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NOTE ON A NONLINEAR EIGENVALUE PROBLEM

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ABSTRACT. This note complements some known facts about the ordinary differential equation $(|u'|^{p-2}u')'+\lambda|u|^{p-2}u$

$= 0$. The eigenvalues exhibit a fascinating dependence on the exponent p, namely, $\sqrt[p]{\lambda_p} = \sqrt[q]{\lambda_q}$ for conjugate exponents. In terms of the Rayleigh quotients,

$$\min_u \frac{||u'||_p}{||u||_p} = \min_v \frac{||v'||_q}{||v||_q}, \qquad \frac{1}{p}+\frac{1}{q} = 1.$$

Corresponding eigenfunctions are related for conjugate exponents. We shall express this dependence in a nice formula.

1. Introduction. The minimum λ_p of the Rayleigh quotient

$$\frac{\int_a^b |u'(x)|^p\,dx}{\int_a^b |u(x)|^p\,dx}, \qquad 1 < p < \infty \tag{1}$$

taken among all real-valued functions $u \in C^1[a,b]$ with $u(a) = u(b) = 0$ is equal to the first eigenvalue λ of the equation

$$\frac{d}{dx}(|u'|^{p-2}u') + \lambda|u|^{p-2}u = 0. \tag{2}$$

(The resulting sharp estimate $\sqrt[p]{\lambda_p}||u||_p \le ||u'||_p$ is called Wirtinger's inequality in the classical case $p = 2$, when the equation reduces to $u'' + \lambda u = 0$.) The existence of eigenvalues and eigenfunctions has been considered in [**1**, Theorem 4.4]. This problem has been thoroughly studied by M. Ôtani. He has explicitly determined all eigenvalues and described the eigenfunctions and their zeros, cf. [**4**]. These results are so exhaustive that it seems difficult to add anything

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relevant to the original problem, and so the trend has been to generalize the equation. For example, in [**2**] Ôtani's approach is applied to the equation $(|u'|^{p-2}u')' + \lambda|u|^{p-2}u = f(u)$, and in [**6**] equations of the type $(|u'|^{p-2}u')' + a(x)|u|^{p-2}u = 0$ are studied. See [**5**] for further generalizations.

However, we will consider the original and more pregnant formulation (2). Our starting point is a simple but striking observation.

Proposition. *For conjugate exponents, we have*

$$\sqrt[p]{\lambda_p} = \sqrt[q]{\lambda_q}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3}$$

This beautiful conjugation is an immediate consequence of Ôtani's formula [**4**, p. 28]

$$\lambda_p = (p-1)\left\{\frac{2}{b-a}\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\right\}^p \tag{4}$$

for the first eigenvalue of Equation (2). A direct evaluation of the integral (the so-called Eulerian integral of the first kind [**9**, 12.4]) yields

$$\sqrt[p]{\lambda_p} = \frac{2\sqrt[p]{p-1}\,\pi}{(b-a)p\sin(\pi/p)}. \tag{4'}$$

Moreover, even the higher eigenvalues appear in conjugate pairs. Thus, the whole spectrum exhibits the same conjugation property!

This interesting behavior reflects a fascinating dependence among the first eigenfunctions, say u_p and u_q, $1/p + 1/q = 1$. Namely, if one of these is known, the other one can be constructed by the aid of a nice formula. See (7). This is remarkable, although Equation (2) can be "completely" integrated, the reason being that for $p \neq 2$, the solution appears in the shape of a very unilluminating implicit function.[1] Moreover, there is a kind of conjugation among the higher eigenfunctions, too.

2. Conjugate eigenfunctions. For the necessary background, we refer the reader to [**4**] (especially Remark 8 on page 28 should be noticed). Section 1 of [**2**] is a resumé of Ôtani's paper [**4**]. Before proving the conjugation formula, we will sketch these preliminaries.

The starting point is to consider all absolutely continuous functions $u : [a,b] \to \mathbf{R}$ with $u(a) = u(b) = 0$. In this considerably wide class of functions, the existence of a positive minimum $\lambda_p > 0$ for the Rayleigh quotient (1) is easily established. Interpreting equation (2) in its weak form to begin with, we see that its smallest eigenvalue evidently is $\lambda = \lambda_p$. The corresponding first eigenfunction (or ground state solution) u_p is unique up to a constant factor, and it has no zeros in the open interval $]a,b[$. According to [**4**, Lemma 3] $u_p \in C^2[a,b]$, if $1 < p \leq 2$; and $u_p \in C^1[a,b] \cap C^2(J)$, $J = [a,b]\backslash\{(a+b)/2\}$, if $2 < p < \infty$ (in the latter case, u_p is not twice differentiable at the midpoint; at this point $u_p' = 0$). Thus, one can work with classical solutions. Actually, u_p is real analytic in the open intervals $]a,(a+b)/2[$ and $](a+b)/2, b[$.

