# Note on the factorization of a square matrix into two hermitian or symmetric matrices 

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## EINDHOVEN UNIVERSITY OF TECHNOLOGY

## Department of Mathematics and Computing Science

## COSOR-Memorandum 84-12

## Note on the factorization of a square matrix into two Hermitian or symmetric matrices

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## OR SYMMETRIC MATRICES

by

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1. Introduction

Although the results already have been published (partially) by Frobenius in 1910 (see [5]), these are still not very known to mathematicians.

I even could not find them in modern textbooks on matrix theory or linear algebra. These results and their proofs (see [1], [2], [3]) are not very accessible for non-mathematicians. But they need the results. Applications can be found in system theory and in problems in mechanics concerning systems of differential equations. The aim of this paper is to give elementary proofs as well as a clear summary of the conditions. The basis of all proofs is the Jordan normal form. As we will see: every square matrix (real or complex) is a product of two symmetric (real resp. complex) matrices. However, not every complex square matrix is a product of two hermitian matrices.

## 2. Notations

A is a complex or real matrix of order $n \times n$;
$\Lambda$ is a diagonal matrix of eigenvalues; $A^{T}$ is the transpose of $A$;
$A^{*}=\bar{A}^{T}$ the conjugate transpose of $A$;
H denotes an hermitian matrix: $\mathrm{H}^{*}=\mathrm{H}$; U a unitary matrix: $\mathrm{UU}^{*}=\mathrm{I}$;
$S$ a real symmetric matrix: $S^{T}=\bar{S}^{T}=S$; $C$ a complex symmetric matrix $C^{T}=C$;
$A \simeq D$ means: $A$ is similar to the matrix $D$ or $A=B D B^{-1}$;
$H>0$ means $H$ is positive definite: for all vectors $x \neq 0: x^{*} H x>0$.
3. Preliminaries (for Theorems 1 and 3 see [4])

Theorem 1: Let $H_{1}$ be an hermitian matrix. Then there exists a unitary matrix $U$ such that $H_{1}=U \Lambda U^{*}$ with $\Lambda$ real.
Moreover, is $H_{1}>0$ then all eigenvalues $\lambda_{i}$ are positive.

Theorem 2: Let $H_{1}>0$. Then there exists an $H>0$ such that $H_{1}=H^{2}$. Proof: $H_{1}=U \Lambda U^{*}=\left(U \Lambda^{\frac{1}{2}} U^{*}\right)\left(U \Lambda^{\frac{1}{2}} U^{*}\right)=: H^{2}$ and $H>0$.

Theorem 3: (Jordan normal form). Let $A$ be an arbitrary $n \times n-m a t r i x$. Then $A=$ B J B $^{-1}$ where


Definition: A Jordan matrix $J$ is called balanced when $J_{k}(\lambda)$ is a Jordanblock in $J, J_{k}(\bar{\lambda})$ is also in $J$. This means that each complex $\lambda$ and $\bar{\lambda}$ have the same "Jordanstructure", and $J \simeq \bar{J}$, or equivalently $\mathrm{A} \simeq \overline{\mathrm{A}}$.

## 4. Lemmas on factorization

Lemma 1: Every complex $n \times n$-matrix $A$ is a product of two complex symmetric matrices: $A=C_{1} C_{2}$, where $C_{1}$ or $C_{2}$ is nonsingular.

Proof:

$$
J=\left(\begin{array}{cccc}
\mathrm{S}_{\mathbf{k}_{1}}{ }^{\mathbf{c}_{\mathbf{k}_{1}}} & & & \\
& & \ddots & \\
& & \ddots & \\
& & \ddots & \\
& & & \mathbf{s}_{\mathbf{k}_{\mathbf{r}}} \mathbf{c}_{\mathbf{k}_{\mathbf{r}}}
\end{array}\right)=: \widetilde{\mathrm{S}} \widetilde{\mathrm{C}}
$$

$$
A=B \tilde{S} \tilde{C}_{B}^{-1}=\left(B \tilde{S}_{B}^{T}\right)\left(B^{-T} \tilde{C}_{B}^{-1}\right)=: C_{1} C_{2} \text { with } C_{1} \text { nonsingular. }
$$

