NOTE ON EQUILIBRIUM CONFIGURATIONS FOR ROTATING WHITE DWARFS

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Summary

The properties of non-rotating white dwarfs have been considered in detail by Chandrasekhar *, who finds that no degenerate equilibrium configuration exists if the mass of a star exceeds 5·75 $M_\odot/\mu_e^2$, where $M_\odot$ is the solar mass and μ_e is defined by the equation

$$\mu_e = \rho/nm_H,$$

where ρ is the density, n is the number of electrons per cm.^{3}, and m_H is the mass of the hydrogen atom. The main purpose of the present paper is to show that this result does not remain valid if the star possesses rotation. It will be shown that a degenerate equilibrium configuration always exists for a star with rotation, no matter how large a value is taken for the mass.

The existence of a degenerate equilibrium configuration (in which the temperature of the material is zero) does not necessarily mean that a collapsing star can attain such a configuration. Indeed it is shown below that when the mass appreciably exceeds 5·75 $M_\odot/\mu_e^2$ the equilibrium configuration could not be attained without a collapsing star passing through rotationally unstable states. It is suggested therefore that collapsing stars of large mass must become rotationally unstable. The observed result that white dwarfs of large mass do not occur is attributed to this process. In the writer's opinion this conclusion can be stated in a stronger form: That rotational instability is the only process capable of explaining the observational data, for it seems that if rotational instability did not occur there would be no process that could intervene to prevent collapsing stars of large mass from attaining their equilibrium configurations.

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1. It will be useful to begin by giving a very brief account of the reason for the existence of a limiting mass in the case of stars without rotation.† The equation of hydrostatic equilibrium for a spherically symmetric star is

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{1}$$

where P is the pressure, r is a radial coordinate measured from the centre, $M(r)$ is the mass contained in a sphere of radius r, ρ is the density and G is the gravitational constant. For a pressure law of the form

$$P = k\rho^n \tag{2}$$

equation (1) becomes

$$\frac{kd[\rho(u)]^n}{du} = -\frac{GM(u)\rho(u)}{Ru^2}, \tag{3}$$

where $r = uR$ and R is the radius of the star.

* S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, p. 412, Chicago, 1938.
† For a full discussion see S. Chandrasekhar, *op. cit.*

It is interesting to consider separately the dependence of the two sides of equation (3) on R and $M(R)$ for configurations satisfying the relations

$$\begin{cases} \rho(u) = \dfrac{M(R)}{R^3} f(u), \\ M(u) = M(R) g(u), \end{cases} \tag{4}$$

where $f(u)$, $g(u)$ depend on u but not on $M(R)$ and R. It will be noticed that all configurations satisfying (4) are built on the same model, the difference from one such configuration to another being only a question of the total mass $M(R)$ and of the scale factor R. Thus the left-hand side of (3) is proportional to $[M(R)/R^3]^n$ while the right-hand side is proportional to $[M(R)/R^2]^2$. In the special case where $n=4/3$ the two sides have the same dependence on R but different dependence on $M(R)$. This simple result leads immediately to the following important conclusions for the case $n=4/3$:—

(1) Since the dependence on R is the same, it follows that if, for a fixed value of $M(R)$, the pressure gradient and the magnitude of the gravitational force are unequal at one particular value of R, then they are unequal at all values of R.

(2) Since both sides of (3) decrease monotonically with increasing $M(R)$, but at different rates, it follows that there can be only one value of $M(R)$ at which they are equal. This gives a critical value of $M(R)$. An accurate calculation shows that * this critical mass is close to $5 \cdot 75\, M_\odot/\mu_e^2$.

(3) Since the pressure gradient varies with $M(R)$ more slowly than the gravitational force, it follows that the pressure gradient is unable to support the star if $M(R)$ exceeds the critical mass. Thus in this case the star will continue to collapse indefinitely. For stars with masses less than the critical value the opposite situation occurs and expansion continues so long as $P=k_2\rho^{4/3}$ remains valid.

It is to be emphasized that these remarkable results apply only to the case $n=4/3$. For $P=k_1\rho^{5/3}$, for example, it is easy to see that no matter what value is chosen for $M(R)$ the radius R can always be adjusted so that equation (3) is satisfied. The equilibrium value of R in this case is proportional to $[M(R)]^{-1/3}$, and the density in the equilibrium configuration is accordingly proportional to $[M(R)]^2$.

