# *Note on Casimir's Method of the Spin Summation in the Case of the Meson.*

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## Abstract.

A new form of the vector meson theory is described, in which six-rowed matrices having properties very similar to those of Dirac's matrices are introduced. Making use of this formalism, we have extended Casimir's method of the spin summation (technique of spur) to calculations for radiative processes of the meson. As an application of this method, a derivation of Laporte's scattering formula is given. The relation to Kemmer's form of the meson theory is also discussed.


## I.

According to the vector theory of the meson,[1] the Hamiltonian for the system consisting of mesons of mass $m_u$ and charge $\pm e$ in the electromagnetic field with the scalar and vector potentials $A_0$ and $\vec{A}$ is given by

$$\overline{H} = m_u c^2 \int \left[ \vec{\widetilde{F}} \left\{ \vec{F} - \frac{1}{\kappa^2} \vec{D} (\vec{D}\vec{F}) + \frac{ie}{m_u c^2} A_0 \vec{U} \right\} + \right.$$

$$\left. + \vec{\widetilde{U}} \left\{ \vec{U} + \frac{1}{\kappa^2} \vec{D} \times (\vec{D} \times \vec{U}) - \frac{ie}{m_u c^2} A_0 \vec{F} \right\} \right] \cdot dv, \qquad (1)$$

where $\vec{U}$, $\vec{\widetilde{U}}$, $\vec{F}$, and $\vec{\widetilde{F}}$ are the field variables satisfying commutation

(1) YUKAWA, SAKATA, and TAKETANI: *Proc. Phys. Math. Soc. Japan*, **20** (1938), 319; KEMMER: *Proc. Roy. Soc. A*, **166** (1938), 127.

relations as follows:*

$$\left.\begin{aligned}&\left[\tilde{F}_i(\vec{r},\ t),\ U_k(\vec{r}',\ t)\right]=i\delta(\vec{r}-\vec{r}')\delta_{ik},\\&\left[F_i(\vec{r},\ t),\ \tilde{U}_k(\vec{r}',\ t)\right]=i\delta(\vec{r}-\vec{r}')\delta_{ik},\\&\left[\tilde{F}_i(\vec{r},\ t),\ F_k(\vec{r}',\ t)\right]=0,\\&\left[\tilde{F}_i(\vec{r},\ t),\ \tilde{U}_k(\vec{r}',\ t)\right]=0,\\&\text{etc.,}\end{aligned}\right\}\tag{2}$$

and we define

$$\left.\begin{aligned}\vec{D}&=\text{grad}-\frac{ie}{\hbar c}\vec{A},\\\kappa&=\frac{m_u c}{\hbar}\end{aligned}\right\}\tag{3}$$

The equations of motion derived from the above Hamiltonian become

$$\left.\begin{aligned}\frac{1}{c}\frac{\partial\vec{F}}{\partial t}&=\frac{1}{\kappa}\vec{D}\times(\vec{D}\times\vec{U})+\kappa U-\frac{ie}{\hbar c}A_0\vec{F},\\\frac{1}{c}\frac{\partial\vec{U}}{\partial t}&=\frac{1}{\kappa}\vec{D}(\vec{D}\vec{F})-\kappa\vec{F}-\frac{ie}{\hbar c}A_0\vec{U},\end{aligned}\right\}\tag{4}$$

and conjugates.

Now, if we define wave functions $\tilde{\Psi}$ and $\Psi$, having six components, by

$$\left.\begin{aligned}\tilde{\Psi}&=(-\vec{\tilde{F}},\ \vec{\tilde{U}}),\\\Psi&=\begin{pmatrix}-\vec{F}\\\vec{U}\end{pmatrix},\end{aligned}\right\}\tag{5}$$

and six-rowed matrices $\rho_1$, $\rho_2$, $\rho_3$, and $\sigma_1$, $\sigma_2$, $\sigma_3$, acting on the $\Psi$, by

