Note on the classification theorems of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g)

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Abstract. In [7], it is proved that all g-natural metrics on tangent bundles of m-dimensional Riemannian manifolds depend on arbitrary smooth functions on positive real numbers, whose number depends on m and on the assumption that the base manifold is oriented, or non-oriented, respectively. The result was originally stated in [8] for the oriented case, but the smoothness was assumed and not explicitly proved. In this note, we shall prove that, both in the oriented and non-oriented cases, the functions generating the g-natural metrics are, in fact, smooth on the set of all nonnegative real numbers.

Keywords: Riemannian manifold, tangent bundle, natural operation, g-natural metric, curvatures

Classification: Primary 53C07; Secondary 53A55

If (M, g) is an m-dimensional Riemannian manifold, then we use the terminology of "g-natural metrics" (cf. [2]) on the tangent bundle TM to describe metrics on TM which come from g by a first order natural operator ([8] and [7]). We have studied these metrics in [1], [2] and [3]. The well-known example of such metrics is the Sasaki metric g^s [11]. All natural metrics are characterized by the following result:

Theorem 1 ([8]). *There is a bijective correspondence between natural (possibly degenerated) metrics G on the tangent bundles of (oriented) Riemannian manifolds and the triples of first order natural F-metrics $(\zeta_1, \zeta_2, \zeta_3)$, where ζ_1 and ζ_3 are symmetric. The correspondence is given by*

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v,$$

where ζ^s, ζ^h and ζ^v denote the Sasaki lift, the horizontal lift and the vertical lift of ζ, respectively.

For the definitions of F-metrics and their lifts, we refer to [8] (see also [7] for more details on the concept of naturality).

It is proved, furthermore, in [7] that all first order natural F-metrics on (oriented) Riemannian manifolds form a family parameterized by some arbitrary smooth function on positive real numbers, where the number of functions depends on the dimensions of manifolds (the result was originally stated in [8] for the oriented case, but the smoothness was assumed and not explicitly proved). Precisely, with the notations of [7], we have

Theorem 2 ([7]). 1) *All first order natural F-metrics ζ on non-oriented Riemannian manifolds of dimension $m > 1$ form a family parametrized by two arbitrary smooth functions α, $\beta : (0,\infty) \to \mathbb{R}$ in the following way: For every Riemannian manifold (M,g) and tangent vectors $u, X, Y \in M_x$*

$$\zeta_{(M,g)}(u)(X,Y) = \alpha(g(u,u))g(X,Y) + \beta(g(u,u))g(u,X)g(u,Y). \tag{1}$$

If $m = 1$, then the same assertion holds, but we can always choose $\beta = 0$.
In particular, all first order natural F-metrics are symmetric.

2) *On oriented Riemannian manifolds, we have the same results for dimensions $m = 1$ and $m > 3$, but for $m = 2$ and $m = 3$, there exist other arbitrary smooth functions φ, γ and $\delta : (0,\infty) \to \mathbb{R}$ such that:*
If $m = 3$, then

$$\begin{aligned} \zeta_{(M,g)}(u)(X,Y) = \alpha(g(u,u))g(X,Y) &+ \beta(g(u,u))g(u,X)g(u,Y) \\ &\varphi(g(u,u))g(u,X \times Y), \end{aligned} \tag{2}$$

where $\times$ means the vector cross-product.
If $m = 2$, then

(3)
$$\begin{aligned} \zeta_{(M,g)}(u)(X,Y) = \alpha(g(u,u))g(X,Y) &+ \beta(g(u,u))g(u,X)g(u,Y) \\ &\gamma(g(u,u))(g(J^g(u),X)g(u,Y) + g(u,X)g(j^g(u),Y)) \\ &\delta(g(u,u))(g(J^g(u),X)g(u,Y) - g(u,X)g(j^g(u),Y)), \end{aligned}$$

where J^g is the canonical almost complex structure on (M,g).

Actually, the arbitrary parameterizing functions are smooth on all the set of nonnegative real numbers:

Theorem 3. *All basic functions from Theorem 2 can be prolonged, in fact, to smooth functions on the set $\mathbb{R}^+$ of all nonnegative real numbers.*

Proof: Note that we will use the technique from [7] throughout the whole proof.

1) Using the same arguments as in [7], we have to discuss all $O(m)$-equivariant maps $\zeta : \mathbb{R}^m \to \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$. Denote by $g^0 = \sum_i dx^i \otimes dx^i$ the canonical Euclidean metric, and by $|\ |$ the induced norm. Each vector $v \in \mathbb{R}^m$ can be transformed in $|v| \frac{\partial}{\partial x^1}|_0$ by an element of $O(m)$. Hence ζ is determined by its values on the one-dimensional subspace spanned by $\frac{\partial}{\partial x^1}|_0$. Moreover, we can also change the orientation of the first axis by an element of $O(m)$, i.e., we have to define ζ only on $\{t\frac{\partial}{\partial x^1}|_0,\ t \geq 0\}$.

