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Note on the Distribution of Ramanujan's Tau Function

By D. H. Lehmer

Abstract. According to a conjecture of Sato and Tate, the angle θ whose cosine is $\frac{1}{2}\tau(p)p^{-11/2}$, where τ is Ramanujan's function and p a prime, is distributed over $[0, \pi]$ according to a $\sin^2\theta$ law. The paper reports on a test of this conjecture for the 1229 primes under 10000. Extreme values of θ are also given.

Ramanujan's function $\tau(n)$ is defined by its generator

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\{(1 - x)(1 - x^2)(1 - x^3) \cdots\}^{24}.$$

For p a prime we can define θ_p by

$$\tau(p) = 2p^{11/2} \cos \theta_p.$$

According to the famous Ramanujan conjecture, θ_p should be real. Assuming this, one can ask how are the values of θ_p distributed in the interval $0 < \theta < \pi$. Sato and Tate [1] have conjectured that the density of those primes p for which

$$a < \theta_p < b$$

is given by

$$\frac{2}{\pi}\int_a^b \sin^2 \theta \, d\theta.$$

In other words, if we define $I(t)$ by

$$I(t) = \frac{2}{\pi}\int_{\arccos t}^{\pi/2} \sin^2 \theta \, d\theta$$

then the density of primes p for which

$$A < \tfrac{1}{2}\tau(p)p^{-11/2} < B$$

is $I(B) - I(A)$.

To test this conjecture we computed $\frac{1}{2}\tau(p)p^{-11/2}$ for each of the 1229 primes $p < 10^4$ from our table of $\tau(n)$ [2] and made a histogram of their values. If we divide the interval $(-1, 1)$ into equal subintervals of length h we can count the number $N_k(h)$ of primes p for which $\cos \theta_p$ satisfies

$$kh \leqq \cos \theta_p < (k + 1)h.$$

According to the conjecture $N(h)$ should be approximately the expected number:

$$E_k(h) = 1229\{I((k + 1)h) - I(kh)\}.$$

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Thus, for $h = 1$ we are merely asking how many $\tau(p)$ are positive and how many are negative. The expected number in each case is just one half of the total number of primes, namely 614.5. The actual counts give 616 positive and 613 negative values.

Breaking $(-1, 1)$ into four parts we have ($h = \frac{1}{2}$)

K	$N_K(\frac{1}{2})$	$E_K(\frac{1}{2})$
-2	228	240
-1	385	374
0	374	374
1	242	240

For $h = \frac{1}{4}$ we find

K	$N_K(\frac{1}{4})$	$E_K(\frac{1}{4})$
-4	89	89
-3	139	152
-2	187	181
-1	198	194
0	189	194
1	185	181
2	155	152
3	87	89

For $h = \frac{1}{8}$ we find

K	$N_K(\frac{1}{8})$	$E_K(\frac{1}{8})$	K	$N_K(\frac{1}{8})$	$E_K(\frac{1}{8})$
-8	29	32	0	91	98
-7	60	57	1	98	96
-6	69	71	2	99	93
-5	70	81	3	86	88
-4	89	88	4	81	81
-3	98	93	5	74	71
-2	87	96	6	63	57
-1	111	98	7	24	32

The agreement here is pretty reassuring.

With reference to the original Ramanujan conjecture we submit a list of those primes p whose $\cos\,\theta_p$ is larger (or smaller) than those of all smaller primes. These extreme primes and their values are as follows.

Extremely large

p	$\lvert\cos \theta_p\rvert$
2	.26516
3	.29936
5	.34560
11	.50043
17	.58982
47	.85458
103	.95940
3371	.97489
3967	.97571
7451	.97751
7589	.98026

Extremely small

p	$\lvert\cos \theta_p\rvert$
2	.26516
7	.18827
31	.16575
43	.00888
617	.00547
907	.00228
2927	.00049
7993	.00031

Department of Mathematics
University of California
Berkeley, California 94720

1. Letter of J. P. Serre to author, June 1, 1964.
2. D. H. LEHMER, *Table of Ramanujan's Function* $\tau(n)$, 1963. Ms. of 164 pages of computer printout. UMT File.