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* We shall use an abbreviation

$$[A,\ B]\equiv AB-BA.$$

$$\left.\begin{array}{l}
\rho_1=\left(\begin{array}{ccc:ccc} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hdashline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{array}\right),\ \rho_2=\left(\begin{array}{ccc:ccc} 0&0&0&-i&0&0\\ 0&0&0&0&-i&0\\ 0&0&0&0&0&-i\\ \hdashline i&0&0&0&0&0\\ 0&i&0&0&0&0\\ 0&0&i&0&0&0 \end{array}\right),\ \rho_3=\left(\begin{array}{ccc:ccc} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ \hdashline 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1 \end{array}\right),\\
\text{and}\\
\sigma_1=\left(\begin{array}{ccc:ccc} 0&0&0&0&0&0\\ 0&0&-i&0&0&0\\ 0&i&0&0&0&0\\ \hdashline 0&0&0&0&0&0\\ 0&0&0&0&0&-i\\ 0&0&0&0&i&0 \end{array}\right),\ \sigma_2=\left(\begin{array}{ccc:ccc} 0&0&i&0&0&0\\ 0&0&0&0&0&0\\ -i&0&0&0&0&0\\ \hdashline 0&0&0&0&0&i\\ 0&0&0&0&0&0\\ 0&0&0&-i&0&0 \end{array}\right),\ \sigma_3=\left(\begin{array}{ccc:ccc} 0&-i&0&0&0&0\\ i&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hdashline 0&0&0&0&-i&0\\ 0&0&0&i&0&0\\ 0&0&0&0&0&0 \end{array}\right),
\end{array}\right\}\quad (6)$$

then (1), (2), and (4) can be written in following forms:

$$\overline{H}=\int\widetilde{\Psi}\rho_2 H\Psi\,dv, \tag{7}$$

$$\left[(\widetilde{\Psi}\rho_2)_\mu,\ \Psi_\nu\right]=\left[\widetilde{\Psi}_\mu,\ (\rho_2\Psi)_\nu\right]=-\delta_{\mu\nu}\delta(\vec{r}-\vec{r}'), \tag{8}$$

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi, \tag{9}$$

where

$$\rho_2 H=m_u c^2\left[1-\frac{\vec{D}^2-\frac{e}{\hbar c}\cdot(\vec{\sigma},\vec{H})}{2\kappa^2}+\rho_2\cdot\frac{eA_0}{m_u c^2}+\rho_3\cdot\left\{\frac{(\vec{\sigma},\vec{D})^2}{\kappa^2}-\frac{\vec{D}^2-\frac{e}{\hbar c}\cdot(\vec{\sigma},\vec{H})}{2\kappa^2}\right\}\right], \tag{10}$$

with

$$\vec{H}=\mathrm{rot}\,\vec{A},\ \text{and}\ \vec{\sigma}=(\sigma_1,\ \sigma_2,\ \sigma_3).$$

Similarly, all other quantities can also be expressed in terms of these operators. For example, the charge density $S_0$ is written as

$$S_0=\widetilde{\Psi}\rho_2\Psi. \tag{11}$$

From definitions (6), $\rho$'s and $\sigma$'s satisfy following algebraic relations

$$\left.\begin{array}{l}\rho_1^2=\rho_2^2=\rho_3^2=1,\\ \rho_1\rho_2=i\rho_3=-\rho_2\rho_1,\ \rho_2\rho_3=i\rho_1=-\rho_3\rho_2,\ \rho_3\rho_1=i\rho_2=-\rho_1\rho_3,\\ \sigma_1\sigma_2-\sigma_2\sigma_1=i\sigma_3,\ \sigma_2\sigma_3-\sigma_3\sigma_2=i\sigma_1,\ \sigma_3\sigma_1-\sigma_1\sigma_3=i\sigma_2,\\ \sigma_1^1+\sigma_2^2+\sigma_3^2=2,\\ \text{etc.,}\end{array}\right\} \tag{12}$$

Further, we can easily prove that $\sigma$'s correspond to the spin operators of meson. These properties are very similar to those of Dirac's matrices.