Corollary $1^{*}$ ) $: A \simeq A^{T}$.
Proof: $A=C_{1} C_{2}$, suppose $C_{1}$ nonsingular

$$
C_{1}^{-1} A C_{1}=C_{2} C_{1}=A^{T}, \text { hence } A \simeq A^{T}
$$

[^0]Lemma 2: A complex matrix $A$ of order $n \times n$ is a product of two hermitian matrices: $A=H_{1} H_{2}$, where $H_{1}$ or $H_{2}$ is nonsingular, iff $A \simeq \bar{A}$.

Proof:
i: $\quad A=H_{1} H_{2} ; A^{*}=H_{2} H_{1}=H_{1}^{-1}\left(\mathrm{H}_{1} \mathrm{H}_{2}\right) \mathrm{H}_{1}=H_{1}^{-1} A H_{1}$.
Hence $A^{*} \simeq A$. With Corollary $1: A^{*} \simeq\left(A^{*}\right)^{T}=\bar{A}$, so $A \simeq \bar{A}$.
ii: $A \simeq \bar{A}$ or $A=B J B^{-1}$ with $J$ balanced: for each $J_{k}(\lambda)$ in $J$ there is a $J_{k}(\bar{\lambda})$ in $J$. By permutation of the columns of $B$, it is always possible that $J_{k}(\bar{\lambda})$ comes directly after $J_{k}(\lambda)$ for each complex $\lambda$.
$\mathbf{J}=\left(\begin{array}{lllll}\mathbf{J}_{\mathbf{k}_{\mathbf{1}}}(\lambda) & & & & \\ & & \mathbf{J}_{\mathbf{k}_{\mathbf{1}}}(\bar{\lambda}) & & \\ & & & & \\ & & \ddots & \\ & & & \ddots & \\ & & & \ddots & \\ & & & & \ddots\end{array}\right) ;$



Hence $J=\tilde{S} \tilde{H}$ and $A=\tilde{B S} \tilde{H}_{B}^{-1}=\left(\tilde{B S}_{B}{ }^{*}\right)\left(B^{*-1} \tilde{H}_{B}^{-1}\right)=: H_{1} H_{2}$ where $H_{1}$ is nonsingular.

Corollary 2: The characteristic polynomial of $\mathrm{H}_{1} \mathrm{H}_{2}$, $\operatorname{det}\left(\mathrm{H}_{1} \mathrm{H}_{2}-\lambda I\right)$, has only real coefficients, and specially $\operatorname{tr}\left(\mathrm{H}_{1} \mathrm{H}_{2}\right)$ and $\operatorname{det}\left(\mathrm{H}_{1} \mathrm{H}_{2}\right)$ are real.

Example:

$$
\begin{aligned}
& A=\left(\begin{array}{ll}
i & 0 \\
0 & i
\end{array}\right) \neq H_{1} H_{2} \quad \text { because } \operatorname{tr} A=2 i \text { is not real. } \\
& A=\left(\begin{array}{ll}
i & i \\
1-i
\end{array}\right) \neq H_{1} H_{2} \quad \text { because } \operatorname{det} A=1-i \text { is not real. } \\
& A=\binom{i-i}{0-i}=H_{1} H_{2}=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)\binom{0-i}{i-0} .
\end{aligned}
$$

Lemma 3: Every complex matrix A with real eigenvalues, is a product of two hermitian matrices: $A=H_{1} H_{2}$, where $H_{1}$ or $H_{2}$ is nonsingular.