When pressure degeneracy first occurs in a contracting star the electron pressure can be written in the non-relativistic form $k_1\rho^{5/3}$, so that a situation occurs similar to that described in the previous paragraph. The star will therefore contract towards an equilibrium configuration, which will be attained provided the pressure remains essentially of the form $k_1\rho^{5/3}$ throughout the contraction. According to Fowler † this condition will be satisfied if the central density in the equilibrium configuration is less than about 10^6 gm. per cm.^{3}. Since the equilibrium density is proportional to $[M(R)]^2$ it follows that an upper limit on the equilibrium density places a corresponding upper limit on $M(R)$. It has been shown ‡ by Chandrasekhar that the change-over to the $k_2\rho^{4/3}$ formula becomes completely operative when the limiting mass $5 \cdot 75 M_\odot/\mu_e^2$ is reached. Thus when $M(R)$ exceeds $5 \cdot 75\, M_\odot/\mu_e^2$ the pressure becomes of the form $k_2\rho^{4/3}$ before any equilibrium configuration can be reached by a collapsing star. This

* S. Chandrasekhar, *op. cit.*

† R. H. Fowler, *Statistical Mechanics*, p. 652, Cambridge, 1936.

‡ S. Chandrasekhar, *op. cit.*

leads to the remarkable result discussed above concerning the existence of a limiting mass.

In the present discussion the change-over from the non-relativistic form $k_1\rho^{\frac{5}{3}}$ to the relativistic form $k_2\rho^{\frac{4}{3}}$ has been regarded as occurring at a particular value of the density. This is useful in giving a broad outline of the problem but must of course be replaced by a gradual transition between the two formulae if a fuller discussion is required.*

2. *The Effect of Rotation on Collapsing Stars.*—The above remarks show that a strictly non-rotating star with mass appreciably exceeding the limiting mass must contract to a point and that the gravitational potential must diverge.† The existence of such a limiting mass led to a controversy between Eddington and Chandrasekhar. Now this controversy seems to the writer to have largely ignored the fact that a star must always possess some degree of rotation and that rotation completely alters the character of the problem.‡ For in the first place indefinite contraction could not occur on account of the onset of rotational instability. But the change in the problem is deeper rooted than this, and can best be understood by considering an imaginary problem in which rotational instability does not occur. The consideration of this imaginary problem exposes the artificial nature of the spherically symmetrical case discussed above, for it will be shown that when rotation is considered there is always an equilibrium configuration in which the gravitational potential is finite no matter what value is taken for the mass of the star.

The rotational energy of a star increases as contraction proceeds, on account of the conservation of angular momentum, and a stage must be reached at which the total rotational energy is comparable with, but less than, the gravitational potential of the star. At this stage the star will become markedly spheroidal (always remembering that rotational instability is being ignored in this imaginary problem) and appreciable further contraction in the equatorial plane is prevented by centrifugal forces. The collapse may continue, however, in a direction perpendicular to the equatorial plane, and the question of whether the gravitational potential diverges or not depends on whether the star collapses to a mathematical plane or to a disk of finite thickness. This question is decided by the following investigation which shows that equilibrium is always possible with a disk of finite thickness.

Let r be a radial coordinate, and let z be measured perpendicular to the plane of the disk with $z=0$ on the equatorial plane. Let $V(r)$ be the velocity of material moving in circular orbits at distance r from the centre (the dependence of V on z can be neglected for a thin disk). Then since radial contraction is prevented by centrifugal forces we have

$$\frac{V^2}{r} = F(r), \tag{5}$$

where $F(r)$ is the radial component of the gravitational field. The second

* S. Chandrasekhar, *op. cit.*, p. 428.

† This statement ignores the effect of general relativity which becomes important when contraction occurs to dimensions comparable with the gravitational radius of the star. This refinement need not be considered here.

‡ The question of rotation is mentioned by Chandrasekhar in *Colloque International D'Astrophysique* 1939—*Novae and White Dwarfs*, III, p. 245, Paris, 1941. It is, however, fair to say that the effects of rotation played little part in the arguments between Chandrasekhar and Eddington.

