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NOTE ON A CONJECTURE FOR THE SUM OF SIGNLESS LAPLACIAN EIGENVALUES

Xiaodan Chen, Nanning, Guoliang Hao, Nanchang,
Dequan Jin, Nanning, Jingjian Li, Nanning

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Abstract. For a simple graph G on n vertices and an integer k with $1 \leqslant k \leqslant n$, denote by $\mathcal{S}_k^+(G)$ the sum of k largest signless Laplacian eigenvalues of G. It was conjectured that $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$, where $e(G)$ is the number of edges of G. This conjecture has been proved to be true for all graphs when $k \in \{1, 2, n-1, n\}$, and for trees, unicyclic graphs, bicyclic graphs and regular graphs (for all k). In this note, this conjecture is proved to be true for all graphs when $k = n-2$, and for some new classes of graphs.

Keywords: sum of signless Laplacian eigenvalues; upper bound; clique number; girth

MSC 2010: 05C50, 15A18

1. Introduction

All graphs considered in this note are finite, undirected and simple. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, and let $e(G) = |E(G)|$. Denote by $d_G(v_i)$ the degree of the vertex v_i in G. The adjacency matrix of G is $A(G) = [a_{ij}]_{n\times n}$, where $a_{ij} = 1$ if the vertices v_i and v_j in G are adjacent, and $a_{ij} = 0$ otherwise. The Laplacian matrix and the signless Laplacian matrix of G are, respectively, defined to be $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. The eigenvalues of $L(G)$ and $Q(G)$, usually called the Laplacian eigenvalues and the signless Laplacian eigenvalues of the graph G, are arranged (in non-increasing order)

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as $\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G) = 0$ and $q_1(G) \geqslant q_2(G) \geqslant \ldots \geqslant q_n(G) \geqslant 0$, respectively. It is known that if G is bipartite, then $L(G)$ and $Q(G)$ are similar, and hence their eigenvalues are identical (see, e.g., [4], p. 217). For more details on the Laplacian eigenvalues and the signless Laplacian eigenvalues of graphs one may refer to [3], [4].

For an integer k with $1 \leqslant k \leqslant n$, let $\mathcal{S}_k(G)$ be the sum of k largest Laplacian eigenvalues of a graph G, that is, $\mathcal{S}_k(G) = \sum\limits_{i=1}^{k} \mu_i(G)$. Grone and Merris [9] conjectured that for any graph G with n vertices and each $k \in \{1, 2, \ldots, n\}$,

$$\mathcal{S}_k(G) \leqslant \sum_{i=1}^{k} |\{v \in V(G) \colon \ d_G(v) \geqslant\}|.$$

This conjecture has been proved to be true by Bai [2] recently and now is called the Grone-Merris theorem. As a variation of the Grone-Merris theorem, Brouwer [3] conjectured that for any graph G with n vertices and each $k \in \{1, 2, \ldots, n\}$,

$$\mathcal{S}_k(G) \leqslant e(G) + \binom{k+1}{2}.$$

Brouwer's conjecture has attracted the attention of many researchers, but has not been settled yet. For the progress on this conjecture one can see [5], [7], [8], [10], [11], [12].

Analogously to the definition of $\mathcal{S}_k(G)$, let $\mathcal{S}_k^+(G)$ be the sum of k largest signless Laplacian eigenvalues of the graph G, that is, $\mathcal{S}_k^+(G) = \sum\limits_{i=1}^{k} q_i(G)$, $k \in \{1, 2, \ldots, n\}$. Motivated by Brouwer's conjecture, Ashraf et al. [1] posed the following conjecture:

Conjecture 1.1. For any graph G with n vertices and each $k \in \{1, 2, \ldots, n\}$,

$$\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}.$$

By a computer search, Ashraf et al. [1] confirmed Conjecture 1.1 for all graphs with at most 10 vertices. They also proved that Conjecture 1.1 is true for all graphs when $k \in \{1, 2, n-1, n\}$, and for regular graphs (for all k). In addition, they pointed out that Conjecture 1.1 holds for trees (for all k), since $\mathcal{S}_k^+(G) = \mathcal{S}_k(G)$ holds when G is bipartite. Recently, Yang and You [13] further showed that Conjecture 1.1 is true for unicyclic and bicyclic graphs (for all k).

In this note, we continue to explore Conjecture 1.1. We will show that Conjecture 1.1 holds for all graphs when $k = n - 2$, and for some new classes of graphs.