From now on, we shall normalize the situation so that $[a,b] = [0,1]$ and $u_p'(0) = 1$. Then $u = u_p$ is positive in $]0,1[$. By symmetry, $u(x) = u(1-x)$. Multiplying the equation by u' and integrating, we obtain

$$|u'|^p = 1 - \frac{\lambda}{p-1} u^p, \qquad u'(0) = 1 \tag{5}$$

and so we have to integrate

$$\pm \frac{du}{dx} = \left(1 - \frac{\lambda}{p-1} u^p\right)^{1/p}, \tag{6}$$

where the plus sign is valid for $0 \leq x \leq 1/2$ and the minus sign for $1/2 \leq x \leq 1$. Note that $u'(1/2) = 0$.

We are not going to explain how to find the fundamental formula (7) below, but, once it is given, the verification is easily done, as we shall see.

Theorem. *The first normalized eigenfunctions in* $[0,1]$ *are related by*

$$\frac{\lambda_p u_p(x)^p}{p-1} + \frac{\lambda_q u_q(y)^q}{q-1} = 1, \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{7}$$

where $y = 1/2 - x$, *when* $0 \leq x \leq 1/2$, *and* $y = 3/2 - x$, *when* $1/2 \leq x \leq 1$. *Moreover,* $\sqrt[p]{\lambda_p} = \sqrt[q]{\lambda_q}$.

Proof. Let us use the abbreviations

$$u(x) = u_p(x), \qquad \lambda = \lambda_p; \qquad v(x) = v_q(x), \qquad \mu = \lambda_q,$$

when $p + q = pq$. By symmetry, we may assume that $0 \leq x < 1/2$. Then $1 > \lambda u^p(x)/(p-1)$ and (6) yields

$$x = + \int_0^x dx = \int_0^{u(x)} \frac{du}{(1 - \frac{\lambda}{p-1} u^p)^{1/p}}, \qquad 0 \leq x \leq 1/2. \tag{8}$$

Anticipating the final result, we substitute

$$\frac{\lambda}{p-1} u^p + \frac{\mu}{q-1} v^q = 1, \qquad \lambda q u^{p-1}\, du + \mu p v^{q-1}\, dv = 0$$

in (8), and after some simplification, we arrive at

$$\int_0^{u(x)} \frac{du}{(1 - \frac{\lambda}{p-1} u^p)^{1/p}} = \frac{\mu^{1/q}}{\lambda^{1/p}} \int_?^{[(q-1)/\mu]^{1/q}} \frac{dv}{(1 - \frac{\mu}{q-1} v^q)^{1/q}}, \tag{9}$$

the lower limit of integration being

$$? = \left\{ \frac{q-1}{\mu} \left(1 - \frac{\lambda}{p-1} u(x)^p \right) \right\}^{1/q}.$$

On the other hand,

$$\frac{1}{2} - y = \int_y^{1/2} dy = \int_{v(y)}^{[(q-1)/\mu]^{1/q}} \frac{dv}{(1 - \frac{\mu}{q-1} v^q)^{1/q}}, \qquad 0 \leq y \leq 1/2 \tag{10}$$

by (6). Using the fact that $\mu^{1/q} = \lambda^{1/p}$, and comparing the integrals, we conclude that

$$v(y) = \left\{ \frac{q-1}{\mu} \left(1 - \frac{\lambda}{p-1} u(x)^p \right) \right\}^{1/q}, \tag{11}$$

if $1/2 - y = x$. Treating the values $1/2 \leq x \leq 1$ in a similar way, we arrive at the desired result

$$\frac{\lambda u(x)^p}{p-1} + \frac{\mu v(y)^q}{q-1} = 1, \qquad |x-y| = 1/2. \qquad \square$$

3. Higher eigenvalues. The equation

$$\frac{d}{dx}(|u'|^{p-2}u') + \lambda |u|^{p-2}u = 0$$

has nontrivial solutions in $[0,1]$ with zero endpoint values only for the following values of λ:

$$\lambda_p^{(k)} = k^p \lambda_p, \qquad k = 1, 2, 3, \dots ,$$

cf. [**4**]. Here $\lambda_p = \lambda_p^{(1)}$ is the first eigenvalue (4). (In the linear case we have the eigenvalues $k^2\pi^2$ corresponding to the normalized eigenfunctions $u_2^{(k)}(x) = \sin(k\pi x)/k\pi$, $k = 1, 2, 3, \dots$). Hence,

$$\sqrt[p]{\lambda_p^{(k)}} = \sqrt[q]{\lambda_q^{(k)}}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{12}$$