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equation of equilibrium gives the support of the disk in a direction perpendicular to the equatorial plane. This equation is

$$k_2 \frac{d}{dz}[\rho(r,z)]^{\frac{4}{3}} = -4\pi G\rho(r,z)\int_0^z \rho(r,z')\,dz'. \tag{6}$$

The present problem is to investigate equation (6), since given sufficient contraction equation (5) can always be satisfied. By differentiation we obtain

$$k_2 \frac{d}{dz}\left(\frac{1}{\rho}\frac{d\rho^{\frac{4}{3}}}{dz}\right) = -4\pi G\rho \tag{7}$$

from (6). To solve (7) put $\rho = u^3(2k_2/\pi G)^{\frac{3}{2}}$, and let $\chi(u) = du/dz$. Then (7) transforms to

$$\chi\, d\chi/du = -2u^3 \tag{8}$$

which integrates to give

$$[\chi^2]_{u_0}^{u} = u_0^4 - u^4, \tag{9}$$

where $\rho_0 = u_0^3(2k_2/\pi G)^{\frac{3}{2}}$ is the density on the equatorial plane (it may be noted that ρ, u, u_0 are all functions of r). Now from (6) it follows that $\chi(u_0) = 0$, since $d\rho/dz$ is zero at $z = 0$. This boundary condition, taken together with the fact that du/dz is less than zero, leads to the equation

$$\chi = du/dz = -(u_0^4 - u^4)^{\frac{1}{2}}. \tag{10}$$

The next step is to introduce the mass per unit area $\sigma(r)$, defined by

$$\sigma(r) = 2\int_0^\infty \rho(r,z)dz. \tag{11}$$

The quantity $\sigma(r)$ is related to the total mass M of the star by the equation

$$M = 2\pi\int_0^\infty r\sigma(r)\,.\,dr. \tag{12}$$

Equation (11) can be written in the form

$$\sigma(r) = 2\left(\frac{2k_2}{\pi G}\right)^{\frac{3}{2}}\int_0^\infty u^3\,.\,dz = -2\left(\frac{2k_2}{\pi G}\right)^{\frac{3}{2}}\int_0^{u_0}\frac{u^3\,.\,du}{(u_0^4 - u^4)^{\frac{1}{2}}} = \left(\frac{2k_2}{\pi G}\right)^{\frac{3}{2}} u_0^2. \tag{13}$$

The result (13) solves our problem since it shows that u_0 and hence ρ_0 is finite for any finite value of $\sigma(r)$. Furthermore equation (12) shows that since the radius of the disk is of stellar dimensions the mass M can be adjusted to have any value we please without requiring infinite values of σ. Thus it follows that no matter what value is assigned to M the equilibrium configuration always possesses finite gravitational potential energy. It is seen therefore that the divergence of the gravitational potential is a special property of a spherical condensation and that as soon as the star possesses some degree of rotation no divergence can occur.

Although the limiting mass loses its meaning in the sense of the divergence of the gravitational potential it nevertheless retains importance in the theory of collapsing stars. The reason for this is clearly seen as soon as we reintroduce the question of rotational instability. We have seen that the equilibrium configuration for a rotating star with a mass appreciably greater than the limiting mass must consist of a thin disk. Now a collapsing star could not reach such a configuration without passing through configurations that are rotationally unstable. As a rough indication, instability may be expected to arise when the polar radius of the star is reduced to about half the equatorial radius. This configuration will evidently be reached before a disk structure can be formed. Thus a star with mass appreciably exceeding the limiting mass must become rotationally unstable. As a result of rotational instability material is thrown off to infinity, thereby reducing the mass of the star. When the mass has been reduced to a value comparable with the limiting mass the equilibrium configuration becomes approximately spherical and the star is then able to attain a permanently stable state.

