NOTE ON THE INTEGRABILITY OF A CERTAIN STRUCTURE ON DIFFERENTIABLE MANIFOLD

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In this short note we consider an n-dimensional differentiable manifold of class C^∞ on which a structure is defined by three tensor fields (generally complex) $\underset{1}{F}{}_i^j$, $\underset{2}{F}{}_i^j$, $\underset{3}{F}{}_i^j$ [1], of class C^∞ satisfying one of the following two systems of conditions:

System A:

$$\mathrm{A}\begin{cases} \mathrm{A}_1: \underset{1}{F}{}_i^j,\ \underset{2}{F}{}_i^j \text{ are non trivial and not proportional.} \\ \mathrm{A}_2: \underset{1}{F}{}_i^j \underset{1}{F}{}_j^k = \lambda_1^2 \delta_i^k,\ \underset{2}{F}{}_i^j \underset{2}{F}{}_j^k = \lambda_2^2 \delta_i^k, \text{ where } \lambda_1, \lambda_2 \text{ are} \\ \qquad \text{fixed non zero complex numbers.} \\ \mathrm{A}_3: \underset{1}{F}{}_i^j \underset{2}{F}{}_j^k = \underset{2}{F}{}_i^j \underset{1}{F}{}_j^k. \end{cases}$$

In this case, if we put

$$(1) \qquad \underset{1}{F}{}_i^j \underset{2}{F}{}_j^k = \underset{2}{F}{}_i^j \underset{1}{F}{}_j^k \equiv - \underset{3}{F}{}_i^k,$$

then we have

$$(2) \qquad \underset{3}{F}{}_i^j \underset{3}{F}{}_j^k = \lambda_1^2 \lambda_2^2 \delta_i^k,$$

$$\underset{1}{F}{}_i^j \underset{3}{F}{}_j^k = \underset{3}{F}{}_i^j \underset{1}{F}{}_j^k = - \lambda_1^2 \underset{2}{F}{}_i^k,\ \underset{2}{F}{}_i^j \underset{3}{F}{}_j^k = \underset{3}{F}{}_i^j \underset{2}{F}{}_j^k = - \lambda_2^2 \underset{1}{F}{}_i^k.$$

System B:

$$\mathrm{B}\begin{cases} \mathrm{B}_1: \underset{1}{F}{}_i^j,\ \underset{2}{F}{}_i^j \text{ are non trivial and not proportional.} \\ \mathrm{B}_2: \underset{1}{F}{}_i^j \underset{1}{F}{}_j^k = \lambda_1^2 \delta_i^k,\ \underset{2}{F}{}_i^j \underset{2}{F}{}_j^k = - \lambda_2^2 \delta_i^k, \text{ where } \lambda_1, \lambda_2 \\ \qquad \text{are fixed non zero complex numbers.} \\ \mathrm{B}_3: \underset{1}{F}{}_i^j \underset{2}{F}{}_j^k = - \underset{2}{F}{}_i^j \underset{1}{F}{}_j^k. \end{cases}$$

In this case if we put

1) The Latin indicies $i, j, k \cdots\cdots$ vary from 1 to n.

(3) $$\underset{1}{F_i^j}\underset{2}{F_j^k} = -\underset{2}{F_i^j}\underset{1}{F_j^k} \equiv \underset{3}{F_i^k},$$

then we have

(4) $$\underset{3}{F_i^j}\underset{3}{F_j^k} = \lambda_1^2 \lambda_2^2 \delta_i^k$$

$$\underset{2}{F_i^j}\underset{3}{F_j^k} = -\underset{3}{F_i^j}\underset{2}{F_j^k} = \lambda_2^2 \underset{1}{F_i^j}, \quad \underset{3}{F_i^j}\underset{1}{F_j^k} = -\underset{1}{F_i^j}\underset{3}{F_j^k} = -\lambda_1^2 \underset{2}{F_i^k}.$$

We call the structure satisfying system A or system B respectively the *structure* A or *structure* B for convenience sake.

Such structures are said to be integrable if at each point of the manifold, there exists a complex coordinate system (i. e. n independent complex valued functions of the local coordinates of the points in the neighborhood) in which the fields $\underset{1}{F_j^i}$, $\underset{2}{F_j^i}$ and $\underset{3}{F_i^j}$ have simultaneously numerical components [1][2].