2. Lemmas and results

As usual, we denote by K_n and $K_{1,\,n-1}$ the complete graph and the star with n vertices, respectively. Let $G_1 \cup G_2$ denote the vertex-disjoint union of two graphs G_1 and G_2, and let kG denote the vertex-disjoint union of k copies of the graph G. For a subgraph H of G, write $G - E(H)$ for the spanning subgraph of G whose edge set is $E(G) \setminus E(H)$. The complement of G is denoted by $\overline{G}$.

Lemma 2.1 ([4]). *Let G be a graph with $n \geqslant 2$ vertices and let G' be an edge-deleted subgraph of G, that is, $G' = G - E(K_2)$. Then $q_1(G) \geqslant q_1(G') \geqslant q_2(G) \geqslant q_2(G') \geqslant \ldots \geqslant q_n(G) \geqslant q_n(G')$.*

It is known that $q_1(K_n) = 2n-2$ and $q_2(K_n) = \ldots = q_n(K_n) = n-2$ (see, e.g., [1]). This, together with Lemma 2.1, yields the next lemma.

Lemma 2.2. *If G is a graph of order $n \geqslant 2$, then $q_1(G) \leqslant 2n-2$ and $q_i(G) \leqslant n-2$ for $i = 2, \ldots, n$.*

For an $n \times n$ Hermitian matrix M, we arrange its eigenvalues (in non-increasing order) as $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \ldots \geqslant \lambda_n(M)$. The following result is the well-known Courant-Weyl inequality (see, e.g., [4], p. 19).

Lemma 2.3 ([4]). *If A and B are $n \times n$ Hermitian matrices, then for $n \geqslant i \geqslant j \geqslant 1$, $\lambda_i(A+B) \leqslant \lambda_j(A) + \lambda_{i-j+1}(B)$.*

The next result gives a relation between the signless Laplacian eigenvalues of G and those of $\overline{G}$, which can be deduced from Lemma 2.3 by bearing in mind that $Q(K_n) = Q(G) + Q(\overline{G})$ and $q_n(K_n) = n-2$.

Lemma 2.4. *If G is a graph with $n \geqslant 2$ vertices and $\overline{G}$ is its complement, then for $i = 1, 2, \ldots, n$, $q_i(\overline{G}) \geqslant n - 2 - q_{n-i+1}(G)$.*

Lemma 2.5 ([13]). *If G is a graph with n vertices and $G_1, G_2, \ldots, G_t$ are its edge-disjoint subgraphs with $E(G) = \bigcup\limits_{i=1}^{t} E(G_i)$, then for any integer k with $1 \leqslant k \leqslant n$, $\mathcal{S}_k^+(G) \leqslant \sum\limits_{i=1}^{t} \mathcal{S}_k^+(G_i)$, where $\mathcal{S}_k^+(G_i) = \mathcal{S}_{n_i}^+(G_i)$ if $k > |V(G_i)| = n_i$.*

Lemma 2.6 ([13]). *If G is a connected graph with n vertices, then for any integer k with*

$$\frac{3n - 4 + \sqrt{8n^2(e(G) - n + 1) + (n-4)^2}}{2n} \leqslant k \leqslant n, \quad \mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}.$$

It is shown [10], [12] that for any acyclic graph (i.e., tree or forest) F of order n, $\mathcal{S}_k(F) \leqslant e(F)+\binom{k+1}{2}$ holds for all $k \in \{1,2,\ldots,n\}$. This, together with the fact that $\mathcal{S}_k^+(G) = \mathcal{S}_k(G)$ holds for any bipartite graph G, yields the following result directly.

Lemma 2.7. *If F is an acyclic graph (i.e., tree or forest) with $n \geqslant 2$ vertices, then for any integer k with $1 \leqslant k \leqslant n$, $\mathcal{S}_k^+(F) \leqslant e(F) + \binom{k+1}{2}$.*

Lemma 2.8 ([6]). *If G is a connected graph with $n \geqslant 2$ vertices, then $q_1(G) \leqslant 2e(G)/(n-1)+n-2$, with equality if and only if $G \cong K_{1,n-1}$, or $G \cong K_n$.*

Now, we are in position to present the main results of this note.