## II.

If we treat $\Psi$ as a classical field, the equation (4) or (9) may be interpreted as a wave equation for single particle with spin 1. For a free particle, this equation has the solution

$$\Psi_\mu=u_\mu\cdot e^{i\frac{\vec{p}\vec{r}}{\hbar c}-i\frac{Et}{\hbar}} \tag{13}$$

where $\vec{p}$ and $E$ are constants satisfying the relation

$$E=\pm\sqrt{\vec{p}^2+m_u^2c^4}, \tag{14}$$

and the amplitude $u$ is the solution of the equation

$$E\rho_2u=\left[\frac{\vec{p}^2}{2m_uc^2}+m_uc^2+\rho_3\left\{\frac{\vec{p}^2}{2m_uc^2}-\frac{(\vec{\sigma},\vec{p})^2}{m_uc^2}\right\}\right]u. \tag{15}$$

For a given momentum $\vec{p}/c$, there are six independent solutions of (15) which we shall denote

$$u(\vec{p},\lambda,\varepsilon), \tag{16}$$

where $\lambda=1, 0, -1$, and $\varepsilon=1, -1$ correspond to three polarization states* and two charge states respectively, and where the latter is defined by

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* $\lambda=0$ denotes the state represented by the longitudinal wave, while $\lambda=\pm1$ denote those represented by the transverse waves polarized in directions perpendicular to each other.

$$\varepsilon = \frac{E}{|E|}. \tag{17}$$

(16) has to be normalized so that

$$\tilde{u}(\vec{p}, \lambda, \varepsilon)\rho_2 u(\vec{p}, \lambda', \varepsilon') = \varepsilon\delta_{\lambda\lambda'}\delta_{\varepsilon\varepsilon'}. \tag{18}$$

From (18), we obtain the converse orthogonality relations

$$\left.\begin{aligned} \sum_{\lambda=1,0,-1}\sum_{\varepsilon=1,-1}\varepsilon\left\{\tilde{u}(\vec{p}, \lambda, \varepsilon)\rho_2\right\}_\mu \cdot u_\nu(\vec{p}, \lambda, \varepsilon) &= \delta_{\mu\nu}, \\ \sum_{\lambda=1,0,-1}\sum_{\varepsilon=1,-1}\varepsilon\cdot\tilde{u}_\mu(\vec{p}, \lambda, \varepsilon)\cdot\left\{\rho_2 u(\vec{p}, \lambda, \varepsilon)\right\}_\nu &= \delta_{\mu\nu}. \end{aligned}\right\} \tag{19}$$

If we define the operators $\Lambda^+(\vec{p})$ and $\Lambda^-(\vec{p})$ by

$$\Lambda^\pm(\vec{p}) = \frac{\rho_2 E \pm\left[\dfrac{\vec{p}^2}{2m_u c^2} + m_u c^2 + \rho_3\left\{\dfrac{\vec{p}^2}{2m_u c^2} - \dfrac{(\vec{\sigma}, \vec{p})^2}{m_u c^2}\right\}\right]}{2E}, \tag{20}$$

they will have the same properties as "annihilation operators" in Dirac's theory of the electron:

$$\Lambda^\pm(\vec{p})\cdot\rho_2\cdot u(\vec{p}, \lambda, \varepsilon) = \frac{\varepsilon\pm 1}{2}u(\vec{p}, \lambda, \varepsilon). \tag{21}$$

## III.

Now, we return to the second quantization. Using plane wave solutions (13), we expand $\Psi$ and $\tilde{\Psi}$ as

$$\left.\begin{aligned} \Psi &= \sum_{\vec{p},\lambda,\varepsilon} a(\vec{p}, \lambda, \varepsilon)\cdot u(\vec{p}, \lambda, \varepsilon)\cdot e^{i\frac{\vec{p}\vec{r}}{\hbar c} - i\frac{Et}{\hbar}}, \\ \tilde{\Psi} &= \sum_{\vec{p},\lambda,\varepsilon} a^*(\vec{p}, \lambda, \varepsilon)\cdot\tilde{u}(\vec{p}, \lambda, \varepsilon)\cdot e^{-i\frac{\vec{p}\vec{r}}{\hbar c} + i\frac{Et}{\hbar}}, \end{aligned}\right\} \tag{22}$$

where $a(\vec{p}, \lambda, \varepsilon)$ and $a^*(\vec{p}, \lambda, \varepsilon)$ are operators satisfying commutation relations

$$\left[a(\vec{p}, \lambda, \varepsilon), a^*(\vec{p}, \lambda', \varepsilon')\right] = \varepsilon\cdot\delta_{\lambda,\lambda'}\delta_{\varepsilon,\varepsilon'}. \tag{23}$$