In the following, after some preparations the conditions for the integrability of structure A and structure B are studied by different ways.

It is evident that structure A contains as special cases the case II ($\lambda_1^2 = 1$, $\lambda_2^2 = \lambda_1^2 \lambda_2^2 = -1$ or $\lambda_1^2 = \lambda_2^2 = -1$, $\lambda_1^2 \lambda_2^2 = 1$) and the case IV ($\lambda_1^2 = \lambda_2^2 = \lambda_1^2 \lambda_2^2 = 1$), the structure B contains the case I ($\lambda_1^2 = -1$, $\lambda_2^2 = 1$) and the case III ($\lambda_1^2 = -1 = \lambda_2^2$ or $\lambda_1^2 = \lambda_2^2 = 1$) of a previous note of the present author [3]. In these cases, all the tensors $\underset{1}{F_i^j}$, $\underset{2}{F_i^j}$, $\underset{3}{F_i^j}$ are real. The results obtained below hold also for these cases.

1. In this section we treat the structure A and obtain its integrability conditions by applying a result in a previous paper of the present author [4].

A manifold is said to be endowed with an *r-π-structnre* if there exist r distributions (differentiable) $\underset{1}{T}$, $\underset{2}{T}, \ldots\ldots \underset{r}{T}$ of (complex) tangent subspaces such that $T_P^C = \underset{1}{T_P} + \ldots\ldots + \underset{r}{T_P}$ (direct sum) holds at each point, where T_P^C is the complexification of the tangent space at P and $\underset{t}{T_P}$ is the subspace at P belonging to the distribution $\underset{t}{T}$ [2] [4]. Then we have

THEOREM 1. *If the manifold has a structure* A, *then the manifold is endowed with a* 4-*π-structure or with a* 3-*π-structure. In the latter situation, there is the following relation*:

$$\frac{1}{\lambda_2}\underset{2}{F_i^j} - \frac{1}{\lambda_1}\underset{1}{F_i^j} - \frac{1}{\lambda_1 \lambda_2}\underset{3}{F_i^j} = \delta_i^j.$$

The converse also holds good.

2) Number in bracket refers to the references at the end of the paper.

PROOF. Let the linear transformation induced by $\underset{a}{F_i^j}(a = 1, 2, 3)$ in T_P^C be denoted as $\underset{a}{\mathfrak{F}}$. The proper subspaces of $\underset{1}{\mathfrak{F}}$ corresponding to the proper value λ_1 and $-\lambda_1$ are respectively denoted as $T_{1'}$ and $T_{1''}$. If we use the adapted basis of T_P^C [i. e. the basis of which the former $n_1(=\dim T_{1'})$ vectors are in $T_{1'}$ and the other $n_2(=\dim T_{1''})$ vectors are in $T_{1''}$], we have

$$\text{(5)} \qquad \underset{1}{\mathfrak{F}} = \lambda_1 \begin{pmatrix} E_{n_1} & 0 \\ 0 & E_{n_2} \end{pmatrix} \text{ and } \underset{2}{\mathfrak{F}} = \begin{pmatrix} A_{n_1} & 0 \\ 0 & A_{n_2} \end{pmatrix}$$