It is probable that the limiting mass is no longer so critical as in Chandrasekhar's theory. Thus stars with masses only slightly exceeding $5{\cdot}75\ M_\odot/\mu_e^2$ may attain equilibrium configurations that are not appreciably different from the spherical form, and it seems doubtful whether such configurations would be rotationally unstable. It is somewhat illogical therefore to continue referring to a "limiting mass". It is convenient, however, to have a standard of reference and Chandrasekhar's value of $5{\cdot}75\ M_\odot/\mu_e^2$ may be used for this purpose. The properties of the limiting mass in the present sense are:—

(*a*) A spherical equilibrium configuration is available when the mass is less than $5{\cdot}75\ M_\odot/\mu_e^2$ and consequently there is no condition that enforces rotational instability in this case. It is of course possible for instability to occur if the initial rotational velocity of the star before collapse is sufficiently large, since in this case a state of rotational instability could evidently arise before the star arrived at the equilibrium configuration.

(*b*) A star with small initial rotation and mass only slightly exceeding $5{\cdot}75\ M_\odot/\mu_e^2$ can probably attain an equilibrium configuration without instability occurring.

(*c*) A star with mass exceeding $5{\cdot}75\ M_\odot/\mu_e^2$ by an appreciable margin *must* become rotationally unstable.

It may be noted in the present connection that the observed masses of white dwarfs are always less than about $3\ M_\odot$. It seems necessary to invoke rotational instability to explain this result, for an equilibrium configuration is available for all values of the mass and rotational instability appears to be the only process that can prevent these equilibrium configurations being attained by stars of large mass.

3. *General Remarks on the Process of Rotational Instability.*—The problem of rotational instability has been considered * by Jeans for a pressure formula $\kappa\rho^\gamma$ where κ, γ are constants throughout the star. These constants can take the values appearing in the relativistic formula $k_2\rho^{\frac{4}{3}}$. According to Jeans a star with the latter pressure formula will develop a sharp edge lying in the equatorial plane when the ratio of the polar and equatorial radii is about one to two. Material can pass through this edge, from an inner region in which the equipotentials are closed, into a region where the equipotentials are open and extend to infinity.

The important question concerning rotational instability is the time required for the process to occur. This problem has been investigated † by Cartan for the case of a rotating fluid mass. This work shows that the process is ordinarily unstable and not secularly unstable, as had been supposed by Jeans. Thus the instability in the case of the fluid mass must develop quickly. It has been

* J. H. Jeans, *Astronomy and Cosmogony*, p. 250, Cambridge, 1928.

† E. Cartan, *Proceedings International Congress*, vol. (ii), Toronto, 1924.

pointed out * by Lyttleton that this requires a fraction of the material of the star to be thrown off to infinity. For if material were not thrown off with sufficient energy to reach infinity it would pursue an orbit that re-entered the star. Thus no permanent relief from instability could be obtained through material being thrown off with insufficient energy to reach infinity.

The remarks of the previous paragraph refer to an incompressible fluid mass and are not directly applicable to stars with a pressure formula $k_2\rho^{\frac{4}{3}}$. The question of whether the process is ordinarily or secularly unstable is consequently not decided in this case by direct investigation. But the indirect evidence in favour of ordinary instability combined with rapid ejection of material to infinity is strong. For this process provides a means whereby the expulsion of material from P Cygni Stars, Wolf-Rayet Stars, novae, and supernovae can be explained. Thus these classes of stars can be fitted into a sequence. At one end of the scale are collapsing stars of large angular momentum that become unstable after a small amount of contraction, while supernovae are stars of small angular momentum that must undergo a considerable degree of collapse before rotational instability arises. The work of the present paper provides a second argument in support of ordinary instability and ejection of material to infinity. Thus if ejection of material to infinity is not admitted the observed absence of white dwarfs with masses large compared with $M_{\odot}$ remains unexplained.

Finally it may be noted that the rapid ejection of material to infinity must depend on the radial pressure gradient being comparable in magnitude with the gravitational and centrifugal forces. This condition is satisfied in a star but not in a nebula where the radial pressure gradient is extremely small compared with the other forces. This means that although a disk-shaped extra-galactic nebula is rotationally unstable in a formal sense, the pressure gradient is so small that the instability terms are negligible compared with perturbations arising from other causes.†

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* R. A. Lyttleton, *M.N.*, **98**, 646, 1938.
† F. Hoyle, *M.N.*, **105**, 298 and 369, 1945.