Note on the factorization of a square matrix into two hermitian or symmetric matrices

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1. Introduction

Although the results already have been published (partially) by Frobenius in 1910 (see [5]), these are still not very known to mathematicians. I even could not find them in modern textbooks on matrix theory or linear algebra. These results and their proofs (see [1] , [2], [3]) are not very accessible for non-mathematicians. But they need the results. Applications can be found in system theory and in problems in mechanics concerning systems of differential equations. The aim of this paper is to give elementary proofs as well as a clear summary of the conditions. The basis of all proofs is the Jordan normal form. As we will see: every square matrix (real or complex) is a product of two symmetric (real resp. complex) matrices. However, not every complex square matrix is a product of two hermitian matrices.

2. Notations

A is a complex or real matrix of order $n \times n$;

Λ is a diagonal matrix of eigenvalues; A^T is the transpose of A;

$A^* = \bar{A}^T$ the conjugate transpose of A;

H denotes an hermitian matrix: $H^* = H$; U a unitary matrix: $UU^* = I$;

S a real symmetric matrix: $S^T = \bar{S}^T = S$; C a complex symmetric matrix $C^T = C$;

$A \simeq D$ means: A is similar to the matrix D or $A = BDB^{-1}$;

$H > 0$ means H is positive definite: for all vectors $x \neq 0$: $x^* Hx > 0$.

3. Preliminaries (for Theorems 1 and 3 see [4])

Theorem 1: Let H_1 be an hermitian matrix. Then there exists a unitary matrix U such that $H_1 = U \Lambda U^*$ with Λ real.
Moreover, is $H_1 > 0$ then all eigenvalues λ_i are positive.

Theorem 2: Let $H_1 > 0$. Then there exists an $H > 0$ such that $H_1 = H^2$.
Proof: $H_1 = U \Lambda U^* = (U \Lambda^{\frac{1}{2}} U^*)(U \Lambda^{\frac{1}{2}} U^*) =: H^2$ and $H > 0$. ∎

Theorem 3: (Jordan normal form). Let A be an arbitrary $n \times n$-matrix.
Then $A = B J B^{-1}$ where

$$J := \begin{pmatrix} J_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_r}(\lambda_r) \end{pmatrix}; \quad J_k(\lambda) \text{ is a } k \times k\text{-matrix} \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

$J_1(\lambda) = \lambda$; The λ_i are not necessarily different.

<u>Definition</u>: A Jordan matrix J is called <u>balanced</u> when $J_k(\lambda)$ is a Jordan-block in J, $J_k(\bar{\lambda})$ is also in J. This means that each complex λ and $\bar{\lambda}$ have the same "Jordanstructure", and $J \simeq \bar{J}$, or equivalently $A \simeq \bar{A}$.

4. Lemmas on factorization

<u>Lemma 1</u>: Every complex $n \times n$-matrix A is a product of two complex symmetric matrices: $A = C_1 C_2$, where C_1 or C_2 is nonsingular.

Proof:

$$J_k(\lambda) = \begin{pmatrix} & & & 1 \\ & O & \cdot & \\ & \cdot & & O \\ 1 & & & \end{pmatrix} \begin{pmatrix} & & & \lambda \\ & O & \lambda & 1 \\ & \lambda & \cdot & O \\ \lambda & 1 & & \end{pmatrix} =: S_k C_k \,;$$

$$J = \begin{pmatrix} S_{k_1} C_{k_1} & & O \\ & \ddots & \\ O & & S_{k_r} C_{k_r} \end{pmatrix} =: \tilde{S}\tilde{C} \,.$$

$$A = B\tilde{S}\,\tilde{C}B^{-1} = (B\tilde{S}B^{T})\,(B^{-T}\tilde{C}B^{-1}) =: C_1 C_2 \text{ with } C_1 \text{ nonsingular.} \qquad \blacksquare$$

<u>Corollary 1</u> *): $A \simeq A^T$.

Proof: $A = C_1 C_2$, suppose C_1 nonsingular

$$C_1^{-1} A C_1 = C_2 C_1 = A^T, \text{ hence } A \simeq A^T. \qquad \blacksquare$$

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*) Thanks to Dr. Laffey, Dublin, for this corollary and as a consequence, the improvement in the proof of Lemma 2,i.

<u>Lemma 2</u>: A complex matrix A of order $n \times n$ is a product of two hermitian matrices: $A = H_1 H_2$, where H_1 or H_2 is nonsingular, iff $A \simeq \bar{A}$.

