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NOTE ON QUASI-UNIFORM SPACES AND REPRESENTABLE SPACES

BY

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1. Introduction. In this paper we define the concept of a representable topological space. With each representable space we associate a compatible quasi-uniformity in a natural way. Using this quasi-uniformity we show that a connected representable space is homogeneous and that the full homeomorphism group of a representable space is a topological semigroup under the topology of quasi-uniform convergence.

A general introduction to quasi-uniform spaces may be found in [5].

Throughout this paper $\circ$ denotes the usual composition of relations.

2. Preliminaries. Let X be a non-empty set and let $\mathscr{U}$ be a filter on $X \times X$ such that

(i) each element of $\mathscr{U}$ is a reflexive relation on X,

(ii) if $U \in \mathscr{U}$, there exists $W \in \mathscr{U}$ such that $W \circ W \subset U$.

Then $\mathscr{U}$ is a *quasi-uniformity on* X.

Let X be a set and let $\mathscr{U}$ be a quasi-uniformity on X. Let $\mathscr{T}_{\mathscr{U}} = \{A \subset X$: if $a \in A$ then there exists $U \in \mathscr{U}$ such that $U(a) \subset A\}$. Then $\mathscr{T}_{\mathscr{U}}$ is a topology on X, called the *quasi-uniform topology on X generated by* $\mathscr{U}$. If $(X, \mathscr{T})$ is a topological space and $\mathscr{U}$ is a quasi-uniformity on X such that $\mathscr{T} = \mathscr{T}_{\mathscr{U}}$, then $\mathscr{U}$ is a *compatible quasi-uniformity.*

We let $H(X)$ denote the group of all homeomorphisms from a space X onto itself and let i denote the identity of $H(X)$. If $A \subset X$, then $A' = \{h \in H(X): h_{|A} = i_{|A}\}$ and if G is a subgroup of $H(X)$, then $G' = \{x \in X: g(x) = x$ for each $g \in G\}$. If $G \subset H(X)$ and $A \subset X$, then $G(A) = \{g(a): g \in G, a \in A\}$. We often write A'' for $(A')'$, x' for $\{x\}'$ and $G(x)$ for $G(\{x\})$.

Definition. *A topological space $(X, \mathscr{T})$ is representable provided that if F is a closed set and $x \in X - F$, then $F'(x)$ is a neighborhood of x.*

3. Representable spaces. We prove

THEOREM 1. *Let $(X, \mathscr{T})$ be a representable space and for each $A \subset X$ let $U_A = \big(\bigcup\{\{x\} \times A'(x): x \in X - A\}\big) \cup (A \times X)$. Let $\beta = \{U_A: A$ is closed$\}$. Then β is a subbase for a compatible quasi-uniformity $\mathscr{U}$ on X.*

Proof. For each $A \subset X$, $\Delta \subset U_A = U_A \circ U_A$ so that β is a subbase for a quasi-uniformity $\mathscr{U}$ on X. Let $x \in A \in \mathscr{T}$. Then $U_{X-A}(x) = (X-A)'(x) \subset A$. Thus $\mathscr{T} \subset \mathscr{T}_{\mathscr{U}}$. Now let A be a proper $\mathscr{T}_{\mathscr{U}}$-open set such that $x \in A$. Then there is a finite collection $\{F_i\}_{i=1}^n$ of closed subsets of X such that

$$x \notin \bigcup_{i=1}^{n} F_i \quad \text{and} \quad x \in \bigcap_{i=1}^{n} [U_{F_i}(x)] \subset A.$$

Since

$$x \notin \bigcup_{i=1}^{n} F_i, \qquad \bigcap_{i=1}^{n} [U_{F_i}(x)] = \bigcap_{i=1}^{n} F_1'(x)$$

and since $(X, \mathscr{T})$ is a representable space, $\bigcap_{i=1}^{n} F_i'(x)$ contains a $\mathscr{T}$-open set about x. Thus $\mathscr{T} \subset \mathscr{T}_{\mathscr{U}}$ and $\mathscr{U}$ is a compatible quasi-uniformity.

COROLLARY. *Let $(X, \mathscr{T})$ be a representable space, let F be a closed set and let $x \in X - F$. Then $F'(x) \in \mathscr{T}$.*

Proof. Let $z \in U_F(x)$. Then $U_F(z) \subset U_F \circ U_F(x) = U_F(x)$. Thus $F'(x) = U_F(x) \in \mathscr{T}_{\mathscr{U}} = \mathscr{T}$.