If we introduce new variables

$$\left.\begin{aligned} N^+(\vec{p}, \lambda) &= a^*(\vec{p}, \lambda, 1)\cdot a(\vec{p}, \lambda, 1), \\ \text{and}\qquad & \\ N^-(\vec{p}, \lambda) &= a(\vec{p}, \lambda, -1)a^*(\vec{p}, \lambda, -1), \end{aligned}\right\} \tag{24}$$

the total charge and the total energy of the field, which consists of mesons only, take diagonal forms:

$$\overline{S}_0 = e\sum_{\vec{p},\lambda}\left\{N^+(\vec{p}, \lambda) - N^-(\vec{p}, \lambda)\right\}, \tag{25}$$

and

$$\overline{H}_u = \sum_{\vec{p},\lambda} E \cdot \left\{N^+(\vec{p}, \lambda) + N^-(\vec{p}, \lambda) + 1\right\}. \tag{26}$$

From these expressions and that for the total momentum, we can consider that the operator $N^\pm(\vec{p}, \lambda)$ denotes the number of the mesons with the charge $\pm e$ in the state of energy $\sqrt{p^2+m_u^2c^4}$, momentum $\pm\vec{p}/c$ and polarization $\lambda$. Hence, $a^*(\vec{p}, \lambda, 1)$ and $a(\vec{p}, \lambda, -1)$ denote the operators which increase the number $N^+(\vec{p}, \lambda)$ and $N^-(\vec{p}, \lambda)$ by one respectively, whereas $a(\vec{p}, \lambda, -1)$ and $a^*(\vec{p}, \lambda, -1)$ denote those which decrease them by one respectively.

The interaction energy between the mesons and the electromagnetic field is given by $\overline{H}'_e$, where

$$\overline{H}'_e = \overline{H} - \overline{H}_u. \tag{27}$$

Inserting (22) in (7), we obtain

$$H_e = \sum_{\vec{p},\lambda,\varepsilon}\sum_{\vec{p}',\lambda',\varepsilon'} a^*(\vec{p}, \lambda, \varepsilon)a(\vec{p}', \lambda', \varepsilon')\cdot\left[e\cdot\left\{\int A_0 e^{-i\frac{(\vec{p}-\vec{p}')\vec{r}}{\hbar c}}dv\right\}\cdot\right.$$

$$\cdot\,\tilde{u}(\vec{p}, \lambda, \varepsilon)\rho_2 u(\vec{p}', \lambda', \varepsilon') + \sum_{\vec{k},\mu}\frac{e}{m_u c^2}\sqrt{\frac{2\pi\hbar^2c^2}{k}}\,\cdot$$

$$\left\{c^*(\vec{k}, \mu)\delta(\vec{p}-\vec{p}'-\vec{k}) + c(\vec{k}, \mu)\delta(\vec{p}-\vec{p}'+\vec{k})\right\}\times\tilde{u}(\vec{p}, \lambda, \varepsilon)\,\cdot$$

$$\cdot\left\{\rho_3\sigma_e(\sigma_p+\sigma_{p'}) - \frac{1+\rho_3}{2}(p_e+p'_e)\right\}u(\vec{p}', \lambda', \varepsilon')$$

$$+\sum_{\vec{k},\mu}\sum_{\vec{k}',\mu'}\frac{e^2}{m_u c^2}\frac{2\pi\hbar^2c^2}{\sqrt{kk'}}\Big\{c^*(\vec{k},\ \mu)c^*(\vec{k}',\ \mu')\delta(\vec{p}-\vec{p}'+\vec{k}+\vec{k}')$$

