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NOTE ON THE DEGREE OF C^0–SUFFICIENCY OF PLANE CURVES

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Abstract

Let f be a germ of plane curve, we define the ∂–degree of sufficiency of f to be the smallest integer r such that for any germ g such that $j^{(r)}f = j^{(r)}g$ then there is a set of disjoint annuli in $\mathbb{S}^3$ whose boundaries consist of a component of the link of f and a component of the link of g. We establish a formula for the ∂–degree of sufficiency in terms of link invariants of plane curves singularities and, as a consequence of this formula, we obtain that the ∂–degree of sufficiency is equal to the C^0–degree of sufficiency

0. Introduction

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a plane curve. Given $\varepsilon > 0$, we shall let S_ε denote the sphere of radius ε centered at the origin of $\mathbb{C}^2$. By [M], there is $\eta > 0$ such that for each $0 < \varepsilon < \eta$, $(S_\varepsilon, S_\varepsilon \cap \{f = 0\})$ is a link ambient isotopic to $(S_\eta, S_\eta \cap \{f = 0\})$. We shall call the ambient isotopy class of the link $(S_\eta, S_\eta \cap \{f = 0\})$ *the link of* f and write it L_f.

If r is an integer, we let $j^{(r)}f$ denote the r–jet determined by f. There is a classical invariant of the germ f that is called the "C^0–degree of sufficiency of f". The definition of this invariant is the following: the integer r is the C^0–degree of sufficiency of f if r is the smallest integer that satisfies the condition: for any germ g such that $j^{(r)}f = j^{(r)}g$, then $L_f = L_g$.

The usual definition of degree of C^0–sufficiency of f is given in terms of the topological type of the germ of f at 0. Remark that the link $(S_\eta, S_\eta \cap \{f = 0\})$ is ambient isotopic to the link $(S_\varepsilon, S_\varepsilon \cap \{g = 0\})$ if and only if there is an orientation preserving homeomorphism $h : S_\eta \to S_\varepsilon$ which carries $S_\eta \cap \{f = 0\}$ to $S_\varepsilon \cap \{g = 0\}$ (see for example [B–Z]). Applying the above result and the ones of [M], one can prove that $L_f = L_g$ if and only if f and g are topologically equivalent germs. Then the definition of C^0–degree of sufficiency that we give and the usual one are equivalent.

Suppose that r is the degree of sufficiency of f, and that g is a germ such that $j^{(r)}f = j^{(r)}g$. Let $\eta > 0$ be a real number such that if $0 < \varepsilon < \eta$, then $(S_\varepsilon, S_\varepsilon \cap \{f = 0\} \cup S_\varepsilon \cap \{g = 0\})$ is a link of constant topological type. By the definition of the C^0–degree of sufficiency we can say that in $S_\varepsilon, 0 < \varepsilon < \eta$,

$S_\varepsilon \cap \{f=0\}$ and $S_\varepsilon \cap \{g=0\}$ have the same topological type, but what can we say about the relative position of the two links,i.e., which is the topological type of the link $S_\varepsilon \cap \{fg=0\}$? and how is $S_\varepsilon \cap \{f=0\}$ linked with $S_\varepsilon \cap \{g=0\}$?.

The purpose of this paper is to describe this relative position of the two links. By one of our results (i. e. corollary 1.2), there is a set of disjoint annuli in $\mathbb{S}^3$ whose boundaries consist of a component of the link of f and a component of the link of g.

Two germs whose links are as above will be called ∂–equivalents. Accordingly we shall define the ∂–degree of sufficiency of f to be the smallest integer r such that: for any germ g such that $j^{(r)}f = j^{(r)}g$, then f is ∂–equivalent to g.

In this paper we establish a formula for the ∂–degree of sufficiency in terms of link invariants of plane curves. This formula takes the same values as the formula for the C^0–degree of sufficiency obtained in [**K–L**] (see also [**T**] and [**Li**]). Consequently we have the equality:

∂–degree of sufficiency = C^0–degree of sufficiency.

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1. Definitions and results

Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two germs of plane curves. We will say that "f is ∂–equivalent to g" if the link of f is isotopic by disjoint annuli to the link of g. More precisely, let S_ε be the sphere with center the origin and radius ε. The above condition means that there is an $\eta > 0$, such that for every $0 < \varepsilon < \eta$ there is a set of disjoint annuli $S \subset S_\varepsilon$ such that for each annulus $A \in S$, ∂A consists of a component of $\{f=0\} \cap S_\varepsilon$ and a component of $\{g=0\} \cap S_\varepsilon$ (the orientation of A does not induce the orientation of each component, see figure 1).