This means that also *the higher eigenvalues appear in conjugate pairs.*

The higher eigenfunctions $u_p^{(1)}, u_p^{(2)}, \dots$ and $u_q^{(1)}, u_q^{(2)}, \dots$ are pairwise related, i.e., given $u_p^{(k)}$ (or only u_p) the conjugate k'th eigenfunction $u_q^{(k)}$, $1/p + 1/q = 1$, can be constructed from an explicit formula. More precisely,

$$u_q^{(k)}(x) = (-1)^j u_q(kx - j), \quad j/k \leq x \leq (j+1)/k \tag{13}$$

for $j = 0, 1, 2, \dots, k-1$ according to [**4**, Theorem 1, Remark 8] or [**2**, Equation (1.4)], and so the missing link is provided by (7). Through this chain any $u_q^{(k)}$ can be constructed from any $u_p^{(m)}$, if $1/p + 1/q = 1$. It should be kept in mind that also the higher eigenfunctions are unique apart from a normalizing constant factor.[2] This is essential for the meaning of the construction.

4. Comparison of eigenfunctions. The first eigenfunction u_p of (2), normalized by $u'(0) = 1$ (hence, $u'(1) = -1$) has maximum

$$u_p(1/2) = \sqrt[p]{\frac{p-1}{\lambda_p}} = \frac{1}{2}\frac{\sin(\pi/p)}{(\pi/p)} \tag{14}$$

in $[0,1]$. Clearly, $0 < u_p(1/2) < 1/2$, when $1 < p < \infty$. The maximum increases with p. The same is true for any $u_p(x)$, as p increases. Indeed,

$$u_s(x) > u_p(x), \qquad 1 < p < s < \infty, \tag{15}$$

when $0 < x < 1$. This follows almost directly from (6), when one uses the fact that, for any fixed t in $]0,1[$, the expression $(1-t^p)^{1/p}$ increases with p.

In particular[3],

$$0 < u_q(x) < \frac{\sin(\pi x)}{\pi} < u_p(x) < \frac{1}{2} - \left| x - \frac{1}{2} \right| \tag{16}$$

when $1 < q < 2 < p < \infty$ and $0 < x < 1$. Here the function $u_\infty(x) = 1/2 - |x - 1/2|$ is the solution to the "minimax" problem

$$\min_u \left\{ \frac{\max_x |u'(x)|}{\max_x |u(x)|} \right\} = 2 \tag{17}$$

obtained when $p \to \infty$ (the admissible functions being merely absolutely continuous). One can show that $\lim_{q\to\infty} u_p(x) = u_\infty(x)$ uniformly in $[0,1]$. By (7) also $\lim_{q\to 1+} u_q(x) = 0$ uniformly in $[0,1]$. (However, the limiting eigenvalue $\lambda_1 = \lim_{q\to 1+} \lambda_q = 2$ is not attained for any reasonable admissible function with zero end point values.)

The curves $y = u_p(x)$, $0 < x < 1$, $1 < p < \infty$, form a field filling up the open triangle

$$\Delta = \{(x,y) \mid 0 < 2y < 1 - |2x-1|,\ 0 < x < 1\}.$$

To see this, one just has to show that for any fixed x in $]0,1[$, the function $p \to u_p(x)$ is continuous, $1 \le p \le \infty$ (we denote $u_1(x) = 0$, although this is not the solution to the problem, when $p = 1$). The desired continuity with respect to p can be read off from (8), λ_p varying continuously with p.

More can be said about this, but we think that the above gives a sufficiently clear picture of the situation.

5. Concluding remarks. In several dimensions little is known about the corresponding problem. Given a bounded domain Ω in the n-dimensional Euclidean space $\mathbf{R}^n$, the minimization of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \tag{18}$$

among all functions u (belonging to a convenient function space) with zero boundary values in Ω leads to the nonlinear eigenvalue problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0. \tag{19}$$

See, for example, [**3, 7, 8**]. The proper counterpart to the conjugation in one dimension is far from obvious: unfortunately, (3) does not hold even when Ω is a ball in $\mathbf{R}^n$ and $n \geq 2$.

We intend to return to this topic in a subsequent work.

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ENDNOTES

1. The simplest way to integrate the equation is indicated in [**6**]: the convex function $z = z(x) = |u'(x)|^{p-2}u'(x)/|u(x)|^{p-2}u(x)$ satisfies the separable equation $z' + (p-1)|z|^q + \lambda_p = 0$.

2. This is not true in general for the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ in several dimensions.

3. The exponents need not be conjugate here.

Note added in proof. Extending u_p as an odd function to the interval $[-1, 0]$ and, then, periodically to the whole real axis, i.e., $u_p(x) = -u_p(-x)$, $u_p(x+2) = u_p(x)$, we can write the higher eigenfunctions as $u_p^{(k)}(x) = u_p(kx)$.

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