$$+c^*(\vec{k},\ \mu)c(\vec{k}',\ \mu')\delta(\vec{p}-\vec{p}'+\vec{k}-\vec{k}')+c(\vec{k},\ \mu)\cdot$$

$$\cdot c^*(\vec{k}',\ \mu')\delta(\vec{p}-\vec{p}'-\vec{k}+\vec{k}')+c(\vec{k},\ \mu)\cdot c(\vec{k}',\ \mu')\cdot$$

$$\cdot\delta(\vec{p}-\vec{p}'-\vec{k}-\vec{k}')\Big\}\times\tilde{u}(\vec{p},\ \lambda,\ \varepsilon)\Big\{\frac{1+\rho_3}{2}(\vec{e}(k,\mu)\vec{e}(\vec{k}',\ \mu'))$$

$$-\rho_3\sigma_e\sigma_{e'}\Big\}u(\vec{p}',\ \lambda',\ \varepsilon')\Bigg],\qquad(28)$$

where

$$\sigma_e=(\vec{\sigma},\ \vec{e}(\vec{k},\ \mu)),\qquad \sigma_{e'}=(\vec{\sigma},\ \vec{e}(\vec{k}',\ \mu')),$$

$$\sigma_p=(\vec{\sigma},\ \vec{p}),\qquad \sigma_{p'}=(\vec{\sigma},\ \vec{p}'),$$

$$p_e=(\vec{p},\ \vec{e}(\vec{k},\ \mu)),\qquad p'_e=(\vec{p}',\ \vec{e}(\vec{k},\ \mu)).$$

In (28) we have expanded $\vec{A}$ as

$$\vec{A}=\sum_{\vec{k}}\sum_{\mu=1,-1}\sqrt{\frac{2\pi\hbar^2c^2}{k}}\Big\{c^*(\vec{k},\ \mu)e^{-i\frac{\vec{k}\vec{r}}{\hbar c}}+c(\vec{k},\ \mu)e^{i\frac{\vec{k}\vec{r}}{\hbar c}}\Big\}\vec{e}(\vec{k},\ \mu),\qquad(29)$$

where $\vec{k}/c$'s are momenta of photons and $\vec{e}(\vec{k},\ \mu)$ denote two unit vectors which are perpendicular to each other and to $\vec{k}$. $c^*(\vec{k},\ \mu)$ or $c(\vec{k},\ \mu)$ is the operator which increases or decreases the number of photons in states of momentum $\vec{k}/c$ and polarization $\mu$.

In all calculations for the radiative processes of the meson, considering the interaction $H'_e$ as a small perturbation, the matrix elements for various transitions can easily be singled out from (27).

## IV.

In practical calculations, we shall find it necessary to evaluate the expressions of the form:

$$\sum_{\lambda}\sum_{\lambda'}\cdots\cdots\sum_{\lambda^{(n)}}(\tilde{u}O_1u')(\tilde{u}'O_2u'')\cdots\cdots(\tilde{u}^{(n-1)}O_nu^{(n)})(\tilde{u}^{(n)}O_{n+1}u),\qquad(30)$$

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where $u$, $u'$, ...... represent the amplitudes of states with momenta $\vec{p}$, $\vec{p}'$, ......, polarization $\lambda$, $\lambda'$, ...... and signs of charge $\varepsilon$, $\varepsilon'$, ...... respectively. $\sum\limits_{\lambda^{(i)}}$ denotes the summation over all three states of polarization $\lambda^{(i)} = 1, 0, -1$. $O_1$, $O_2$, ...... are operators which are defined as products of $\rho$'s and $\sigma$'s.

