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NOTE ON A CHARACTERIZING PROPERTY OF THE EXPONENTIAL DISTRIBUTION

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0. Summary. In this note we discuss a characteristic property of the exponential distribution which is closely related to the "lack-of-memory" characterization.

1. Introduction. We begin with a random variable (rv) X having an exponential distribution:

$$\Pr(X > x) = U(x) = \exp(-kx), \qquad k > 0, \qquad x \geqq 0 \tag{1}$$

and note that

$$U(y+z) - U(y)U(z) = 0 \qquad y \geqq 0, z \geqq 0. \tag{2}$$

The lack-of-memory characterization is obtained by showing that the only solution of (2), where $1-U(\cdot)$ is the cdf of some rv X, is of the form (1) (Feller [1]).

Now if Y and Z are arbitrary rv's, it follows from (2) that

$$\textstyle\int_0^\infty \int_0^\infty [U(y+z) - U(y)U(z)]\, dG(y)\, dH(z) = 0 \tag{3}$$

where $G(\cdot)$ and $H(\cdot)$ are the cdf's of Y and Z respectively. If we further assume that Y and Z are nonnegative, independent and independent of X, it is easy to see that (3) implies that

$$\Pr(X > Y+Z) = \Pr(X > Y)\Pr(X > Z). \tag{4}$$

In the following section we give conditions under which the validity of (4) implies that X is exponential. We can see immediately that if (4) is valid for all Y and Z in any class of rv's that includes all the nonnegative degenerates then (2) is implied and X must be exponential.

2. Characterizations. We introduce families $\mathscr{F}$ of nonnegative rv's that have the property

(a) For each interval $I \subset [0, \infty)$ with positive Lebesgue measure $\Pr(Y \in I) > 0$ for some $Y \in \mathscr{F}$.

In what follows we write $\Pr(X > x) = U(x)$ and

$$D(y, z) = U(y+z) - U(y)U(z).$$

THEOREM. *If* (i) X *is a nonnegative* rv *and* $U(x)$ *is continuous on* $(0, \infty)$ *and either* (b.1) $D(y, z) \geqq 0$, $y \geqq 0$, $z \geqq 0$ *or* (b.2) $D(y, z) \leqq 0$, $y \geqq 0$, $z \geqq 0$;

(ii) $\mathscr{A}$ *and* $\mathscr{B}$ *are families satisfying condition* (a) *such that every member of* $\mathscr{A}$ *is independent of every member of* $\mathscr{B}$ *and equation* (4) *holds whenever* $Y \in \mathscr{A}$ *and* $Z \in \mathscr{B}$, *then* X *is exponential.*

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PROOF. Suppose (b.1) holds and $D(y_0, z_0) = \delta > 0$ for some $y_0 \geqq 0, z_0 \geqq 0$. Then, since D is continuous there is a square neighborhood S of (y_0, z_0) with sides of positive Lebesgue measure such that $D > \delta/2$ in S. In accordance with condition (a) we can then choose Y and Z with cdf's $G(\cdot)$ and $H(\cdot)$ respectively such that the product GXH assigns a positive measure to S; for this choice the contribution of S to the integral in (3) is positive. But this is impossible because by hypothesis (3) holds for this pair of G and H also. The case (b.2) can also be treated in a similar way. This implies (2) and since $1 - U(x)$ is a cdf, X is exponential.

By properly choosing the families $\mathscr{A}$ and $\mathscr{B}$ we can easily prove the following.

COROLLARY. *If X, Y and Z are mutually independent nonnegative* rv's *such that X has a continuous* cdf *and satisfies either* (b.1) *or* (b.2) *and Y and Z assign positive measures to every interval $I \subset [0, \infty)$ of positive Lebesgue measure, then*

(i) (4) *holds* iff *X is exponential and*
(ii) $\Pr(X - Y > t) = \Pr(X > Y)\Pr(X > t)$, $t \geqq 0$ iff *X is exponential.*

Lastly we state a result which does not require the conditions (a), (b.1) and (b.2).

PROPOSITION. *If X and Y are independent and nonnegative such that* $\Pr(X - Y > 0) > 0$, *then*

$$\Pr(X - Y > x + y) = \Pr(X - Y > x)\Pr(X > y), \qquad x \geqq 0, y \geqq 0 \tag{5}$$

iff *X is exponential.*

PROOF. The "if" part is obvious from Section 1. Writing $W(x) = \Pr(X - Y > x)$ (5) implies that $W(x+y) = W(x)U(y)$, $x \geqq 0$, $y \geqq 0$ and hence $W(s) = W(0)U(s)$, $s \geqq 0$, so that $W(0)U(x+y) = W(0)U(x)U(y)$ for $x \geqq 0$, $y \geqq 0$ which implies (2) since $W(0) > 0$.

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REFERENCE

[1] FELLER, W. (1957). *An Introduction to Probability Theory and its Applications* **1**. Wiley, New York.