Let us define a smooth map $\xi : \mathbb{R} \to \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$ by $\xi(t) = \zeta(t\frac{\partial}{\partial x^1}|_0) \in \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$, for all $t \in \mathbb{R}$, and consider the group K_m of all linear orthogonal transformations keeping $\frac{\partial}{\partial x^1}|_0$ fixed. So for $t \in \mathbb{R}^+$ (or generally $\mathbb{R}$), the tensor $\xi(t)$ is K_m-invariant. On the other hand, every such smooth ξ on $\mathbb{R}^+$ determines a natural F-metric.

So let us assume $s_{ij} dx^i \otimes dx^j$ is K_m-invariant. Since we can change the orientation of any coordinate axis, except the first one, by elements of K_m, then $s_{ij} = 0$ for $i \neq j$. Further we can exchange any couple of coordinate axes different from the first one by elements of K_m, and so $s_{ii} = s_{jj}$, for all $i \neq 1$ and $j \neq 1$. Hence all K_m-invariant tensors are of the form

$$\bar{\nu} dx^1 \otimes dx^1 + \bar{\mu} g^0, \tag{4}$$

the reals $\bar{\mu}$ and $\bar{\nu}$ being independent, if $m > 1$. In dimension 1, all K_1-invariant tensors are of the form $\bar{\mu} g^0 = \bar{\mu} dx^1 \otimes dx^1$.

Thus, our mapping ξ is defined by

$$\xi(t) = \bar{\nu}(t) dx^1 \otimes dx^1 + \bar{\mu}(t) g^0, \tag{5}$$

for all $t \in \mathbb{R}$, where $\bar{\mu}$ and $\bar{\nu}$ are arbitrary smooth functions on $\mathbb{R}$ (and they reduce to one function if $m = 1$).

For $t = 0$, since ζ is $O(m)$-invariant, then the tensor $\xi(0)$ is $O(m)$-invariant and so it is a multiple of g^0 (cf. [6, I; p. 277]). It follows, by virtue of (5) that $\bar{\nu}(0) = 0$. On the other hand, if we consider the linear orthogonal transformation A_m which changes the orientation of the first coordinate axis, then the equivariance of ζ by A_m implies that for every $t \in \mathbb{R}$, $\bar{\mu}(-t) = \bar{\mu}(t)$ and $\bar{\nu}(-t) = \bar{\nu}(t)$, i.e., $\bar{\mu}$ and $\bar{\nu}$ are even.

Now, given $v = t \frac{\partial}{\partial x^1}|_0$, $t > 0$, we can write

$$\begin{aligned}\zeta_{(\mathbb{R}^m, g^0)}(v)(X, Y) &= \xi(|v|)(X, Y) \\ &= \bar{\mu}(|v|) g^0(X, Y) + \bar{\nu}(|v|)\, |v|^{-2}\, g^0(v, X) g^0(v, Y).\end{aligned}$$

To complete the proof, we need the following lemma.

Lemma 4 ([4]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.*

(a) *If f is even, then there exists a smooth function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f(t) = f(0) + t^2.g(t^2)$ for any t.*
(b) *If f is odd, then there exists a smooth function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f(t) = t.(f'(0) + t^2.g(t^2))$ for any t.*

Let us define the functions $\mu(t)$ and $\nu(t)$ by $\nu(t) = t^{-1} \bar{\nu}(\sqrt{t})$ and $\mu(t) = \bar{\mu}(\sqrt{t})$, for all $t > 0$. The functions μ and ν being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on $\mathbb{R}^+$. For this, applying (a) of Lemma 4 to $\bar{\mu}$ and $\bar{\nu}$, there exist two smooth functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}$, such that $\bar{\mu}(t) = \bar{\mu}(0) + t^2 \alpha(t^2)$ and $\bar{\nu}(t) = \bar{\nu}(0) + t^2 \beta(t^2) = t^2 \beta(t^2)$

(since $\bar{\nu}(0) = 0$), for all $t \in \mathbb{R}^+$. We deduce that $\mu(t) = \bar{\mu}(\sqrt{t}) = \bar{\mu}(0) + t\alpha(t)$ and $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t}) = \beta(t)$, for all $t > 0$. In other words, μ and ν coincide on $\mathbb{R}^+_*$ with two smooth functions on $\mathbb{R}^+$, and the formula (1) of Theorem 2 is extended to $\mathbb{R}^+$. Obviously, every such operator is natural and 1) of the Theorem is proved.

2) For the oriented situation, when $m > 3$ and $m = 1$, the same proof remains valid if we replace K_m by $K_m^+ := K_m \cap SO(m)$ and A_m by the element B_m of $SO(m)$ which changes the orientations of the first and the second axes.

It remains to extend the formulas (2) and (3) from Theorem 2 to $\mathbb{R}^+$. We can use a similar procedure as before.

