

# Note on real hypersurfaces of complex space forms with recurrent Ricci tensor

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## Abstract

It is known that there are no real hypersurfaces with parallel Ricci tensor in complex projective space  $P_n(\mathbb{C})$  equipped with the Kähler metric. We proved in a previous paper that there are no real hypersurfaces with recurrent Ricci tensor in  $P_n(\mathbb{C})$  and such that the structure vector field of  $M$  is a principal curvature vector everywhere. In the present paper we showed that there are no real hypersurfaces  $M$  in complex space forms  $M_n(c)$  with recurrent Ricci tensor.

**M.S.C. 2000:** 53C40.

**Key words:** real hypersurface, complex space form, Ricci tensor.

## §1. Introduction.

The study of real hypersurfaces in complex space forms has an active field over the past decade. They do not admit umbilic real hypersurfaces and their geodesic hyperspheres do not have constant curvature. They also do not admit Einstein real hypersurfaces. Let  $M$  be a connected real hypersurface of complex space forms  $M_n(c)$ ,  $c \neq 0$  with the Kähler metric of constant holomorphic sectional curvature  $c$ . Then  $M$  has an almost contact metric structure  $(\phi, \xi, \eta, g)$  induced from the Kähler structure of  $M_n(c)$ . T. E. Cecil and P. J. Ryan proved that there are no Einstein real hypersurfaces of  $P_n(\mathbb{C})$  [3]. U-H. Ki showed that the nonexistence of real hypersurfaces of complex space forms with parallel Ricci tensor [5]. We considered the condition that the Ricci tensor  $S$  is *recurrent*, i.e., there exists a 1-form  $\alpha$  such that  $\nabla S = S \otimes \alpha$ . We may regard the parallel condition as a special case. This condition means that the eigenspaces of the Ricci tensor  $S$  of  $M$  are parallel along any curve  $\gamma$  in  $M$ . C. Baikoussis, S-M. Lyu and Y-J. Suh [1], [2] and the author [4] investigated the real hypersurfaces of  $M_n(c)$ ,  $n \geq 2$  with recurrent Ricci tensor under the condition that  $\xi$  is a principal curvature vector. In this paper, we prove the following theorem.

**Theorem 1.1** *There are no real hypersurfaces with recurrent Ricci tensor of complex space forms  $M_n(c)$ ,  $c \neq 0$ ,  $n \geq 3$ .*

## §2. Preliminaries

A complex  $n$ -dimensional Kähler manifold of constant holomorphic sectional curvature  $c$  is called a complex space form, which is denoted by  $M_n(c)$ . A complete and simply connected complex space form is a complex projective space  $P_n(\mathbb{C})$ , a complex Euclidean space  $\mathbb{C}^n$  or a complex hyperbolic space  $H_n(\mathbb{C})$ , according as  $c > 0$ ,  $c = 0$  or  $c < 0$ . Let  $M$  be a real hypersurface of complex space forms  $M_n(c)$ . In a neighborhood of each point, we take a unit normal vector field  $N$  in  $M_n(c)$ . The Riemannian connections  $\tilde{\nabla}$  in  $M_n(c)$  and  $\nabla$  in  $M$  are related by the following formulas for arbitrary vector fields  $X$  and  $Y$  on  $M$ .

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + g(AX, Y)N, \\ \tilde{\nabla}_X N &= -AX,\end{aligned}$$

where  $g$  denotes the Riemannian metric of  $M$  induced from the Kähler metric  $G$  of  $M_n(c)$  and  $A$  is the second fundamental tensor of  $M$  in  $M_n(c)$ . We denote by  $TM$  the tangent bundle of  $M$ . An eigenvector  $X$  of the second fundamental tensor  $A$  is called a *principal curvature vector*. Also an eigenvalue  $\lambda$  of  $A$  is called a *principal curvature*. We know that  $M$  has an almost contact metric structure induced from the Kähler structure  $(J, G)$  of  $M_n(c)$ : We define a  $(1, 1)$ -tensor field  $\phi$ , a vector field  $\xi$ , and a 1-form  $\eta$  on  $M$  by  $g(\phi X, Y) = G(JX, Y)$  and  $g(\xi, X) = \eta(X) = G(JX, N)$ . Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0.$$

It follows that

$$\nabla_X \xi = \phi AX.$$

Let  $\tilde{R}$  and  $R$  be the curvature tensors of  $M_n(c)$  and  $M$ , respectively. From the expression of the curvature tensor  $\tilde{R}$  of  $M_n(c)$ , we have the following equations of Gauss and Codazzi:

$$\begin{aligned}R(X, Y)Z &= \frac{c}{4}(g(Y, Z)X - g(X, Z)Y \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \frac{c}{4}(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi).\end{aligned}$$