Theorem 2.9. *For $n \geqslant 3$, let p be an integer with $1 \leqslant p \leqslant n/3$. If Conjecture* 1.1 *holds for all graphs when $k = p$, then Conjecture* 1.1 *holds for all graphs when $k = n-p$ as well.*

Proof. Suppose that G is any graph with $n \geqslant 3$ vertices and $\overline{G}$ is its complement. The hypothesis of the theorem implies that

$$\mathcal{S}_p^+(G) \leqslant e(G) + \binom{p+1}{2} \quad \text{and} \quad \mathcal{S}_p^+(\overline{G}) \leqslant e(\overline{G}) + \binom{p+1}{2}.$$

We now just need to show that

$$\mathcal{S}_{n-p}^+(G) \leqslant e(G) + \binom{n-p+1}{2}.$$

Indeed, bearing in mind the well-known fact that $\sum\limits_{i=1}^{n} q_i(G) = 2e(G)$, we have

$$\begin{aligned}
\mathcal{S}_{n-p}^+(G) &= 2e(G) - \sum_{i=n-p+1}^{n} q_i(G) \\
&\leqslant 2e(G) - \sum_{i=1}^{p}(n-2-q_i(\overline{G})) \quad \text{(by Lemma 2.4)} \\
&= 2e(G) - p(n-2) + \mathcal{S}_p^+(\overline{G}) \\
&\leqslant 2e(G) - p(n-2) + e(\overline{G}) + \binom{p+1}{2} \\
&= e(G) + \binom{n}{2} - p(n-2) + \binom{p+1}{2} \quad (\text{as } e(G)+e(\overline{G}) = \tbinom{n}{2}) \\
&= e(G) + \frac{n^2-(2p+1)n+p^2+5p}{2} \\
&\leqslant e(G) + \binom{n-p+1}{2} \quad (\text{as } n \geqslant 3p)
\end{aligned}$$

as desired. □

It is known [1] that Conjecture 1.1 holds for all graphs when $n \leqslant 10$ or $k = 2$. This, together with Theorem 2.9, yields the following corollary, which asserts that Conjecture 1.1 holds for all graphs when $k = n - 2$.

Corollary 2.10. *If G is a graph with $n \geqslant 3$ vertices, then $\mathcal{S}_{n-2}^{+}(G) \leqslant e(G)+\binom{n-1}{2}$.*

It is also worth pointing out that Theorem 2.9 suggests that to prove Conjecture 1.1, it is sufficient to prove Conjecture 1.1 for all graphs when $1 \leqslant k \leqslant 2n/3$. As an application of this idea, we may derive the following result, which, in some sense, can be regarded as a partial solution to Conjecture 1.1.

Theorem 2.11. *Let G be a graph with $n > 3$ vertices. If*

$$e(G) \geqslant \frac{(n-1)(4n^2-15n)}{9(n-3)},$$

then $\mathcal{S}_k^{+}(G) \leqslant e(G) + \binom{k+1}{2}$ holds for $1 \leqslant k \leqslant 2n/3$.

Proof. For $1 \leqslant k \leqslant 2n/3$, by Lemmas 2.2 and 2.8, we have

$$\mathcal{S}_k^{+}(G) \leqslant \frac{2e(G)}{n-1} + n - 2 + (k-1)(n-2) \leqslant e(G) + \binom{k+1}{2},$$

provided that

$$k^2 - (2n-5)k + \frac{2(n-3)}{n-1}e(G) \geqslant 0. \tag{2.1}$$

To complete this proof, we just need to prove that (2.1) holds for $1 \leqslant k \leqslant 2n/3$ when

$$e(G) \geqslant \frac{(n-1)(4n^2-15n)}{9(n-3)}.$$

Now, consider the quadratic equation

$$f(x) = x^2 - (2n-5)x + \frac{2(n-3)}{n-1}e(G) = 0,$$

with the discriminant being

$$\Delta = (2n-5)^2 - \frac{8(n-3)}{n-1}e(G).$$

It is easy to see that if $\Delta \leqslant 0$, that is,

$$e(G) \geqslant \frac{(n-1)(2n-5)^2}{8(n-3)} \quad \Big(> \frac{(n-1)(4n^2-15n)}{9(n-3)}\Big),$$

then $f(x) \geqslant 0$ holds for any real number x and hence, (2.1) follows. Otherwise, the least root of $f(x) = 0$ is