as $\underset{1}{\mathfrak{F}}$, $\underset{2}{\mathfrak{F}}$ commute. Here $\underset{a}{\mathfrak{F}}$ also represents the corresponding matrix, E_{n_1} denotes the $n_1 \times n_1$ unit matrix, whereas A_{n_1} denotes a $n_1 \times n_1$ matrix. Since $\underset{2}{\mathfrak{F}}^2 = \lambda_2^2 \mathfrak{I}$ ($\mathfrak{I}$: identily transformation), we have $A_{n_1}^2 = \lambda_1^2 E_{n_1}$, $A_{n_2}^2 = \lambda_2^2 E_{n_2}$. Hence A_{n_1} and A_{n_2} corresponds respectively to the linear transformation $\underset{2}{\mathfrak{F}}'$ and $\underset{2}{\mathfrak{F}}''$ induced by $\underset{2}{\mathfrak{F}}$ in $T_{1'}$ and $T_{1''}$. If $\underset{2}{\mathfrak{F}}'$ and $\underset{2}{\mathfrak{F}}''$ are non trivial on $T_{1'}$ and $T_{1''}$ respectively, then we denote the proper subspaces of $\underset{2}{\mathfrak{F}}'$ in $T_{1'}$ corresponding to λ_2 or $-\lambda_2$ respectively as $T_{(1'2')}$, $T_{(1'2'')}$ and the proper subspaces of $\underset{2}{\mathfrak{F}}''$ in $T_{1''}$ corresponding to λ_2 or $-\lambda_2$ as $T_{(1''2')}$ and $T_{(1'2'')}$. It is now evident that the manifold is endowed with a 4-π-structure defined by the four distributions: $T_{(1'2')}$, $T_{(1'2'')}$, $T_{(1''2')}$, and $T_{(1'2'')}$. If we denote the projection operations from T_P^C to $T_{(1'2')}$, $T_{(1'2'')}$, $T_{(1''2')}$ and $T_{(1''2'')}$ respectively as P_1, P_2, P_3 and P_4, then we have

$$\text{(6)} \qquad \mathfrak{I} = P_1 + P_2 + P_3 + P_4, \quad \underset{1}{\mathfrak{F}} = \lambda_1(P_1 + P_2 - P_3 - P_4)$$
$$\underset{2}{\mathfrak{F}} = \lambda_2(P_1 - P_2 + P_3 - P_4) \quad \underset{3}{\mathfrak{F}} = -\lambda_1\lambda_2(P_1 - P_2 - P_3 + P_4).$$

Next, if $\underset{2}{\mathfrak{F}}'$ is trivial on $T_{1'}$ and $\underset{2}{\mathfrak{F}}''$ is trivial on $T_{1''}$, then $\underset{1}{\mathfrak{F}}$, $\underset{2}{\mathfrak{F}}$ are proportional and this case is excepted. So if $\underset{2}{\mathfrak{F}}'$ is trivial on $T_{1'}$, then $\underset{2}{\mathfrak{F}}''$ is non trivial on $T_{1''}$. In this case $\underset{2}{\mathfrak{F}}'$ has only one proper value and its proper subspace is $T_{1'}$, itself. Whereas $\underset{2}{\mathfrak{F}}''$ has two proper subspaces $T_{(1''2')}$ and $T_{(1''2'')}$ in $T_{1''}$ corresponding to the proper valuer λ_2 and $-\lambda_2$. Thus the manifold is endowed with a 3-π-structure defined by $T_{1'}$, $T_{(1''2')}$ and $T_{(1''2'')}$. Denote the projection operations from T_P^C to $T_{1'}$, $T_{(1''2')}$ and $T_{(1''2'')}$ respectively as P_1, P_2 and P_3, then we have

$$\text{(7)} \qquad \mathfrak{I} = P_1 + P_2 + P_3, \qquad \underset{1}{\mathfrak{F}} = \lambda_1(P_1 - P_2 - P_3),$$
$$\underset{2}{\mathfrak{F}} = \lambda_2(P_1 + P_2 - P_3), \ \underset{3}{\mathfrak{F}} = -\lambda_1\lambda_2(P_1 - P_2 + P_3).$$

From which we have

$$(8) \qquad \frac{1}{\lambda_2}\underset{2}{\mathfrak{F}} - \frac{1}{\lambda_1}\underset{1}{\mathfrak{F}} - \frac{1}{\lambda_1\lambda_2}\underset{3}{\mathfrak{F}} = \mathfrak{I},$$

$$\text{i. e.} \quad \frac{1}{\lambda_2}\underset{2}{F^j_i} - \frac{1}{\lambda_1}\underset{1}{F^j_i} - \frac{1}{\lambda_1\lambda_2}\underset{3}{F^j_i} = \delta^j_i.$$

Conversely, if the manifold is endowed with a 4-π-structure and the projection operations from T^C_P to the four subspaces induced in T^C_P by the distributions are denoted as P_1, P_2, P_3 and P_4, then the tensor fields corresponding to the linear transformations $\underset{a}{\mathfrak{F}}$ in (6) define a structure A. In case the manifold is endowed with a 3-π-structure, the tensor fields corresponding to the linear transformation $\underset{a}{\mathfrak{F}}$ in (7) define a structure A for which the relation (8) holds.