Proof:

i: $A = H_1 H_2$; $A^* = H_2 H_1 = H_1^{-1}(H_1 H_2) H_1 = H_1^{-1} A H_1$.
Hence $A^* \simeq A$. With Corollary 1: $A^* \simeq (A^*)^T = \bar{A}$, so $A \simeq \bar{A}$.

ii: $A \simeq \bar{A}$ or $A = B J B^{-1}$ with J balanced: for each $J_k(\lambda)$ in J there is a $J_k(\bar{\lambda})$ in J. By permutation of the columns of B, it is always possible that $J_k(\bar{\lambda})$ comes directly after $J_k(\lambda)$ for each complex λ.

$$J = \begin{pmatrix} J_{k_1}(\lambda) & & & \bigcirc \\ & J_{k_1}(\bar{\lambda}) & & \\ & & \ddots & \\ \bigcirc & & & \ddots \end{pmatrix};$$

$$\begin{pmatrix} J_k(\lambda) & \bigcirc \\ \bigcirc & J_k(\bar{\lambda}) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & & & & & \bigcirc \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda & 0 & & \\ & & & & \bar{\lambda} & 1 & \\ & & & & & \ddots & \ddots \\ & \bigcirc & & & & & 1 \\ & & & & & & \bar{\lambda} \end{pmatrix} =$$

$$\begin{pmatrix} \bigcirc & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \bigcirc \end{pmatrix} \begin{pmatrix} \bigcirc & & & & & & \bar{\lambda} \\ & & & & & \cdot^{\cdot^{\cdot}} & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & & & \bar{\lambda} & 1 & & \\ & & \lambda & 0 & & & \\ & \cdot^{\cdot^{\cdot}} & 1 & & & & \\ \lambda & 1 & & & & & \bigcirc \end{pmatrix} = S_k H_k .$$

Hence $J = \tilde{S}\tilde{H}$ and $A = B\tilde{S}\tilde{H}B^{-1} = (B\tilde{S}B^{*})(B^{*-1}\tilde{H}B^{-1}) =: H_1H_2$ where H_1 is nonsingular. ∎

<u>Corollary 2</u>: The characteristic polynomial of H_1H_2, $\det(H_1H_2 - \lambda I)$, has only real coefficients, and specially $tr(H_1H_2)$ and $\det(H_1H_2)$ are real.

Example:

$$A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \neq H_1H_2 \quad \text{because } tr\, A = 2i \text{ is not real.}$$

$$A = \begin{pmatrix} i & i \\ 1 & -i \end{pmatrix} \neq H_1H_2 \quad \text{because } \det A = 1 - i \text{ is not real.}$$

$$A = \begin{pmatrix} i & -i \\ 0 & -i \end{pmatrix} = H_1H_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & -0 \end{pmatrix} .$$

<u>Lemma 3</u>: Every complex matrix A with <u>real</u> eigenvalues, is a product of two hermitian matrices: $A = H_1H_2$, where H_1 or H_2 is nonsingular.

Proof: This follows directly from Lemma 2. Λ is real, so J is real and $J \simeq \bar{J}$. The condition for Lemma 2 is fulfilled ∎

<u>Lemma 4</u>: Every real $n \times n$-matrix A is a product of two real matrices: $A = S_1S_2$ where S_1 or S_2 is nonsingular.

Proof 1: $A = BJB^{-1}$. A is real, $A = \bar{A}$. The condition of Lemma 2 holds. By permutation of the columns of B, it is always possible that

$$J = \begin{pmatrix} J_{k_1}(\lambda) & & & & \\ & J_{k_2}(\lambda) & & \bigcirc & \\ & & \ddots & & \\ & & & J_{k_1}(\bar{\lambda}) & \\ & \bigcirc & & & J_{k_2}(\bar{\lambda}) \\ & & & & & \ddots \end{pmatrix}, \text{ with all real } J_k(\lambda) \text{ in the middle of } J.$$

From AB = BJ we see that

$Ab_i = \lambda_i b_i + \delta_i b_{i-1}$ ($\delta_i = 0$ or 1) and

$A\bar{b}_i = \bar{\lambda}_i \bar{b}_i + \bar{\delta}_i \bar{b}_{i-1}$.

This means that, if b_i is a column of B, $\bar{b}_i$ also.

Then $B = (b_1 \ldots b_p b_{p+1} \ldots b_q \bar{b}_1 \ldots \bar{b}_p)$; The b_i, $i = p+1, \ldots, q$ are real columns corresponding with the real eigenvalues of B (this set can be empty as well as the set $\{b_1, \ldots, b_p\}$).

As in the proof of Lemma 1, $J = \tilde{S}\tilde{C}$.