$$\alpha = \frac{2n-5-\sqrt{(2n-5)^2-\frac{8(n-3)}{n-1}e(G)}}{2},$$

which shows that $f(x) \geqslant 0$ holds for $x \leqslant \alpha$. Thus, (2.1) holds for $1 \leqslant k \leqslant 2n/3$ if

$$\frac{2n-5-\sqrt{(2n-5)^2-\frac{8(n-3)}{n-1}e(G)}}{2} \geqslant \frac{2n}{3},$$

that is,

$$(2n-15)^2 \geqslant 9\Big[(2n-5)^2 - \frac{8(n-3)}{n-1}e(G)\Big],$$

which follows when

$$e(G) \geqslant \frac{(n-1)(4n^2-15n)}{9(n-3)}.$$

This completes the proof. □

Remark. By the above proof, one may draw a stronger conclusion (from a somewhat stronger assumption): for any graph G with $n > 3$ vertices, if

$$e(G) \geqslant \frac{(n-1)(2n-5)^2}{8(n-3)},$$

then $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$ holds for $1 \leqslant k \leqslant n$; in other words, Conjecture 1.1 holds for the graphs with at least

$$\frac{(n-1)(2n-5)^2}{8(n-3)}$$

edges. In particular, a direct calculation shows that

$$\frac{(n-1)(2n-5)^2}{8(n-3)} \leqslant \binom{n}{2} - (n-2),$$

which implies that Conjecture 1.1 holds for the graphs obtained from K_n by deleting at most $n-2$ edges.

Recall that the clique number of a graph G, denoted by $\omega(G)$, is the number of vertices of a maximum complete subgraph contained in G. By Theorem 2.11, we may obtain the next corollary.

Corollary 2.12. *Let G be a connected graph with $n > 3$ vertices. If $\omega(G) \geqslant (2\sqrt{2}n-1)/3$, then $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$ holds for $1 \leqslant k \leqslant 2n/3$.*

P r o o f. Note that for a connected graph G we have $e(G) \geqslant \frac{1}{2}\omega(G)(\omega(G)-1)+n-\omega(G)$. Thus, by Theorem 2.11, it suffices to show that

$$\frac{1}{2}\omega(G)(\omega(G)-1)+n-\omega(G) \geqslant \frac{(n-1)(4n^2-15n)}{9(n-3)},$$

that is,

$$9(n-3)\omega(G)^2-27(n-3)\omega(G)-4n(2n^2-14n+21) \geqslant 0.$$

Consider the following quadratic equation:

$$h(x)=9(n-3)x^2-27(n-3)x-4n(2n^2-14n+21)=0.$$

It is easy to check that the largest root of $h(x)=0$ is

$$\beta=\frac{9(n-3)+\sqrt{81(n-3)^2+16n(n-3)(2n^2-14n+21)}}{6(n-3)},$$

which shows that $h(x) \geqslant 0$ holds for $x \geqslant \beta$. Thus, to complete this proof, we just need to prove that

$$\frac{9(n-3)+\sqrt{81(n-3)^2+16n(n-3)(2n^2-14n+21)}}{6(n-3)} \leqslant \frac{2\sqrt{2}n-1}{3},$$

that is,

$$81(n-3)+16n(2n^2-14n+21) \leqslant (n-3)\big(4\sqrt{2}n-11\big)^2,$$

that is,

$$\big(16-11\sqrt{2}\big)n^2+\big(33\sqrt{2}-37\big)n-15 \geqslant 0,$$

which holds for $n>3$ (since its largest root is 1.4543), completing the proof. $\square$

Recall that the girth of a graph G (of order n), denoted by $g(G)$, is the length (i.e., the number of edges) of a shortest cycle contained in G. Clearly, $3 \leqslant g(G) \leqslant n$. We here make a convention that a graph G is acyclic if and only if $g(G)>n$.

Theorem 2.13. *If G is a graph with n vertices and $n \geqslant g(G) \geqslant g \geqslant 4$, then $\mathcal{S}_k^+(G) \leqslant e(G)+\binom{k+1}{2}$ holds for $1 \leqslant k \leqslant \lfloor g/4 \rfloor$.*

P r o o f. Some idea of the proof comes from Lemma 15 in [1]. Let G be a counterexample for the theorem having a minimum number of edges. Then we have that

$$\mathcal{S}_k^+(G) > e(G)+\binom{k+1}{2} \tag{2.2}$$

holds for some k with $1 \leqslant k \leqslant \lfloor g/4 \rfloor$.