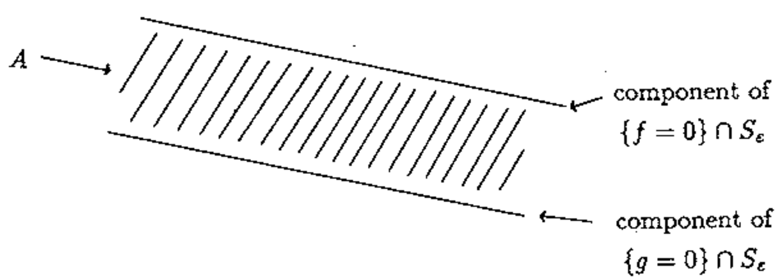


Figure 1

Example 1. Consider the germs at the origin given by the polynomials $y^2 + x^3 + x^7$ and $y^2 + x^3 + 4x^7$. They are ∂–equivalent and in figure 2 we show the annulus between the two corresponding knots.

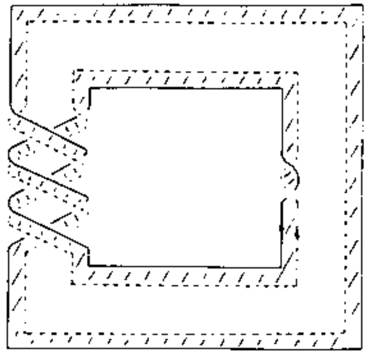


Figure 2

Example 2. Consider now the germs given by the polynomials $y^2 + x^3$ and $y^3 + x^2$. They have the same topological type but they are not ∂–equivalent (see figure 3).

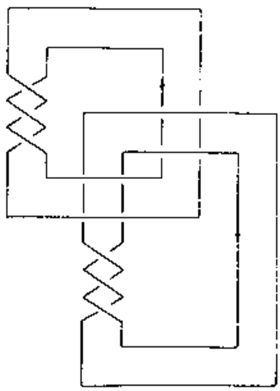


Figure 3

The link $\{f = 0\} \cap S_\varepsilon$ is an iterated torus link. That is to say, $\{f = 0\} \cap S_\varepsilon$ is obtained by successive satellizations of torus links (see [M–W]). Let $\{T_i\}$ be the minimal collection of satellization tori (unique up isotopy by the results of Jaco–Shalen and Johannson, see [E–N]). We split along $\{T_i\}$ the link exterior to obtain a finite set of pieces $\{P_i\}$. Each of one P_i has a Seifert fibered structure. More precisely, each piece P_i can be considered as $\mathbf{S}^3$ with a Seifert fibration with two exceptional fibers and where we have suppressed a finite number of fibers.

We denote the resolution tree of the germ f by $\Gamma(f)$ and we label the strict transforms of the branches by an arrow (↑). If two arrows start on a same vertex then the corresponding components of the link $\{f=0\}\cap S_\varepsilon$ are general fibers in a same piece P_i. Then there is an annulus such that its boundary consists of the two considered components. Let $\Gamma(fg)$ be the resolution tree of the product fg, we label the strict transforms of the branches of f with an arrow (↑) and the strict transforms of the branches of g with a star (⫯). If the same number of arrows and stars are attached at every vertex of $\Gamma(fg)$ then f and g are ∂–equivalent (see fig. 4).

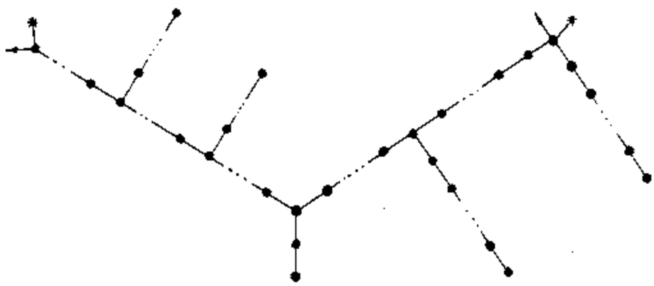


Figure 4. Exemple of f ∂–equivalent to g

Let φ be a branch of the germ f. If $\Gamma(f)$ is the resolution tree of f we call v the vertex of $\Gamma(f)$ where the arrow of φ start. We define $\widetilde{\varphi}$ as the germ such that the resolution tree, $\Gamma(f\widetilde{\varphi})$, of $f\widetilde{\varphi}$ is obtained from $\Gamma(f)$ by adding a star at the vertex v (see fig. 5).

