quantity and provide explicit expressions for the optimal order quantity in three special cases.
(INVENTORY/PRODUCTION/STOCHASTIC MODELS)

In this note we consider the so-called static or one-period model (see, for example, Wagner [1], Hillier and Liebermann [2]) in which only a single inventory decision is made in anticipation of demand which is treated as a random variable. We assume that the initial inventory is zero. The problem is to determine the order quantity, I, which maximizes the probability of reaching or exceeding a given profit R when the other parameters of the system like selling price, cost price, shortage cost and scrap price, are known. The profit Z, being dependent on the random demand, is itself a random variable. The probability distribution of Z will naturally depend on the other quantity I and the probability distribution of the demand. Hence the probability that the profit will be greater than the pre-determined level R will be a function of I once the distribution of demand is given. It is reasonable to expect that the chances of realizing the profit R will be more if we choose I so that this probability is a maximum. We have given the methodology for arriving at the optimal order quantity using the criterion, Probability[Profit $\geqslant R$] is a maximum, and we have arrived at explicit expressions for the order quantity in a few special cases. We give below the notation used in this note.

X: quantity demanded,
$f(x)$: probability density function of X,
p: unit selling price,
c: unit cost price,
s: unit shortage cost (i.e., ill-will cost over and above lost sale opportunity cost),
q: unit scrap price,
I: the order quantity which is the decision variable,
Z: the profit which depends on the demand X and hence is a random variable.

The expression for Z, when a quantity I is ordered and when X is the demand, is easily seen to be

$$\begin{aligned} Z &= pX + q(I - X) - cI, \qquad X < I \\ &= pI - s(X - I) - cI, \qquad X > I. \end{aligned} \tag{1}$$

Z increases for $X < I$ and decreases for $X > I$. A predetermined level of profit R will

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be realized when X is equal to either x_1 or x_2 where

$$x_1 = \frac{R + (c - q)I}{p - q} \quad \text{and} \tag{2}$$

$$x_2 = \frac{(p + s - c)I - R}{s}. \tag{3}$$

The profit Z will be greater than R whenever $x_1 < X < x_2$. So the probability that Z is greater than or equal to R is given by

$$P[Z \geqslant R] = P[x_1 \leqslant X \leqslant x_2] = \int_{x_1}^{x_2} f(x)\,dx, \tag{4}$$

the right-hand side integral being a function of I. The optimal order quantity is obtained from the condition that $\int_{x_1}^{x_2} f(x)\,dx$ is a maximum, i.e.,

$$\frac{d}{dI}\int_{x_1}^{x_2} f(x)\,dx = 0 \quad \text{or} \tag{5}$$

$$f(x_2)\frac{dx_2}{dI} - f(x_1)\frac{dx_1}{dI} = 0. \tag{6}$$

Using (2) and (3), the condition (6) can be written as

$$f(x_2)\frac{(p + s - c)}{s} - f(x_1)\frac{(c - q)}{p - q} = 0. \tag{7}$$

We now consider three special cases and give explicit expressions for I, the optimal order quantity.

Case I. *Exponential Demand.* $f(x) = \mu e^{-\mu x}$, $\mu > 0$, $x \geqslant 0$. We get by (7)

$$\mu e^{-\mu x_2}\frac{(p + s - c)}{s} = \mu e^{-\mu x_1}\frac{c - q}{p - q}. \tag{8}$$

Setting

$$\frac{(p + s - c)(p - q)}{s(c - q)} = e^{\mu\delta} \tag{9}$$

(8) can be written as

$$e^{-\mu x_2}e^{\mu\delta} = e^{-\mu x_1} \quad \text{or} \quad x_2 - \delta = x_1. \tag{10}$$

Substituting the values of x_1 and x_2 from (2) and (3) in (10), we get, after some algebraic simplification, the optimal order quantity I^* as

$$I^* = \frac{1}{(p - c)}\left[\frac{(p - q)s\delta}{p - q + s} + R\right]. \tag{11}$$

Case II. When the commodity marketed is a luxury item, it is reasonable to assume that the demand is proportional to the income y of an individual, i.e., demand $= ky$: Therefore

$$P[\text{Number of demands} > x] = P\left[\text{Number of people whose income} > \frac{x}{k}\right]$$

$$= \frac{c}{(x/k)^m},$$

assuming Pareto's law (Jan Pen [3], Allen [4]) for income distribution where $m > 1.5$.

The density function for the demand $f(x)$ can, then, be taken as

$$f(x) = A/x^{\alpha} \quad \text{where} \quad \alpha > 2.5. \tag{12}$$

We get from (7)

$$\frac{1}{x_2^{\alpha}} \frac{p+s-c}{s} = \frac{1}{x_1^{\alpha}} \frac{c-q}{p-q}. \tag{13}$$

Setting

$$\frac{(p+s-c)(p-q)}{s(c-q)} = \lambda^{\alpha} \tag{14}$$

(13) reduces to

$$\lambda x_1 = x_2. \tag{15}$$

Substituting the values of x_1 and x_2 from (2) and (3) we see that the probability of achieving the desired profit is maximum when

$$\frac{\lambda[R+(c-q)I]}{p-q} = \frac{(p+s-c)I-R}{x}. \tag{16}$$

This yields the optimal order quantity I^* as

$$I^* = \frac{R[\lambda s + p - q]}{(p+s-c)(p-q) - \lambda s(c-q)}. \tag{17}$$

Case III. Here the demand follows a uniform distribution,

$$f(x) = \frac{1}{b-a}, \qquad a \leqslant x \leqslant b$$
$$= 0, \qquad \text{otherwise.}$$

In this case it can be seen that (7) will not directly yield the optimal order quantity. However we observe that the probability of obtaining the desired profit R, viz., $\int_{x_1}^{x_2} f(x)\,dx$, being a piecewise linear increasing function of the order quantity I, has unity as the least upper bound. Thus the optimal order quantity I^* is obtained from the condition

$$\int_{x_1}^{x_2} f(x)\,dx = 1, \quad \text{i.e.,} \tag{18}$$

$$(x_2 - x_1) = (b-a). \tag{19}$$

Substituting the value of x_1 and x_2 from (2) and (3) in (19), the optimal order quantity I^* can be seen to be given by

$$I^* = \frac{1}{(p-c)}\left[\frac{s(b-a)(p-q)}{p-q+s} + R\right].^{1} \tag{20}$$

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References

1. Wagner, Harvey M., *Principles of Operations Research*, Prentice-Hall of India Pvt. Ltd., New Delhi, 1974.
2. Hillier, F. W., and Liebermann, C. J., *Introduction to Operations Research*, Holden-Day, Inc., 1972.
3. Jan Pen, *Income Distributions*, Penguin Publishers.
4. Allen, R. G. D., *Mathematical Analysis for Economists*, Macmillan and Co. Ltd., New York, 1962.

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