For $m = 3$, let us assume $s_{ij}dx^i \otimes dx^j$ is K_3^+-invariant. If we change the orientation of any coordinate axis, different from the first one, by an element of K_3^+, then we must change the orientation of the other. It follows that $s_{12} = s_{21} = s_{13} = s_{31} = 0$. Further the element of K_3^+ which exchanges the couple of second and third coordinate axes must change the orientation of one of them, and so $s_{22} = s_{33}$ and $s_{23} = -s_{32}$. Hence all K_3^+-invariant tensors are of the form

$$\bar{\nu}dx^1 \otimes dx^1 + \bar{\mu}g^0 + \bar{\kappa}(dx^2 \otimes dx^3 - dx^3 \otimes dx^2), \tag{6}$$

the reals $\bar{\mu}$, $\bar{\nu}$ and $\bar{\kappa}$ being independent. Thus, our mapping ξ is defined by

$$\xi(t) = \bar{\nu}(t)dx^1 \otimes dx^1 + \bar{\mu}(t)g^0 + \bar{\kappa}(t)(dx^2 \otimes dx^3 - dx^3 \otimes dx^2), \tag{7}$$

for all $t \in \mathbb{R}$, where $\bar{\mu}$, $\bar{\nu}$ and $\bar{\kappa}$ are arbitrary smooth functions on $\mathbb{R}$. By similar arguments as in 1) we have $\bar{\nu}(0) = \bar{\kappa}(0) = 0$ and also, if we consider the equivariance of ζ by B_3, then we deduce that the functions $\bar{\nu}$ and $\bar{\nu}$ are even and that the function κ is odd.

As in 1), let us define $\mu(t)$, $\nu(t)$ and $\kappa(t)$ by $\mu(t) = \bar{\mu}(\sqrt{t})$, $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t})$ and $\kappa(t) = t^{-1/2}\bar{\kappa}(\sqrt{t})$ for all $t > 0$. The functions μ, ν and κ being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on $\mathbb{R}^+$. But we can just apply (a) of Lemma 4 to $\bar{\mu}$ and $\bar{\nu}$ and (b) of Lemma 4 to $\bar{\kappa}$, and the result follows.

For $m = 2$, we have $K_2^+ := K_2 \cap SO(2) = \{I_2, -I_2\}$, where I_2 denotes the identity matrix in $GL(2)$. Since every tensor in $\mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$ is K_2^+-invariant, all K_2^+-invariant tensors are of the form

$$\bar{\nu}dx^1 \otimes dx^1 + \bar{\mu}g^0 + \bar{\lambda}(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) + \bar{\tau}(dx^1 \otimes dx^2 - dx^2 \otimes dx^1), \tag{8}$$

the reals $\bar{\mu}$, $\bar{\nu}$, $\bar{\lambda}$ and $\bar{\tau}$ being independent. Thus, our mapping ξ is defined by

$$\begin{aligned} \xi(t) = {} & \bar{\nu}(t)dx^1 \otimes dx^1 + \bar{\mu}(t)g^0 \\ & + \bar{\tau}(t)(dx^2 \otimes dx^1 + dx^1 \otimes dx^2) \\ & + \bar{\lambda}(t)(dx^2 \otimes dx^1 - dx^1 \otimes dx^2), \end{aligned} \tag{9}$$

for all $t \in \mathbb{R}$, where $\bar{\mu}$, $\bar{\nu}$, $\bar{\lambda}$ and $\bar{\tau}$ are arbitrary smooth functions on $\mathbb{R}$. By similar arguments as in 1) we have $\bar{\nu}(0) = \bar{\lambda}(0) = \bar{\tau}(0) = 0$ and also all the functions $\bar{\mu}$, $\bar{\nu}$, $\bar{\lambda}$ and $\bar{\tau}$ are even (it suffices to take the equivariance of ζ by $-I_2$).

As in 1), let us define $\mu(t)$, $\nu(t)$, $\lambda(t)$ and $\tau(t)$ by $\mu(t) = \bar{\mu}(\sqrt{t})$, $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t})$, $\lambda(t) = t^{-1}\bar{\lambda}(\sqrt{t})$ and $\tau(t) = t^{-1}\bar{\tau}(\sqrt{t})$ for all $t > 0$. The functions μ, ν, λ and τ being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on $\mathbb{R}^+$. But we can just apply (a) of Lemma 4 to the functions $\bar{\mu}$, $\bar{\nu}$, $\bar{\lambda}$ and $\bar{\tau}$ and the result follows. □

Combining Theorems 1–3, we obtain for the non-oriented case (an analogous result can be stated for the oriented case):

Corollary 5. *Let (M, g) be a non-oriented Riemannian manifold and G be a g-natural metric on TM. Then there are smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every $u, X, Y \in M_x$, we have*

$$(10) \qquad \begin{cases} G_{(x,u)}(X^h, Y^h) = & (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\ & +(\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = & \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) = & \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = & \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u), \end{cases}$$

where $r^2 = g_x(u, u)$.
For $m = 1$, the same holds with $\beta_i = 0$, $i = 1, 2, 3$.

Acknowledgment. The author would like to thank Professor O. Kowalski for his valuable comments and suggestions on this paper.

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(*Received* February 8, 2004)