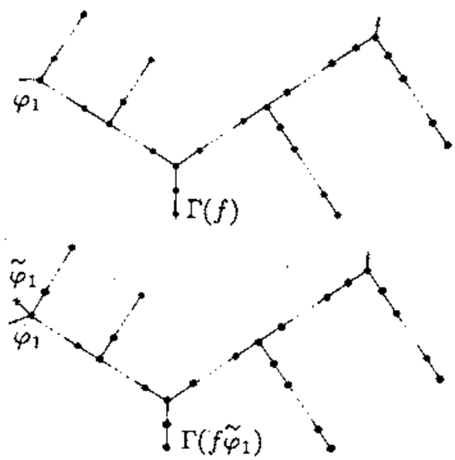


Figure 5

In other terms, assume that the Puiseux expansion of φ is the following:

$$\Sigma_{i=1}^{k_1} a_{1i} t^i + b_1 t^{q_1/p_1} + \Sigma_{i=1}^{k_2} a_{2i} t^{(q_1+i)/p_1} + b_2 t^{q_2/p_1 p_2} + \ldots$$
$$\ldots + b_{g-1} t^{q_{g-1}/p_1 p_2 \cdots p_{g-1}} + \Sigma_{i=0}^{\infty} c_i t^{(q_g+i)/p_1 \cdots p_g}$$

Replacing c_j by $c_j + \varepsilon$, $\varepsilon > 0$, we get another branch which we denote by $\varphi(j, \varepsilon)$. Then $\widetilde{\varphi}$ will be the branch $\varphi(k, \varepsilon)$ where $k \geq 0$ is the minimal integer such that there exists ε such that we have $I(\varphi(k, \varepsilon), \varphi') = I(\varphi, \varphi')$ for every branch φ' of f, where I denotes the intersection number.

In view of a topological interpretation of $\widetilde{\varphi}$, let us consider the iterated torus link $L = \{f = 0\} \cap S_\varepsilon$ and let $N = \{\varphi = 0\} \cap S_\varepsilon$ be one of its components. Then the link $\{f\widetilde{\varphi} = 0\} \cap S_\varepsilon$ is obtained from L by adding a general fiber of the piece which contains N in the Jaco–Shalen–Johannson splitting (cf. [**E–N**] and [**M–W**]). For an easy example see fig. 6.

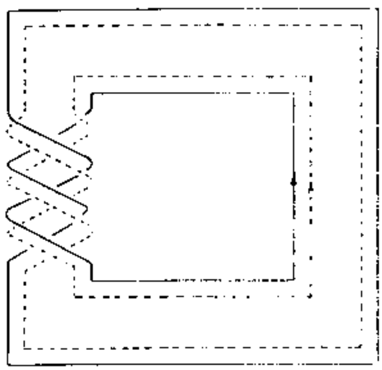


Figure 6

In this paper we establish a formula relating the ∂-degree of sufficiency of f with the intersection numbers of the $\widetilde{\varphi}$ with f.

Theorem 1.1. *Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a germ of plane curve with branches $\varphi_i, i \in I$. Let m_i be the multiplicity of the branch φ_i. Then the degree of ∂-sufficiency r is equal to:*

$$r = \ max \ \left[\frac{1}{m_i} I(f, \widetilde{\varphi}_i) \right].$$

In other words r is equal to the integral part of the largest polar quotient of f in the sense of Lê [**Lê**].

The proof will be given in section 2.

If we apply the result of [**K–L**], then we obtain

Corollary 1.2. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of plane curve. Then: ∂–degree of sufficiency of f = C°-degree of sufficiency of f.*

2. Proof of the theorem 1.1.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of plane curve with branches $\varphi_i, i \in I$, and let $m_i, i \in I$, be the corresponding multiplicities.

We let:

$$r = \max \left[\frac{1}{m_i} I(f, \varphi_i) \right]$$

and

$$s = \partial\text{–degree of sufficiency of } f.$$

By a rotation in $\mathbb{C}^2$ (that does not modify s) we may suppose that the tangent cone of f does not contain $x = 0$.

1. Proof that $s \leq r$.

By definition of s there is a germ g of plane curve such that $j^{(s-1)}f = j^{(s-1)}g$ and f is not ∂–equivalent to g, therefore $j^s f \neq j^s g$.