Proof: This follows directly from Lemma 2. $\Lambda$ is real, so $J$ is real and $\mathbf{J} \simeq \overline{\mathbf{J}}$. The condition for Lemma 2 is fulfilled

Lemma 4: Every real $n \times n$-matrix $A$ is a product of two real matrices: $A=S_{1} S_{2}$ where $S_{1}$ or $S_{2}$ is nonsingular.
Proof 1: $A=B_{B^{-1}}$. A is real, $A=\bar{A}$. The condition of Lemma 2 holds. By permutation of the columns of $B$, it is always possible that

$$
J=\left(\begin{array}{lllll}
J_{\mathbf{k}_{1}}(\lambda) & & & & \\
& J_{\mathbf{k}_{2}}(\lambda) & & & \\
& & \ddots & & \\
& & & J_{\mathbf{k}_{1}}(\bar{\lambda}) & \\
& & & & \\
& & & J_{\mathbf{k}_{\mathbf{2}}}(\bar{\lambda}) & \\
& & & & \\
& & & & \\
& & & &
\end{array}\right) \text { in the middle of } \mathrm{J} .
$$

From $A B=B J$ we see that
$A b_{i}=\lambda_{i} b_{i}+\delta_{i} b_{i-1} \quad\left(\delta_{i}=0\right.$ or 1$)$ and
$A \bar{b}_{i}=\bar{\lambda}_{i} \bar{b}_{i}+\bar{\delta}_{i} \bar{b}_{i-1}$.
This means that, if $b_{i}$ is a column of $B, \bar{b}_{i}$ also.
Then $B=\left(b_{1} \ldots b_{p} b_{p+1} \ldots b_{q} \bar{b}_{1} \ldots \bar{b}_{p}\right) ;$ The $b_{i}, i=p+1, \ldots, q$ are real columns corresponding with the real eigenvalues of $B$ (this set can be empty as well as the set $\left\{b_{1}, \ldots, b_{p}\right\}$ ).
As in the proof of Lemma $1, J=\widetilde{\mathbf{S}} \tilde{\mathrm{C}}$.
$A=B \tilde{S}^{C_{B}}{ }^{-1}=\left(\mathcal{B S B}^{T}\right)\left(B^{-T} \widetilde{C}_{B}^{-1}\right)=: S_{1} S_{2}$ where $S_{1}$ is nonsingular. Indeed:

$$
\begin{aligned}
& S_{1}:=\tilde{B S}_{B}^{T}=\left(b_{1} \ldots b_{p} b_{p+1} \ldots b_{q} \bar{b}_{1} \ldots \bar{b}_{p}\right)\left(\bar{b}_{p} \ldots \bar{b}_{1} b_{q} \ldots b_{p+1} b_{p} \ldots b_{1}\right)^{T}= \\
& p \\
& \sum_{1} b_{i} \bar{b}_{p+1-i}^{T}+\sum_{p+1}^{q} b_{i} b_{p+q+1-i}^{T}+\sum_{1} \bar{b}_{i} b_{p+1-i}^{T} \text { is real. } \\
& S_{2}=S_{1}^{-1} A \text { is, a product of two real matrices, also real. }
\end{aligned}
$$

Proof 2: (suggested by Dr. Laffey, Dublin).
With Lemma 1: $A=C_{1} C_{2}$, suppose $C_{1}=S_{0}+i S_{3}$ nonsingular $\left(S_{0}, S_{3}\right.$ real symmetric) $A C_{1}=C_{1} C_{2} C_{1}=C_{1} A^{T} ; A\left(S_{0}+i S_{3}\right)=\left(S_{0}+i S_{3}\right) A^{T}$;
$A S_{0}=S_{0} A^{T}$ and $A S_{3}=S_{3} A^{T}$.
So, for all real numbers $r: A\left(S_{0}+r S_{3}\right)=\left(S_{0}+r S_{3}\right) A^{T}$.
If $S_{3}=0$, then $C_{1}=S_{0}$ real and $A=S_{1} S_{2}$. So suppose $S_{3} \neq 0$.
Define $f(z):=\operatorname{det}\left(S_{0}+z S_{3}\right) ; \operatorname{det}\left(S_{0}+i S_{3}\right)=\operatorname{det} C_{1} \neq 0$.
Hence $f(z)$ is not the zero-polynomial, or $\exists r \in R$ with $f(r) \neq 0$
or $\operatorname{det}\left(S_{0}+r S_{3}\right) \neq 0, S_{0}+r S_{3}=: S_{1}$ nonsingular.
$A S_{1}=S_{1} A^{T} ; A=S_{1} A^{T} S_{1}^{-1}=: S_{1} S_{2}$ with $S_{2}=: A^{T} S_{1}^{-1} ; S_{2}^{T}=S_{1}^{-1} A=A^{T} S_{1}^{-1}=S_{2}$.