Let the tensor associated to the π-structure is denoted as $\underset{1}{F^j_i}$ and the linear transformation induced by F^j_i in T^C_P be denoted as $\mathfrak{F} \equiv \overset{1}{\mathfrak{F}}$. If a 4-$\pi$-structure corresponds to the considered structure A, then we have

$$(9) \qquad \mathfrak{F} = \lambda(P_1 + \omega_1^3 P_2 + \omega_1^2 P_3 + \omega_1 P_4)$$

where λ is a non zero complex number and ω_1 is a fourth power root of unity. If we solve P's from (6) and put them in (9) we have

$$(10) \qquad \mathfrak{F} = \frac{\lambda}{2\lambda_1}(1 + \omega_1^3)\underset{1}{\mathfrak{F}} + \frac{\lambda}{2\lambda_2}(1 + \omega_1^2)\underset{2}{\mathfrak{F}} - \frac{\lambda}{2\lambda_1\lambda_2}\underset{1}{\mathfrak{F}}$$

$$\text{i. e.} \qquad F^j_i = \frac{1}{2\lambda_1}(1 + \omega_1^3)\underset{3}{F^j_i} + \frac{\lambda}{2\lambda_2}(1 + \omega_1^2)\underset{2}{F^j_i} - \frac{\lambda}{2\lambda_1\lambda_2}\underset{3}{F^j_i}.$$

From which we have

$$(11) \qquad \overset{2}{\mathfrak{F}} = \frac{\lambda^2}{2}(1 + \omega_1^2)\mathfrak{I} + \frac{\lambda^2}{2\lambda_2}(1 - \omega_1^2)\underset{2}{\mathfrak{F}},$$

$$\overset{3}{\mathfrak{F}} = \frac{\lambda^2}{2\lambda_1}(1 + \omega_1)\underset{1}{\mathfrak{F}} + \frac{\lambda^2}{2\lambda_2}(1 + \omega_1^2)\underset{2}{\mathfrak{F}} -- \frac{\lambda^3}{2\lambda_1\lambda_2}(1 + \omega_1^3)\underset{3}{\mathfrak{F}},$$

where $\overset{2}{\mathfrak{F}} \equiv \mathfrak{F}^2$ and $\overset{3}{\mathfrak{F}} \equiv \mathfrak{F}^3$ denotes respectively the linear transformation induced by $\overset{2}{F^k_i} \equiv F^j_i F^k_j$ and $\overset{3}{F^l_i} \equiv F^j_i F^k_j F^l_k$. From (10) and (11) we can solve $\underset{1}{\mathfrak{F}}$, $\underset{2}{\mathfrak{F}}$, $\underset{3}{\mathfrak{F}}$ and express them as linear combinations of $\mathfrak{F} = \overset{1}{\mathfrak{F}}$, $\overset{2}{\mathfrak{F}}$ and $\overset{3}{\mathfrak{F}}$.

If a 3-π-structure corresponds to the structure A, then we have

$$(12) \qquad \mathfrak{F} = \lambda(P_1 + \omega_1^2 P_2 + \omega_1 P_3),$$

where λ is any non zero complex number and ω_1 is a cubic power root of unity. Solving P's from (7) and put them in the above expression (12) we have

$$(13)\quad \overset{1}{\mathfrak{F}} = \frac{\lambda}{2\lambda_1}\underset{1}{\mathfrak{F}} - \frac{\lambda\omega_1}{2\lambda_2}\underset{2}{\mathfrak{F}} + \frac{\lambda\omega_1^2}{2\lambda_1\lambda_2}\underset{3}{\mathfrak{F}},$$

$$\text{i. e. } F_i^j = \frac{1}{2\lambda_1}\underset{1}{F_i^j} - \frac{\lambda\omega_1}{2\lambda_2}\underset{2}{F_i^j} + \frac{\lambda\omega_1^2}{2\lambda_1\lambda_2}\underset{3}{F_i^j}.$$

