NOTE ON SOME MAPPING SPACES

HIDEKAZU WADA

(Received February 8, 1958)

1. In [2], the author obtained a result:

Let G_n be the mapping space of an n-sphere S^n on itself, and let F_n be a subspace of G_n, whose every element fixes a reference point of S^n. Then G_n is of the same homotopy type as $S^n \times F_n$ if and only if $\pi_{2n+1}(S^{n+1})$ contains an element, whose Hopf invariant is 1.

From this result, we can see that G_1, G_3 and G_7 are homotopically equivalent with $S^1 \times F_1$, $S^3 \times F_3$ and $S^7 \times F_7$ respectively [2, Corollary (6.5)]. In the present note, the author will notice that the homeomorphisms hold instead of the homotopy equivalences in the above three cases.

2. We shall say that a space X is an H_*-space if the following conditions are satisfied:

(i) The bi-continuous product $x \cdot y \in X$ is defined for every pair of points x, y of X.

(ii) There is a fixed point $e \in X$, which satisfies the condition

$$x \cdot e = x,$$

for every point x of X. We shall call e the *right identity* of X.

(iii) There exists a point x^{-1} of X, continuously defined by x of X such that

$$x \cdot x^{-1} = e,$$

for every x of X. We shall call x^{-1} the *right inverse* of x.

(iv) For every pair of points x, y of X, the following identity holds:

$$x^{-1} \cdot (x \cdot y) = y.$$

If we put $y = e$ in (iv), we obtain

(iii)′ $$x^{-1} \cdot x = e,$$

using (ii).

Now, for an x, if there is another z such that $x \cdot z = e$, then, by multiplying x^{-1} to the left in this equation, we get $z = x^{-1}$ using (iv) and (ii), which shows the uniqueness of x^{-1}.

On the other hand, if there is a y for a given x such that $y \cdot x = e$, then $x = y^{-1}$ from the uniqueness of the right inverse. In general, $y^{-1} \cdot y = e$ holds from (iii)′, therefore $x \cdot y = e$, which proves $y = x^{-1}$. Therefore the right inverse is the left inverse, which is unique.

Next, if there is a z such that $x \cdot z = x$ for any x, then, multiplying x^{-1} to the left in this equation, we obtain $z = e$ using (iv) and (iii)′, which proves the uniqueness of e.

Now, from (iii), (iii)$'$ and from the uniqueness of the right inverse, we obtain $(x^{-1})^{-1} = x$, from which and from (iv) we get

(iv)$'$ $$x \cdot (x^{-1} \cdot y) = y,$$

for every pair of points x and y.

3. Now, let Y be an H_*-space. Let G be the space of mappings of X in itself with the compact-open topology, and let F be its subspace, whose every mapping fixes e unchanged. We shall define two mappings

$$\lambda : G \to X \times F$$
$$\mu : X \times F \to G$$

as follows:

$$\lambda(g) = (g(e), g_*) \qquad \text{for every } g \in G,$$
$$\mu(x,f) = f_x \qquad \text{for every } x \in X,\ f \in F,$$

where $g_* \in F$ and $f_x \in G$ are defined by

$$g_*(x) = (g(e))^{-1} \cdot g(x) \qquad \text{for } x \in X,$$
$$f_x(y) = x \cdot f(y) \qquad \text{for } x,\ y \in X.$$

The continuities of λ and μ can be seen as follows:

LEMMA. *Let x be a point of X, let C be a compact set of X, and let U be an open set of X such that $x \cdot C \subset U$, then there are open sets $V\,(\ni x)$ and W $(\supset C)$ such that $V \cdot W \subset U$.*

In fact, let $c_\alpha \in C$ be any point, then there are open sets $V_\alpha(\ni x)$ and $W_\alpha(\ni c_\alpha)$ such that $V_\alpha \cdot W_\alpha \subset U$. As C is compact, there are finite number of W_α which cover C, which we shall denote as $\{W_i\}$. Then $V = \bigcap V_i$ and $W = \bigcup W_i$ satisfies the conclusion of the Lemma.