If the multiplicity of f is different from the multiplicity of g then s is equal to the multiplicity of f and hence f is ∂–equivalent to his tangent cone. Thus $I(f, \widetilde{\varphi}_i) = \sum_{j \in I} m_i m_j$, and

$$r = \max_{i \in I} \left[\frac{1}{m_i} \sum_{j \in I} m_i m_j \right] = \sum_{j \in I} m_j = \text{ multiplicity of } f = s.$$

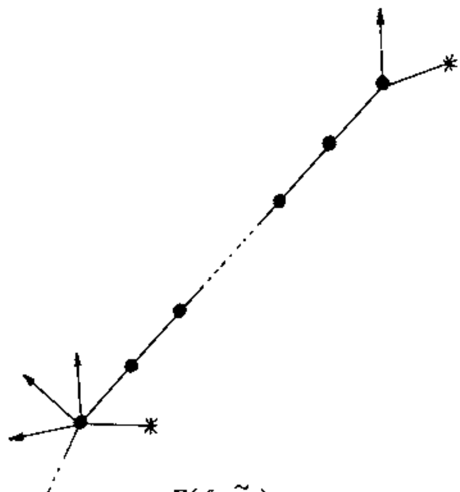
Hence we may assume that the multiplicity of f is equal to the multiplicity of g.

Claim. *If $I(g, \widetilde{\varphi}_i) = I(f, \widetilde{\varphi}_i)$ for every $i \in I$, there is a germ of irreducible curve ψ such that for some $i \in I$:*

a.- The multiplicity of ψ is equal to m_i.

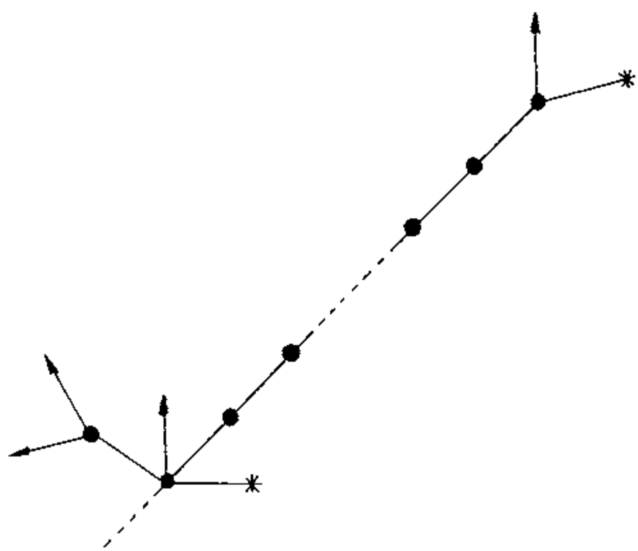
b.- $I(g, \widetilde{\varphi}_i) = I(g, \psi) < I(f, \psi)$.

Proof: Consider the resolution tree, $\Gamma(fg)$, of fg. We label the branches of f with an arrow ($\uparrow$) and the branches of g with a star ($\dagger$). As f is not ∂–equivalent to g there is some vertex of $\Gamma(fg)$ where the number of arrows and stars is different. Moreover, as f and g have the same multiplicity, there is a vertex P in $\Gamma(fg)$ where there are more arrows than stars. Let φ_i be a branch corresponding to an arrow starting from P. If we consider now the resolution tree of $fg\widetilde{\varphi}_i$, the arrows corresponding to φ_i and $\widetilde{\varphi}_i$ start from the same vertex of $\Gamma(fg\widetilde{\varphi}_i)$ (see figure 7). This is a consequence of the choice of P. We choose ψ to be a branch such that φ_i and ψ have a new common point in $\Gamma(fg\psi)$ (see figure 8). Then it is clear that $I(g, \psi) = I(g, \widetilde{\varphi}_i)$ and $I(f, \psi) = I(f, \widetilde{\varphi}_i) + 1$. ∎



$\Gamma(f\, g\widetilde{\varphi}_i)$

Figure 7



$\Gamma(f\, g\psi)$

Figure 8

We shall argue case by case.

1st. case: there is $i \in I$, such that $I(g, \widetilde{\varphi}_i) > I(f, \widetilde{\varphi}_i)$.

Let $\begin{cases} X = t^{m_i} \\ \\ Y = \widetilde{\beta}_i(t) \end{cases}$ be a parametrization of $\widetilde{\varphi}_i$.

Then we have: order $f(t^{m_i}, \widetilde{\beta}_i(t)) <$ order $g(t^{m_i}, \widetilde{\beta}_i(t))$.