From which we have moreover

$$(14)\quad \overset{2}{\mathfrak{F}} = \frac{\lambda^2}{2\lambda_1}\underset{1}{\mathfrak{F}} - \frac{\lambda^2\omega_1^2}{2\lambda_2}\underset{2}{\mathfrak{F}} + \frac{\lambda^2\omega_1}{2\lambda_1\lambda_2}\underset{3}{\mathfrak{F}},$$

where $\overset{2}{\mathfrak{F}}$ is defined by the same way as above. From (8), (13) and (14) we can solve $\underset{1}{\mathfrak{F}}$, $\underset{2}{\mathfrak{F}}$ and $\underset{3}{\mathfrak{F}}$ expressing them as linear combinations of $\mathfrak{F}$, $\overset{1}{\mathfrak{F}}$ and $\overset{2}{\mathfrak{F}}$.

Now let us consider the relation between the integrability of the structure A and that of corresponding π-structure.

By definition, an r-π-structure defined by r distributions $T(t = 1, \ldots\ldots, r)$ is said to be ***integrable*** if at each point of the manifold there is a complex coordinate system(i. e. n independent complex valued functions of class C^∞ $z^1, \ldots\ldots, z^n$ of local coordinates) such that the subspace $\underset{t}{T}$ is represented as $dz^{\bar{a}_t} = 0$, i. e. $dz^i = 0$ except dz^{a_t} where a_t varies from $n_1 + \ldots\ldots + n_{t-1} + 1$ to $n_1 + \ldots\ldots + n_t$ ($n_t = \dim \underset{t}{T}$) $t = 1, \ldots\ldots, r$ [2] [4]. Then we have

THEOREM 2. *A structure* A *on the differentiable manifold is integrable if and only if the corresponding π-structure is integrable.*

PROOF. Suppose the considered structure A is integrable, then there exists a complex coordinate system in which $\underset{1}{F_i^j}$, $\underset{2}{F_i^j}$ and $\underset{3}{F_i^j}$ have simultaneously numerical components. If the corresponding π-structure is a 4-π-structure, we can obtain a new coordinate system z^i (each z^i is a linear combination of the old ones with constant coefficients) in which the three tensor fields are expressed as follows:

$$(15)\quad (\underset{1}{F_i^j}) = \lambda_1\begin{pmatrix} E_r & & & 0 \\ & E_s & & \\ & & -E_t & \\ 0 & & & -E_u \end{pmatrix},\ (\underset{2}{F_i^j}) = \lambda_2\begin{pmatrix} E_r & & & 0 \\ & -E_s & & \\ & & E_t & \\ 0 & & & -E_u \end{pmatrix},$$

$$(\underset{3}{F_i^j}) = -\lambda_1\lambda_2\begin{pmatrix} E_r & & & 0 \\ & -E_s & & \\ & & -E_t & \\ 0 & & & E_u \end{pmatrix},$$

where E_r denotes the $r \times r$ unit matrix, $r = \dim T_{(1'2')}$, $s = \dim T_{(1'2'')}$, $t = \dim T_{(1''2')}$, $u = \dim T_{(1''2'')}$ and $n_1 = r + s$, $n_2 = t + u$, $n_1 + n_2 = n$. From (10) and (15) we have

$$(16) \qquad (F_i^j) = \lambda \begin{pmatrix} E_r & & & 0 \\ & \omega_1^3 E_s & & \\ & & \omega_1^2 E_t & \\ 0 & & & \omega_1 E_u \end{pmatrix}.$$

From the above expression it is evident that $\frac{\partial}{\partial z^{a_t}}$ form a basis of the subspace $\underset{t}{T}$, i. e. $\underset{t}{T}$ is expressed by $dz^{\bar{a}_t} = 0$. Thus the 4-π-structure is integrable.

Conversely if the correspondi..g 4-π-structure is integrable, in the coordinate system in which $\underset{t}{T}$ is expressed by $dz^{\bar{a}_t} = 0$, we have (16) and consequently $(\underset{1}{F}{}_i^j)$, $(\underset{2}{F}{}_i^j)$, $(\underset{3}{F}{}_i^j)$ have simultaneouly numerical components as these tensors can be expressed as linear combinations of (δ_i^j), (F_i^j), $(\overset{2}{F}{}_i^j)$ and $(\overset{3}{F}{}_i^j)$ with constant coefficients.

