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NOTE ON THE E. ISING'S PAPER „BEITRAG ZUR THEORIE DES FERROMAGNETISMUS" [Zs. Physik, 31, 253 (1925)]

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Abstract

In this brief note some inaccuracy which occured in well-known paper by E. Ising [1] is discussed.

It is well-known that the Ising Model (IM) is of fundamental importance for Statistical Physics and its role for developing this branch of science can hardly be overestimated. E. Ising in his work [1], carried out under direct influence of his supervisor W. Lenz [2], considered the model of one-dimensional ferromagnet. It was a pioneering work which inspired and brought about to existence the torrent of papers (see [3]) and that is why, as a matter of fact, the model proposed in [1] is connected forever to the name of its author [4].

For that reason it seems interesting, instructive and curious enough from a historical point of view, to point out the fact that in his paper E. Ising made some mathematical incorrectness.

Namely, in the formula (7) of [1]

$$F(x) = \sum_{n=0}^{\infty} Z(n) x^n \tag{1}$$

it would be more correct to start the sum from $n = 1$ instead of $n = 0$. Indeed, in the paper [1] Ising took the coefficients $Z(n)$ in (1) from the generating function $F(x)$ given by the expression

$$F(x) = \frac{2x[\cosh\alpha - (1 - \exp(-\frac{\varepsilon}{kT}))x]}{1 - 2x\cosh\alpha + (1 - \exp(-\frac{2\varepsilon}{kT}))x^2}, \tag{2}$$

where $\varepsilon \geq 0$ and $\alpha = \frac{\mu H}{kT}$ (here we retain all the notations of [1]). Then, for the coefficients $Z(n)$ in the expansion (1), E. Ising wrote down the formula

$$Z(n) = c_1 \left(\cosh\alpha + \sqrt{\sinh^2\alpha + e^{-\frac{2\varepsilon}{kT}}}\right)^n + \\ + c_2 \left(\cosh\alpha - \sqrt{\sinh^2\alpha + e^{-\frac{2\varepsilon}{kT}}}\right)^n, \tag{3}$$

where $c_{1,2} = const.$ It is clear, that in accordance with (2), the expansion of $F(x)$ into a power series of x, should start from the term of the first order with respect to x, and hence, from (1) we have $Z(n = 0) = 0$. It implies, that $c_1 = -c_2 \equiv c$, i.e.

$$Z(n) = c\left[\left(\cosh\alpha + \sqrt{\sinh^2\alpha + e^{-\frac{2\varepsilon}{kT}}}\right)^n + \\ - \left(\cosh\alpha - \sqrt{\sinh^2\alpha + e^{-\frac{2\varepsilon}{kT}}}\right)^n\right], \quad (4)$$

for $n = 0, 1, 2, ...$, which is, strictly speaking, incorrect: the sign 'minus' in the last formula should be substituted by the sign 'plus'.

This can be checked directly, by means of calculating the first terms of $F(x)$-expansion into power series. However, it can be done also in other, more elegant way. Let us consider the funcion $\Phi(x)$, formally defined by

$$\Phi(x) = \lim_{\varepsilon\to\infty} F(x,\varepsilon) = \frac{2x[\cosh\alpha - x]}{1 - 2x\cosh\alpha + x^2}, \quad (5)$$

and correct from the mathematical point of view. Then, it is clear that for x and α such that $|xe^{\pm\alpha}| < 1$, one can write down the following sequence of equalities:

$$\Phi(x) = \lim_{\varepsilon\to\infty} F(x,\varepsilon) = \frac{2x[\cosh\alpha - x]}{1 - 2x\cosh\alpha + x^2} = \frac{xe^{\alpha}}{1 - xe^{\alpha}} + \frac{xe^{-\alpha}}{1 - xe^{-\alpha}} \\ = \sum_{n=1}^{\infty}[(xe^{\alpha})^n + (xe^{-\alpha})^n] = \sum_{n=1}^{\infty}[e^{n\alpha} + e^{-n\alpha}]x^n. \quad (6)$$

It follows immediately from (6) that the coefficients $Z(n)$ of the generating function $\Phi(x)$ are:

$$Z(n) = e^{n\alpha} + e^{-n\alpha}, \quad c = 1, \quad (7)$$

which obviously contradics to (4) for the chosen values of the parameters $(x, \alpha, \varepsilon \to \infty)$. This means that in (1) the summation over n has to start from $n = 1$, that is

$$F(x) = \sum_{n=1}^{\infty} Z(n)x^n \quad (8)$$

The incorrectness mentioned above does not play any role in the subsequent discussion presented in [1].

References

[1] E. Ising, Zs. Physik, **31**, 253 (1925).

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[4] S. G. Brush, Rev. Mod. Phys., **39**, 883 (1967).