Note that G contains a cycle of length at least g and hence, it contains a subgraph $\lfloor g/2\rfloor K_2$ (belonging to the cycle) as well. It is easy to check that for $1 \leqslant k \leqslant \lfloor g/4\rfloor$,

$$\mathcal{S}_k^+(\lfloor g/2\rfloor K_2) = 2k \leqslant \lfloor g/2\rfloor = e(\lfloor g/2\rfloor K_2). \tag{2.3}$$

Now, let $G' = G - E(\lfloor g/2\rfloor K_2)$. By Lemma 2.5, together with (2.2) and (2.3), we obtain

$$e(G) + \binom{k+1}{2} < \mathcal{S}_k^+(G) \leqslant \mathcal{S}_k^+(\lfloor g/2\rfloor K_2) + \mathcal{S}_k^+(G') \leqslant e(\lfloor g/2\rfloor K_2) + \mathcal{S}_k^+(G'),$$

from which, as well as the fact that $e(G') = e(G) - e(\lfloor g/2\rfloor K_2)$, we conclude that

$$\mathcal{S}_k^+(G') > e(G') + \binom{k+1}{2} \tag{2.4}$$

holds for some k with $1 \leqslant k \leqslant \lfloor g/4\rfloor$.

To complete the proof, it remains to be shown that $n \geqslant g(G') \geqslant g$, which, together with (2.4), would yield that G' is also a counterexample for the theorem but has fewer edges than G, contradicting the minimality of G. Indeed, on one hand, by the definition of girth, we have $g(G') \geqslant g(G) \geqslant g$. On the other hand, if $g(G') > n$, then G' is acyclic and consequently, by Lemma 2.7, we have that $\mathcal{S}_k^+(G') \leqslant e(G') + \binom{k+1}{2}$ holds for all k, contradicting (2.4).

This completes the proof. $\square$

Theorem 2.14. *Let G be a connected graph with $n \geqslant 4$ vertices. If*

$$g(G) \geqslant 6 + 2\sqrt{8(e(G) - n + 1) + 1}, \tag{2.5}$$

then $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$ holds for $1 \leqslant k \leqslant n$.

P r o o f. Since G is connected, we have $e(G) - n + 1 \geqslant 0$ and hence, by (2.5), we obtain $g(G) \geqslant 6$. Consequently, for any integer k with $1 \leqslant k \leqslant g(G)/4$, Theorem 2.13 yields directly that $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$. On the other hand, for $g(G)/4 < k \leqslant n$, again by (2.5) we get

$$n \geqslant k > \frac{g(G)}{4} > \frac{3n - 4 + \sqrt{8n^2(e(G) - n + 1) + (n-4)^2}}{2n},$$

which, together with Lemma 2.6, also yields that $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$. Combining these two cases, we obtain the desired result, completing the proof. $\square$

Remark. Theorem 2.14 asserts that Conjecture 1.1 holds for the connected graphs having sufficiently large girth relative to the number of edges. Moreover, recall that a connected graph with n vertices and c cycles, usually called a c-cyclic graph, has $n-1+c$ edges. We can now restate Theorem 2.14 as follows: if G a c-cyclic graph with $g(G) \geqslant 6 + 2\sqrt{8c+1}$, then $\mathcal{S}_k^+(G) \leqslant e(G) + \binom{k+1}{2}$ holds for $1 \leqslant k \leqslant n$, from which one can easily conclude that Conjecture 1.1 is true for the 3-cyclic graphs with girth at least 16, and for the 4-cyclic graphs with girth at least 18, and so on. It should be mentioned that for the cases of $c = 0, 1$ and 2, the theorem still holds when the restriction $g(G) \geqslant 6 + 2\sqrt{8c+1}$ is removed (see [1], [13] for details).

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Authors' addresses: Xiaodan Chen, College of Mathematics and Information Science, Guangxi University, No. 100, Daxue Road, Nanning, Guangxi, P. R. China, e-mail: `x.d.chen@live.cn`; Guoliang Hao (corresponding author), College of Science, East China University of Technology, No. 418, Guanglan Road, Nanchang 330013, Jiangxi, P. R. China, e-mail: `guoliang-hao@163.com`; Dequan Jin, Jingjian Li, College of Mathematics and Information Science, Guangxi University, No. 100, Daxue Road, Nanning, Guangxi, P. R. China, e-mail: `dqjin@yahoo.cn`, `lijjhx@163.com`.