If the corrsponding π-structure is a 3-π-sturcture, then there is a coordinate system (z^t) in which

$$(17) \qquad (\underset{1}{F}{}_i^j) = \lambda_1 \begin{pmatrix} E_{n_1} & & 0 \\ & -E_t & \\ 0 & & -E_u \end{pmatrix}, (\underset{2}{F}{}_i^j) = \lambda_2 \begin{pmatrix} E_{n_1} & & 0 \\ & E_t & \\ 0 & & -E_u \end{pmatrix},$$

$$(\underset{3}{F}{}_i^j) = -\lambda_1 \lambda_2 \begin{pmatrix} E_n & & 0 \\ & -E_t & \\ 0 & & E_u \end{pmatrix}.$$

Putting there expressions in (13) we have

$$(18) \qquad (F_i^j) = \lambda \begin{pmatrix} E_{n_1} & & 0 \\ & \omega_1^2 E_t & \\ 0 & & \omega_1 E_u \end{pmatrix}.$$

Then the remaining reasoning is the same as in the case of 4-π-structure. Q. E. D.

It is shown in [4] that if the manifold is analytic and both the real and imaginary parts of F_i^j are real analytic functions of the local coordinates, then the π-structure is integrable if and only if the torsion tensor of the π-strucutre vanishes identically.

For 4-π-structure, the torsion tensor of the π-structure is the following:

$$(19) \qquad t_{jk}^m = \frac{1}{4^2 \lambda^4} \Big\{ -3 \sum_{a=1}^{3} \overset{a}{M}{}_{jk}^{pq} \overset{x}{f}{}_{pq}^m + \frac{1}{\lambda^4} \overset{23}{N}{}_{jk}^{pq} \overset{3}{f}{}_{pq}^m$$

$$+\left(\overset{1}{C}{}^{pq}_{jk} + \frac{1}{\lambda^4}\overset{3}{C}{}^{pq}_{jk}\right)\overset{2}{f}{}^{m}_{pq} + \overset{12}{N}{}^{pq}_{jk}\overset{1}{f}{}^{m}_{pq}\Big\}$$

where

$$(20) \qquad \overset{a}{M}{}^{pq}_{jk} = \delta^p_j \overset{a}{F}{}^q_k + \delta^q_k \overset{a}{F}{}^p_j, \qquad \overset{a}{f}{}^m_{pq} = \partial_p \overset{a}{F}{}^m_q - \partial_q \overset{a}{F}{}^m_p,$$

$$\overset{ab}{N}{}^{pq}_{jk} = \overset{a}{F}{}^q_j \overset{b}{F}{}^q_k + \overset{b}{F}{}^p_j \overset{a}{F}{}^q_k, \qquad \overset{a}{C}{}^{pq}_{jk} = \overset{a}{F}{}^p_j \overset{a}{F}{}^q_k.$$

Putting (10) and (11) in (19), we have by simple calculation the following:

$$(21) \qquad t^m_{jk} = \frac{-1}{4^2}\left\{\frac{3}{2\lambda_1^2}(1-\omega_1^2)\underset{1}{M}{}^{pq}_{jk}\underset{1}{f}{}^m_{pq} + \frac{1}{2\lambda_2^2}(7+\omega_1^2)\underset{2}{M}{}^{pq}_{jk}\underset{2}{f}{}^m_{pq}\right.$$

$$+\frac{3}{2\lambda_1^2\lambda_2^2}(1-\omega_1^2)\underset{3}{M}{}^{pq}_{jk}\underset{3}{f}{}^m_{pq} + \frac{1}{2\lambda_1^2\lambda_2^2}(1-\omega_1^2)$$

$$\left.(\underset{12}{N}{}^{pq}_{jk}\underset{3}{f}{}^m_{pq} + \underset{23}{N}{}^{pq}_{jk}\underset{1}{f}{}^m_{pq} + \underset{13}{N}{}^{pq}_{jk}\underset{2}{f}{}^m_{pq})\right\},$$

where

$$\underset{a}{f}{}^m_{pq} = \partial_p \underset{a}{F}{}^m_q - \partial_q \underset{a}{F}{}^m_p,$$

$$(22) \qquad \underset{a}{M}{}^{pq}_{jk} = \delta^p_j \underset{a}{F}{}^q_k + \delta^q_k \underset{a}{F}{}^p_j, \qquad \underset{ab}{N}{}^{pq}_{jk} = \underset{b}{F}{}^p_j \underset{a}{F}{}^q_k + \underset{a}{F}{}^p_j \underset{b}{F}{}^q_k.$$