By using the annihilation operator $\Lambda^{\pm}(\vec{p})$ defined by (20), (30) can be written in the following form:

$$\sum^{p}\sum^{p'}......\sum^{p^{(n)}}\varepsilon\cdot\varepsilon'......\varepsilon^{(n)}\cdot(\tilde{u}O_1\Lambda'\rho_2u')(\tilde{u}'O_2\Lambda''\rho_2u'')$$
$$......(\tilde{u}^{(n-1)}O_n\Lambda^{(n)}\rho_2u^{(n)})\cdot(\tilde{u}^{(n)}O_{n+1}\Lambda\rho_2u), \qquad (31)$$

where $\Lambda^{(i)}$ represents $\Lambda^{+\text{ or }-}(\vec{p}^{(i)})$. $+$ or $-$ must be taken according as $\varepsilon^{(i)}$ in (30) is 1 or $-1$. $\sum^{p^{(i)}}$ denotes the summation over all six states having same momentum $\vec{p}^{(i)}/c$. These summation can easily be carried out by means of the converse orthogonality relations (19), and finally we obtain the following formula:

$$\sum_{\lambda}\sum_{\lambda'}......\sum_{\lambda^{(n)}}(\tilde{u}O_1u')(\tilde{u}'O_2u'')......(\tilde{u}^{(n-1)}O_nu^{(n)})(\tilde{u}^{(n)}O_{n+1}u)$$
$$= Sp(O_1\Lambda'O_2\Lambda''......O_n\Lambda^{(n)}O_{n+1}\Lambda), \qquad (32)$$

which represents "Casimir's method of the spin summation[(2)]" in the case of the meson. Here $Sp$ denotes the spur (diagonal sum) of the operator and we found following rules for the evalution of the spur:

$$\left.\begin{aligned}
&Sp\rho_i = Sp\rho_i\sigma_k = Sp\rho_i\sigma_k\sigma_l = ...... = 0\\
&Sp\sigma_i = 0\\
&Sp(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{b}) = 6\,(\vec{a}\vec{b})\\
&Sp(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{b})(\vec{\sigma}\vec{c}) = 3i(\vec{a},\ \vec{b}\times\vec{c})\\
&Sp(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{b})(\vec{\sigma}\vec{c})(\vec{\sigma}\vec{d}) = 3(\vec{a}\vec{b})(\vec{c}\vec{d})+3(\vec{a}\vec{d})(\vec{b}\vec{c})\\
&\text{etc.}
\end{aligned}\right\} \qquad (33)$$

(2) CASIMIR: *Helv. Phys. Acta*, 6 (1933), 287; HEITLER: *The Quantum Theory of Radiation*, (1936), Oxford.

## V.

As an application of the method which was described in the preceding section, we shall consider the elastic scattering of the meson by the electric field of a nucleus with charge $Ze$. Using the Born approximation, we obtain the differential cross section

$$d\phi = \frac{Z^2e^4}{4m_u^2v^4}\operatorname{cosec}^4\frac{\theta}{2}d\Omega \times \left[\frac{1}{3}\sum_\lambda\sum_{\lambda'}|\tilde{u}(\vec{p}',\lambda',1)\rho_2 u(\vec{p},\lambda,1)|^2\right], \quad (34)$$

for the scattering of the meson through an angle $\theta$ into the solid angle $d\Omega$, where $v$ denotes the velocity of the meson and $\vec{p}/c$, $\vec{p}'/c$ and $\lambda$, $\lambda'$ are the momenta and the states of polarization of the incident and the scattered meson respectively. The first factor represents the well known Rutherford scattering formula, whereas the latter factor contains the relativistic effects due to the Dirac-Proca equations, and can be evaluated by means of the spur techniques as follows:

$$\left.\begin{aligned}
&\frac{1}{3}\sum_\lambda\sum_{\lambda'}|(\tilde{u}'\rho_2 u)|^2 = \frac{1}{3}\sum_\lambda\sum_{\lambda'}(\tilde{u}\rho_2 u')(\tilde{u}'\rho_2 u) = Sp(\rho_2\Lambda'\rho_2\Lambda)\\
&= \frac{1}{12(p^2+m_u^2c^4)}\cdot\\
&\quad\cdot Sp\rho_2\left[m_uc^2+\frac{p^2}{2m_uc^2}+\rho_3\left\{\frac{p^2}{2m_uc^2}-\frac{(\vec{\sigma}\vec{p}')^2}{m_uc^2}\right\}+\rho_2\sqrt{p^2+m_u^2c^4}\right]\\
&\quad\cdot\rho_2\left[m_uc^2+\frac{p^2}{2m_uc^2}+\rho_3\left\{\frac{p^2}{2m_uc^2}-\frac{(\vec{\sigma}\vec{p})^2}{m_uc^2}\right\}+\rho_2\sqrt{p^2+m_u^2c^4}\right]\\
&= 1+\frac{p^4\sin^2\theta}{6m_u^2c^4(p^2+m_u^2c^4)},
\end{aligned}\right\} \quad (35)$$

which agrees completely with the formula derived by Laporte.[3]