Lemma 5: A complex $n \times n$-matrix $A$ is a product of two hermitian matrices: $A=H_{1} H_{2}$, where $H_{1}$ or $H_{2}$ is positive definite, iff $A$ is similar to $\Lambda$ real or $A \simeq \Lambda$ real.
Proof: i: Only if: suppose $H_{1}>0 . H_{1}=H^{2}$ (Theorem 2); $H_{1} H_{2}=H\left(H_{2} H\right) H^{-1}$. $\mathrm{HH}_{2} \mathrm{H}$ is hermitian, so $=\mathrm{U} \Lambda \mathrm{U}^{*}$ with $\Lambda$ real (Theorem 1). $A=H\left(U \Lambda U^{*}\right) H^{-1}=(H U) \Lambda(H U)^{-1}=: B \wedge B^{-1}$ or $A \simeq \Lambda$ real.
ii: If $: A \simeq \Lambda$ real; $A=B \wedge B^{-1} ; A=\left(B^{*}\right)\left(B^{*-1} \Lambda B^{-1}\right)=: H_{1} H_{2}$ with $H_{1}>0$.

Remark: If $H_{1}$ is semi-positive definite, then the "only if" part does not hold:

Example: $H_{1} H_{2}=\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)=\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right)$ is defective, hence not similar to a diagonal matrix.
$a$
Lemma 6: Every real $n \times n$-matrix $A$ is a product of two real symmetric matrices: $A=S_{1} S_{2}$, where $S_{1}$ or $S_{2}$ is positive definite, iff $A$
is similar to a $\Lambda$ real.

Proof: This follows directly from the proof in Lemma 5:
i: Replace each $H$ by $S$ and $U$ by an orthogonal matrix $G$.
ii: $A$ and $\Lambda$ real, hence $B$ is real. $S o H_{i}=S_{i}$ and $A=S_{1} S_{2}$.

Remark: If we weaken the iff-condition and cancel the word (in the 4th column) "real", then of course $A \neq H_{1} H_{2}$ (see Lemma 5), but $A=H_{1} N_{2}$ with $H_{1}>0$ and $N_{2}$ such that $N_{2} H_{1} N_{2}^{*}=N_{2}^{*} H_{1} N_{2}$.

Summary

| 1 emma | A | $\Lambda$ | iff condition | factorization |
| :---: | :---: | :---: | :---: | :---: |
| 1 | complex | complex | - | $A=C_{1} C_{2} \quad C_{1}$ or $C_{2}$ |
| 2 | complex | complex | $\mathrm{A} \simeq \overline{\mathrm{A}}$ | $\mathrm{A}=\mathrm{H}_{1} \mathrm{H}_{2} \quad \mathrm{H}_{1}$ or $\mathrm{H}_{2}$ |
| 3 | complex | real | - | $\mathrm{A}=\mathrm{H}_{1} \mathrm{H}_{2} \quad \mathrm{H}_{1}$ or $\mathrm{H}_{2} \quad \begin{aligned} & \text { non- } \\ & \text { singular }\end{aligned}$ |
| 4 | real | complex | - | $A=S_{1} S_{2} \quad S_{1}$ or $S_{2}$ |
| 5 | complex | complex | $A \simeq \Lambda$ real | $A=\mathrm{H}_{1} \mathrm{H}_{2} \quad \mathrm{H}_{1}$ or $\mathrm{H}_{2}>0$ |
| 6 | real | complex | $\mathrm{A} \simeq \Lambda$ real | $\mathrm{A}=\mathrm{S}_{1} \mathrm{~S}_{2} \quad \mathrm{~S}_{1}$ or $\mathrm{S}_{2}>0$ |

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[^0]:    *) Thanks to Dr. Laffey, Dublin, for this corollary and as a consequence, the improvement in the proof of Lemma $2, i$.