There is a term, $T(X,Y)$, of degree at least s in $f(X,Y)$ (or in $g(X,Y)$) such that $T(t^{m_i}, \widetilde{\beta}_i(t))$ has a term of degree equal to the order of $f(t^{m_i}, \widetilde{\beta}_i(t))$. Set $T(X,Y) = X^\alpha Y^\beta$ with $\alpha + \beta \geq s$.

Then

$$I(f, \widetilde{\varphi}_i) = \text{ order } f(t^{m_i}, \widetilde{\beta}_i(t)) \geq \alpha m_i + \beta(\text{ order } \widetilde{\beta}_i(t)) \geq (\alpha + \beta) m_i \geq s m_i.$$

That is:

$$s \leq \left[\frac{(f, \widetilde{\varphi}_i)}{m_i} \right] \leq r.$$

2nd case. $I(g, \widetilde{\varphi}_i) < I(f, \widetilde{\varphi}_i)$ for some $i \in I$.

Then the order of $f(t^{m_i}, \widetilde{\beta}_i(t))$ is greater than the order of $g(t^{m_i}, \widetilde{\beta}_i(t))$. There is a term, $T(X,Y)$, of degree at least s in $f(X,Y)$ or $g(X,Y)$ such that $T(t^{m_i}, \widetilde{\beta}_i(t))$ has a term of degree equal to the order of $g(t^{m_i}, \widetilde{\beta}_i(t))$. Then:

$$s \leq \left[\frac{I(f, \widetilde{\varphi}_i)}{m_i} \right] < \left[\frac{I(f, \widetilde{\varphi}_i)}{m_i} \right] \leq r$$

3rd case. $I(f, \widetilde{\varphi}_i) = I(g, \widetilde{\varphi}_i)$ for every $i \in I$.

Let ψ be the branch given by the claim. By construction the multiplicity of ψ is equal to m_i for some $i \in I$.

Let $\begin{cases} X = t^{m_i} \\ \\ Y = \widetilde{\beta}(t) \end{cases}$ be a parametrization of ψ.

As $I(g, \psi) < I(f, \psi)$ we have:

$$s \leq \left[\frac{I(g, \psi)}{m_i} \right] = \left[\frac{I(g, \widetilde{\varphi}_i)}{m_i} \right] = \left[\frac{I(f, \widetilde{\varphi}_i)}{m_i} \right] \leq r.$$

2. Proof that $r \leq s$.

By the remark that we made in the introduction this inequality is a consequence of the result of Kuo and Lu. Here is an easy direct argument:

Assume that $s < r$. Then, if $j^{(r-1)}f = j^{(r-1)}g$, we have that f and g are ∂–equivalent, and let S be a set of disjoint annuli verifying the conditions of the definition of ∂–equivalence. If φ_i is a branch of f we may choose $\widetilde{\varphi}_i$ in such a way that the knot corresponding to $\widetilde{\varphi}_i, \widetilde{N}_i$, does not cut the annuli of S. Then the linking number of $\widetilde{N}_i$ with each component N_j of the link of f, is equal to the linking number of the component of the link g in the annulus which contains N_j. Then $I(f,\widetilde{\varphi}_i) = I(g,\widetilde{\varphi}_i)$.

In particular, if we call f_{r-1} the set of terms of degree at most $r-1$ in f, we have:

$I(f_{r-1},\widetilde{\varphi}_i) = I(f_{r-1} + aX^r,\widetilde{\varphi}_i)$, for all $a \in \mathbf{C} - \{0\}$, and for every $i \in I$.

$$\text{Let } \begin{cases} X = t^{m_i} \\ \\ Y = \widetilde{\beta}(t) \end{cases} \quad \text{be a parametrization of } \widetilde{\varphi}_i.$$

Then: order $f_{r-1}(t^{m_i},\widetilde{\beta}(t)) =$ order $(f_{r-1}(t^{m_i},\widetilde{\beta}(t))$ a$t^{rm_i})$ for all $a \in \mathbf{C} - \{0\}$. This implies that order $f_{r-1}(t^{m_i},\widetilde{\beta}(t)) < rm_i$ for every $i \in I$.

Then $r = \max \left[\frac{I(f,\widetilde{\varphi}_i)}{m_i}\right] = \max \left[\frac{I(f_{r-1},\widetilde{\varphi}_i)}{m_i}\right] < r$, which is a contradiction.

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