By the Gauss equation, the Ricci tensor of  $(1, 1)$  type of  $M$  is given by

$$SX = \frac{c}{4}((2n+1)X - 3\eta(X)\xi) + hAX - A^2X,$$

where  $h$  denotes the trace of the shape operator  $A$ . We have the differential of the Ricci tensor,

$$\begin{aligned}(\nabla_X S)Y &= \frac{c}{4}(-3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX) + (Xh)AY \\ &\quad + h(\nabla_X A)Y - A(\nabla_X A)Y - (\nabla_X A)AY.\end{aligned}$$

Now we prepare without proof the following in order to prove our results.

**Lemma 2.2** ([9]) *If  $\xi$  is a principal curvature vector, then the corresponding principal curvature  $a$  is locally constant.*

**Lemma 2.3** ([9]) *Assume that  $\xi$  is a principal curvature vector and the corresponding principal curvature is  $a$ . If  $AX = \lambda X$  for  $X \perp \xi$ , then we have  $(2\lambda - a)A\phi X = (a\lambda + \frac{c}{2})\phi X$ .*

There are no real hypersurfaces with parallel Ricci tensor in  $M_n(c)$ . M. Kimura and S. Maeda, U-H. Ki, S. Nakagawa and Y. J. Suh proved the following result of real hypersurfaces in  $M_n(c)$  under the weaker condition.

**Proposition 2.4** ([6], [7]) *Let  $M$  be a real hypersurface in complex space forms  $M_n(c)$ , where  $n \geq 3$  of constant holomorphic sectional curvature  $c \neq 0$ . There are no real hypersurfaces satisfying  $RS = 0$ .*

### §3. Proof of the theorem

S. Nakajima proved the non-existence of real hypersurfaces with birecurrent second fundamental tensor [8], i.e. there exists a covariant tensor field  $\beta$  of order 2 such that  $\nabla^2 A = A \otimes \beta$ . The method of his proof is applicable to our case. We assume that the Ricci tensor  $S$  is recurrent, i.e. there is a 1-form  $\alpha$  satisfying  $\nabla S = S \otimes \alpha$ . Hence, we have

$$(\nabla_{X,Y}^2 S)Z = (X\alpha)(Y)SZ + \alpha(\nabla_X Y)SZ + \alpha(Y)\alpha(X)SZ$$

for  $X, Y, Z \in TM$ . We set the covariant tensor field  $\beta$

$$\beta(X, Y) = (X\alpha)(Y) + \alpha(\nabla_X Y) + \alpha(Y)\alpha(X),$$

and we get

$$(3.1) \quad (\nabla_{X,Y}^2 S)Z = \beta(X, Y)SZ.$$

The following equation holds for any  $Y \in TM$ :

$$\nabla_Y S^2 = (\nabla_Y S)S + S(\nabla_Y S).$$

Differentiating the above equation by  $X \in TM$ , we have

$$\begin{aligned} \nabla_{X,Y}^2 S^2 &= (\nabla_{X,Y}^2 S)S + (\nabla_X S)(\nabla_Y S) \\ &\quad + (\nabla_Y S)(\nabla_X S) + S(\nabla_{X,Y}^2 S). \end{aligned}$$

From the equation (3.1), we get

$$\nabla_{X,Y}^2 S^2 = 2\beta(X, Y)S^2 + (\nabla_X S)(\nabla_Y S) + (\nabla_Y S)(\nabla_X S).$$

Commutativity of the trace and the derivation we obtain

$$\nabla_{X,Y}^2 (tr S^2) = 2\beta(X, Y)(tr S^2) + 2tr(\nabla_X S)(\nabla_Y S).$$

Replacing  $X$  by  $Y$ , and subtracting from the above equation, we get

$$(\beta(X, Y) - \beta(Y, X))(tr S^2) = 0.$$

Since there exists no real hypersurfaces with  $S = 0$ , we have  $\beta(X, Y) = \beta(Y, X)$  for any  $X, Y \in TM$ . Then we obtain  $(R(X, Y)S)Z = 0$  for  $X, Y, Z \in TM$ . From Proposition 2.4, the proof is concluded.

**Acknowledgment.** The author would like to express his sincere gratitude to Prof. J. Inoguchi for his valuable suggestions and continuous encouragement during the preparation of this paper.

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