THEOREM 2. *Every connected representable space is homogeneous.*

Proof. Let $(X, \mathscr{T})$ be a representable space. Let $U_\varphi = \bigcup \{\{x\} \times H(X)(x) : x \in X\}$. Let $x \in X$. For each $y \in X - U_\varphi(x)$, $U_\varphi(x) \cap U_\varphi(y) = \emptyset$.

It follows from the Corollary to Theorem 1, that $U_\varphi(x)$ is both open and closed. Hence $H(X)(x) = U_\varphi(x) = X$.

Since the entourages of the subbase β of Theorem 1 are reflexive and transitive, β is not a subbase for a uniformity whenever $(X, \mathscr{T})$ is not 0-dimensional ([1], Theorem 1). Nevertheless, if for each $U \in \mathscr{U}$ we let

$$W(U) = \{(f, g) \in H(X) \times H(X) \colon \big(f(x), g(x)\big) \in U \text{ for each } x \in X\}$$

and let

$$\mathscr{W} = \{W(U) \colon U \in \mathscr{U}\},$$

then $\mathscr{W}$ is a quasi-uniformity on $H(X)$. The quasi-uniformity $\mathscr{W}$ is called the *quasi-uniformity of quasi-uniform convergence with respect to $\mathscr{U}$ and $\mathscr{T}_{\mathscr{W}}$ is the topology of quasi-uniform convergence for $H(X)$*. The topology of quasi-uniform convergence has been studied in [6].

THEOREM 3. *Let $(X, \mathscr{T})$ be a representable homogeneous Hausdorff space, let $\mathscr{U}$ be the compatible quasi-uniformity for X described in Theorem 1, and let $H(X)$ have the topology of quasi-uniform convergence with respect to $\mathscr{U}$. Then $H(X)$ is a topological semigroup.*

Proof. By Theorem 1 of [2], it suffices to show that if $h \in H(X)$, then h is a $\mathscr{U}$-quasi-uniformly continuous function. Let $h \in H(X)$ and let A be a closed subset of X so that $U_A \in \mathscr{U}$. Then $U_{h^{-1}(A)} \in \mathscr{U}$. Let $(x, y) \in U_{h^{-1}(A)}$.

If $h(x) \in A$, then $(h(x), h(y)) \in U_A$. If $h(x) \notin A$, then $x \notin h^{-1}(A)$ so that $y \in (h^{-1}(A))'(x)$. There exists $g \in (h^{-1}(A))'$ so that $g(x) = y$. Then $h \circ g \circ h^{-1} \in A'$ and $h \circ g \circ h^{-1}(h(x)) = h(y)$. It follows that $(h(x), h(y)) \in U_A$.

Definition [4]. A space X is S.L.H. (*strong local homogeneity*) if for every neighborhood of any point x, there exists a subneighborhood $U(x)$ such that for any $z \in U(x)$ there exists a homeomorphism $g \in (X - U(x))'$ with $g(x) = z$.

Definition [3]. A space X is a *Galois space* provided that for each closed set F, $F = F''$.

THEOREM 4. *Every S.L.H. space without isolated points is a representable (Galois) space.*

Proof. Let X be an S.L.H. space without isolated points, let F be a closed subset of X and let $x \in X - F$. There is a neighborhood U of x such that $U \subset (X - F) \cap (X - U)'(x)$. Then $F \subset X - U$ so that $x \in U \subset (X - U)(x) \subset F'(x)$. Thus X is representable.

Clearly $F \subset F''$. Suppose that $y \in F'' \cap (X - F)$. Then $\{y\} = F'(y)$ is an open set, since X is representable. This contradicts the assumption that X has no isolated points. Consequently, $F = F''$ and X is a Galois space.

REFERENCES

[1] B. Banaschewski, *Über nulldimensionale Räume*, Mathematische Nachrichten 13 (1955), p. 129-140.

[2] P. Fletcher, *Homeomorphism groups with the topology of quasi-uniform convergence*, Archiv der Mathematik 22 (1971), p. 88-93.

[3] P. Fletcher and R. L. Snider, *Topological Galois spaces*, Fundamenta Mathematicae 68 (1970), p. 143-148.

[4] L. R. Ford, Jr., *Homeomorphism groups and coset spaces*, Transactions of the American Mathematical Society 77 (1954), p. 490-497.

[5] M. G. Murdeshwar and S. A. Naimpally, *Quasi-uniform topological spaces*, Noordhoff 1966.

[6] S. A. Naimpally, *Function spaces of quasi-uniform spaces*, Indagationes Mathematicae 68 (1965), p. 768-771.

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