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TECHNICAL NOTE

NOTE ON THE COMPLEXITY OF THE SHORTEST PATH MODELS FOR COLUMN GENERATION IN VRPTW

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In this note we prove that the relaxation approach in designing the subproblem of pricing out only the feasible routes for the set partition formulation of the **VRPTW** is justified on complexity grounds. That is, the first dynamic programming model presented in M. Desrochers, J. Desrosiers and M. Solomon (1992), that is able to price out all feasible routes, is NP-hard in the strong sense.

In a recent paper by Desrochers, Desrosiers and Solomon (1992), on "A New Optimization Algorithm for the Vehicle Routing Problem With Time Windows", the authors describe a number of important subproblems in the context of generating columns for their set partition formulation of the vehicle routing with time windows (**VRPTW**) problem. Their first model for pricing out new columns is a dynamic programming model whose solution space contains only feasible routes but "is very time consuming to solve." Since this first model is the focus of our note, we repeat its statement.

Given a graph $G = (N, A)$ together with a cost c_{ij} and time $t_{ij} \geq 0$ for each arc $(i, j) \in A$. Let $F_i(S, t)$ denote the minimal (marginal in the original paper) cost of the partial route going from the depot to node i, visiting only once all nodes in set S and ready to leave node i at time t or later. The dynamic programming recurrence equations for computing $F_i(S, t)$ are given by:

$$F_{depot}(\emptyset, 0) = 0;$$

$$F_j(S, t) = \min_{(i,j) \in A} \Big\{ F_i(S - \{j\}, t') + c_{ij} \,|\, t' + t_{ij} \leq t,\ a_i \leq t' \leq b_i, \text{ and } \cdot \sum_{k \in S} q_k \leq Q \Big\}$$

for all j, S, t such that $j \in N$, $S \subseteq N$, $a_j \leq t \leq b_j$, where Q denotes the capacity of a vehicle, $[a_i, b_i]$ the time window for the start of service at node i, and q_i the supply at node i which needs to be picked up. We will denote this problem by **P**. **P** is, in fact, an *elementary* (no node appears more than once) shortest path problem with time window and capacity constraints.

In what follows, Desrochers, Desrosiers and Solomon state that the problem **P** is NP-hard because the shortest weight-constrained path problem, which is a special case of **P**, is NP-hard. This complexity result for the shortest weight-constrained path problem is due to Megiddo (1977), cited as "private communication" by Garey and Johnson (1979), and is based on a reduction from PARTITION which is ordinary NP-complete. At this point Desrochers, Desrosiers and Solomon state "Thus, our dynamic programming problem is NP-hard. Furthermore, no pseudopolynomial algorithm is known for this problem." These authors used two relaxations of **P**, each allowing for nonelementary paths through the set N. For the two relaxations, they provided pseudopolynomial algorithms.

To formally prove that the above elementary shortest path, time window, weight-constrained problem, does not have a pseudopolynomial optimal algorithm (provided NP $\neq$ P), we need to establish that it is an NP-hard problem in the strong sense, and would justify the next two dynamic programming models, which represent relaxations of the first model but can

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be solved to optimality by pseudopolynomial algorithms. The objective of this note is to do just that, i.e., prove that the problem **P** is NP-hard in the strong sense.

To prove that problem **Π** is NP-hard in the strong sense, we first need to check that the problem **Π** is in NP, and then select a candidate **Π**′ NP-hard problem in the strong sense for which we can establish a polynomial (or pseudopolynomial) transformation from **Π**′ to **Π** (or to a special case of **Π**). It is easy to see that **P** is in NP. As a candidate NP-hard problem in the strong sense (**Π**′) we choose the problem denoted in Garey and Johnson (1979), as Sequencing Within Intervals (**SWI**). The problem is restated as:

Given a set N of tasks, each task $i \in N$ having a length $l(i) \in Z^+$ and a time interval $[r(i), d(i)]$ within which it is to be executed.

Question: Is there a feasible task sequencing with at most one task being executed at a time?

Given a general instance of the **SWI** problem we will construct (in a polynomial number of steps) an instance of the **P** problem as follows: Construct a graph $G = (V, A)$ with one node in the set V for each task $i \in N$, plus two additional nodes for source and sink nodes denoted by o and d, respectively (i.e., $V = N \cup \{o, d\}$). The set of arcs A contains an arc (i, j) for every $i, j \in N$; $i, j \neq o$ or d, a set of arcs from the node o to all nodes $i \in V$; $i \neq d$, and a set of arcs from all $i \in V$ to the node d, $i \neq o$. In fact, $A = (N \times N) \cup (\{o\} \times N) \cup (N \times \{d\})$. Denote by $L = \sum_{i \in N} l(i)$. For each node $i \in N$ set a time window $[a_i, b_i]$, where $a_i = r(i)$, $b_i = d(i) - l(i)$. For the node o set the time window $[a_o, b_o] = [0, 0]$ and for node d, $[a_d, b_d] = [0, L]$. Capacity intervals are $[0, 0]$ at node o and $[0, Q]$ for all nodes in V. Capacity parameters are given by $q_i = l(i)$ for all $i \in N$ and $q_i = 0$, $i \in \{o, d\}$. The costs, $c_{ki} = -l(i)$ for $(k, i) \in A$, $i \neq d$, and $c_{kd} = 0$ for all $k \in V \backslash \{o\}$. Travel times are given by $t_{ij} = +l(j)$, $i \in N \cup \{o\}$, $j \in N$, and $t_{ij} = 0$, $i \in N$, $j = d$.

Theorem. *If there is a feasible task sequencing for* **SWI**, *then there is a shortest path solution from o to d satisfying the conditions of problem* **P** *on the graph G with total cost $-L$, and if there is a feasible solution to the shortest path problem* **P** *from o to d on G with total cost of $-L$, then there is a feasible solution to the* **SWI** *problem.*

The proof is clear from the construction of G. If there exists a feasible task sequencing of jobs for the **SWI** problem, then we just follow that task sequencing in the form of a flow from the node o to node d on G. Clearly, this flow traces a path from o to d which is feasible in the context of the time windows for each node, visits each node exactly once, and the cost of this path is $-L$. On the other hand, stating the shortest path problem in its decision version as: Is there a feasible elementary path (a path which does not violate the time window constraints) on G from o to d, which visits each node in G exactly once of total cost $-L$? If the answer is **yes**, then the answer for the corresponding **SWI** problem is also **yes**.

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