Let C be a compact set of X, and U be an open set of X. We shall denote by U^C the set of mappings of G such that $C \to U$. Then, U^C is an open set of G.

Proof of the continuity of λ. Let W be an open set of $\lambda(g) = (g(e),\ g_*)$. Then there are an open set U_1 of X containing $g(e)$, and an open set U_2^C of F containing g_* such that $U_1 \times U_2^C \subset W$. As $g_*(C) = (g(e))^{-1} \cdot g(C) \subset U_2$, there are open sets V_1 and V_2 of X such that $(g(e))^{-1} \in V_1$, $g(C) \subset V_2$ and $V_1 \cdot V_2 \subset U_2$ from the Lemma. Then, we see easily $\lambda((U_1 \cap V_1^{-1})^e \cap V_2^C) \subset W$, which proves the continuity of λ.

Proof of the continuity of μ. Let U^C be an open set containing $\mu(x,f) = f_x$. Then, from $f_x(C) = x \cdot f(C) \subset U$, there are an open set V_1 containing x and an open set V_2 containing $f(C)$ such that $V_1 \cdot V_2 \subset U$. Then, we can see easily that $\mu(V_1 \times (V_2^C \cap F)) \subset U^C$, which proves the continuity of μ.

Next, for any $g \in G$, we see

$$\mu\lambda(g) = \mu(g(e),\ g_*)$$
$$= (g_*)_{g(e)}.$$

On the other hand, for every $x \in X$, we get

$$\begin{aligned}(g_*)_{g(e)}(x) &= g(e)\cdot g_*(x)\\ &= g(e)\cdot((g(e))^{-1}\cdot g(x))\\ &= g(x) \qquad \text{from (iv)}',\end{aligned}$$

which proves $\mu\lambda = 1$ in G.

For $x \in X$ and $f \in F$, we see

$$\begin{aligned}\lambda\mu(x, f) &= \lambda(f_x)\\ &= (f_x(e), (f_x)_*).\end{aligned}$$

On the other hand, as $f(e) = e$, we see $f_x(e) = x \cdot f(e) = x$ from (ii), and for every $y \in X$, we get

$$\begin{aligned}(f_x)_*(y) &= (f_x(e))^{-1}\cdot f_x(y)\\ &= (x\cdot f(e))^{-1}\cdot(x\cdot f(y))\\ &= x^{-1}\cdot(x\cdot f(y))\\ &= f(y) \qquad \text{from (iv)},\end{aligned}$$

which proves $\lambda\mu = 1$ in $X \times F$. Therefore, we obtain

THEOREM 1. *For an H_*-space X, G and $X \times F$ are homeomorphic.*

Now, S^1, S^3 and S^7 are H_*-spaces regarded as complex numbers, quaternions and Cayley numbers of norm 1 respectively [1, p. 108]. Therefore, we conclude

THEOREM 2. *G_1, G_3 and G_7 are homeomorphic to $S^1 \times F_1, S^3 \times F_3$ and $S^7 \times F_7$ respectively.*

4. S^1, S^3 and S^7 are H_*-spaces with the 2-sided identity by the multiplications cited above. Namely, for every x, e of (ii) satisfies

(ii)′ $$e\cdot x = x.$$

But the following example shows that the condition (ii)′ is independent with the conditions of the H_*-space.

$$H_* = \{e,\ x,\ y\},$$
$$e\cdot e = e,\ x\cdot e = x,\ y\cdot e = y,\ e\cdot x = y,\ e\cdot y = x,$$
$$x\cdot x = y,\ y\cdot y = x,\ x\cdot y = y\cdot x = e.$$

This system satisfies the conditions of H_*-space, but e is not the left identity.

I thank Prof. H. Kuniyoshi for his algebraic advice during the preparation of this note.

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TÔHOKU UNIVERSITY.