## VI.

In this section we shall shortly discuss the relation to Kemmer's

(3) LAPORTE: *Phys. Rev.*, **54** (1938), 905.

theory,[4] in which another form of the matrix representation for Dirac-Proca equations is given. According to him, the wave equations for the meson takes the form

$$\sum_{\mu=1,2,3,4} \partial_\mu \beta_\mu \Psi + \kappa \Psi = 0, \tag{36}$$

where $\beta_\mu$'s are operators defined by the commutation relations first given by Duffin:

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\rho} + \beta_\rho \delta_{\nu\mu}, \tag{37}$$

and where

$$\partial_4 = \frac{1}{ic} \frac{\partial}{\partial t} - \frac{e}{\hbar c} A_0, \qquad \partial_k = D_k \qquad (k = 1, 2, 3). \tag{38}$$

There exist three inequivalent, irreducible representations of $\beta_\mu$ and one of them, which is represented by ten-rowed matrices, correspond to Dirac-Proca equations.

In this formulation, the total energy for the system consisting of mesons in the electromagnetic field is given by

$$\overline{H} = \int \widetilde{\Psi} \beta_4 H \Psi dv, \tag{39}$$

where

$$\beta_4 H = \beta_4^2 \left\{ m_u c^2 + e A_0 \beta_4 - \frac{\hbar c}{\kappa} (\vec{D}\vec{\beta})(\vec{D}\vec{\beta}) \right\} \beta_4^2 \tag{40*}$$

with

$$\vec{D}\vec{\beta} = D_1 \beta_1 + D_2 \beta_2 + D_3 \beta_2. \tag{41}$$

It is to be noted that in this expression and those for the total momentum and total charge there appear only operators of the following type

$$\beta_4^2 C \beta_4^2 \tag{42}$$

where $C$ denotes any operator which is defined as products of $\beta_\mu$'s.

On the other hand, $\beta_\mu$'s and their multiple products form a ring.

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(4) KEMMER: *Proc. Roy. Soc. A*, **173** (1939), 91.

* This expression is not identical with that originally given by Kemmer, but it can be obtained from the latter by using the accessory condition (73) in his paper.

As the $\beta_4^2$ is an idempotent of this ring, the $\Psi$-space can be divided into two sub-space

$$\Psi = \Psi_{\mathrm{I}} + \Psi_{\mathrm{II}}, \tag{43}$$

so that

$$\left.\begin{aligned} \beta_4^2 \Psi_{\mathrm{I}} &= \Psi_{\mathrm{I}}, \\ \beta_4^2 \Psi_{\mathrm{II}} &= 0 \end{aligned}\right\} \tag{44}$$

(Peirce's decomposition). In a coordinate system adapted to this decomposition, $\beta_4^2$ and $\beta_4^2 C \beta_4^2$ are represented by

$$\beta_4^2 = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \tag{45}$$

and

$$\beta_4^2 C \beta_4^2 = \begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix} \tag{46}$$

respectively, where $E$ and $C'$ are matrices acting only on $\Psi_{\mathrm{I}}$ and where $E$ denotes the unit matrix. Hence, as far as we are dealing with the operators of type (42) alone, we may limit ourselves to the sub-space $\Psi_{\mathrm{I}}$. By means of this reduction, we can easily prove that Kemmer's form of the meson theory is identical with ours.

Detailed discussions on this subject will also be made by us in another paper.(5)

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(5) TAKETANI and SAKATA: *Proc. Phys.-Math. Soc. Japan*, 22 (1941). the press.

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