$A = B\tilde{S}\tilde{C}B^{-1} = (B\tilde{S}B^T)(B^{-T}\tilde{C}B^{-1}) =: S_1 S_2$ where S_1 is nonsingular. Indeed:

$$S_1 := B\tilde{S}B^T = (b_1 \ldots b_p b_{p+1} \ldots b_q \bar{b}_1 \ldots \bar{b}_p)(\bar{b}_p \ldots \bar{b}_1 b_q \ldots b_{p+1} b_p \ldots b_1)^T =$$

$$\sum_1^p b_i \bar{b}_{p+1-i}^T + \sum_{p+1}^q b_i b_{p+q+1-i}^T + \sum_1^p \bar{b}_i b_{p+1-i}^T \text{ is real.}$$

$S_2 = S_1^{-1}A$ is, a product of two real matrices, also real. ∎

Proof 2: (suggested by Dr. Laffey, Dublin).

With Lemma 1: $A = C_1 C_2$, suppose $C_1 = S_0 + i S_3$ nonsingular (S_0, S_3 real symmetric) $AC_1 = C_1 C_2 C_1 = C_1 A^T$; $A(S_0 + iS_3) = (S_0 + iS_3)A^T$;

$AS_0 = S_0 A^T$ and $AS_3 = S_3 A^T$.

So, for all real numbers r : $A(S_0 + rS_3) = (S_0 + rS_3)A^T$.

If $S_3 = 0$, then $C_1 = S_0$ real and $A = S_1 S_2$. So suppose $S_3 \neq 0$.

Define $f(z) := \det(S_0 + zS_3)$; $\det(S_0 + iS_3) = \det C_1 \neq 0$.

Hence f(z) is not the zero-polynomial, or $\exists r \in \mathbb{R}$ with $f(r) \neq 0$ or $\det(S_0 + rS_3) \neq 0$, $S_0 + rS_3 =: S_1$ nonsingular.

$AS_1 = S_1 A^T$; $A = S_1 A^T S_1^{-1} =: S_1 S_2$ with $S_2 =: A^T S_1^{-1}$; $S_2^T = S_1^{-1} A = A^T S_1^{-1} = S_2$.

∎

<u>Lemma 5</u>: A complex $n \times n$-matrix A is a product of two hermitian matrices: $A = H_1 H_2$, where H_1 or H_2 is positive definite, iff A is similar to Λ real or $A \simeq \Lambda$ real.

Proof: i: Only if: suppose $H_1 > 0$. $H_1 = H^2$ (Theorem 2); $H_1 H_2 = H(HH_2H)H^{-1}$. HH_2H is hermitian, so $= U \Lambda U^*$ with Λ real (Theorem 1).
$A = H(U \Lambda U^*)H^{-1} = (HU)\Lambda(HU)^{-1} =: B \Lambda B^{-1}$ or $A \simeq \Lambda$ real.

ii: If: $A \simeq \Lambda$ real; $A = B \Lambda B^{-1}$; $A = (BB^*)(B^{*-1} \Lambda B^{-1}) =: H_1 H_2$ with $H_1 > 0$. ∎

Remark: If H_1 is <u>semi</u>-positive definite, then the "only if" part does not hold:

Example: $H_1 H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is defective, hence not similar to a diagonal matrix.

<u>Lemma 6</u>: ~~Every~~ a real $n \times n$-matrix A is a product of two real symmetric matrices: $A = S_1 S_2$, where S_1 or S_2 is positive definite, iff A is similar to a Λ real.

Proof: This follows directly from the proof in Lemma 5:

i: Replace each H by S and U by an orthogonal matrix G.

ii: A and Λ real, hence B is real. So $H_i = S_i$ and $A = S_1 S_2$. ∎

Remark: If we weaken the iff-condition and cancel the word (in the 4th column) "real", then of course $A \neq H_1 H_2$ (see Lemma 5), but $A = H_1 N_2$ with $H_1 > 0$ and N_2 such that $N_2 H_1 N_2^* = N_2^* H_1 N_2$.

Summary

lemma	A	Λ	iff condition	factorization		
1	complex	complex	-	$A = C_1C_2$	C_1 or C_2	
2	complex	complex	$A \simeq \bar{A}$	$A = H_1H_2$	H_1 or H_2	
3	complex	real	-	$A = H_1H_2$	H_1 or H_2	non-singular
4	real	complex	-	$A = S_1S_2$	S_1 or S_2	
5	complex	complex	$A \simeq \Lambda$ real	$A = H_1H_2$	H_1 or $H_2 > 0$	
6	real	complex	$A \simeq \Lambda$ real	$A = S_1S_2$	S_1 or $S_2 > 0$	

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