It is evident that the Nijenhuis tensor $N^m_{jk}(\underset{a}{F})$ of $\underset{a}{F}{}^i_j$ is a constant multiple of $\underset{a}{M}{}^{pq}_{jk}\underset{a}{f}{}^m_{pq}$. Since t^m_{jk} is a tensor, it follows that the following $M^m_{jk}(\underset{1}{F}, \underset{2}{F}, \underset{3}{F})$ is also a tensor:

$$M^m_{jk}(\underset{1}{F}, \underset{2}{F}, \underset{3}{F}) \equiv \underset{12}{N}{}^{pq}_{jk}\underset{3}{f}{}^m_{pq} + \underset{23}{N}{}^{pq}_{jk}\underset{1}{f}{}^m_{pq} + \underset{13}{N}{}^{pq}_{jk}\underset{2}{f}{}^m_{pq}$$

$$= (\underset{2}{F}{}^p_j \underset{3}{F}{}^q_k + \underset{3}{F}{}^p_j \underset{2}{F}{}^q_k)(\partial_p \underset{1}{F}{}^m_q - \partial_q \underset{1}{F}{}^m_p)$$

$$+ (\underset{1}{F}{}^p_j \underset{3}{F}{}^q_k + \underset{3}{F}{}^p_j \underset{1}{F}{}^q_k)(\partial_p \underset{2}{F}{}^m_q - \partial_q \underset{2}{F}{}^m_p)$$

$$+ (\underset{1}{F}{}^p_j \underset{2}{F}{}^q_k + \underset{2}{F}{}^p_j \underset{1}{F}{}^q_k)(\partial_p \underset{3}{F}{}^m_q - \partial_q \underset{3}{F}{}^m_p).$$

For the 3-π-structure the torsion tensor of the π-structure is as follows:

$$(24) \qquad t^m_{jk} = \frac{1}{3^2\lambda^3}\left\{-2\sum_{a=1}^{2}\overset{a}{M}{}^{pq}_{jk}\overset{a}{f}{}^m_{pq} + \frac{1}{\lambda^3}\overset{2}{C}{}^{pq}_{jk}\overset{2}{f}{}^m_{pq} + \overset{1}{C}{}^{pq}_{jk}\overset{1}{f}{}^m_{pq}\right\}.$$

Before transforming this formula, we first obtain some relations to be used later:

From (8) we have

(25) $$\frac{1}{\lambda_2}\underset{2}{f}{}^{m}_{pq} - \frac{1}{\lambda_1}\underset{1}{f}{}^{m}_{pq} - \frac{1}{\lambda_1\lambda_2}\underset{3}{f}{}^{m}_{pq} = 0$$

and

(26) $$I^{pq}_{jk} = \frac{1}{\lambda_1^2}\underset{1}{C}{}^{pq}_{jk} + \frac{1}{\lambda_2^2}\underset{2}{C}{}^{pq}_{jk} + \frac{1}{\lambda_1^2\lambda_2^2}\underset{3}{C}{}^{pq}_{jk}$$
$$- \frac{1}{\lambda_1\lambda_2}\underset{12}{N}{}^{pq}_{jk} + \frac{1}{\lambda_1^2\lambda_2}\underset{13}{N}{}^{pq}_{jk} - \frac{1}{\lambda_1\lambda_2^2}\underset{23}{N}{}^{pq}_{jk},$$

in which we have put

(27) $$I^{pq}_{jk} = \delta^p_j\,\delta^q_k, \qquad \underset{a}{C}{}^{pq}_{jk} = \underset{a}{F}{}^p_j\,\underset{a}{F}{}^q_k.$$

We can also reduce the following formula from (8).

(28) $$2\,I^{pq}_{jk} = \frac{1}{\lambda_2}\underset{2}{M}{}^{pq}_{jk} - \frac{1}{\lambda_1}\underset{1}{M}{}^{pq}_{jk} - \frac{1}{\lambda_1\lambda_2}\underset{3}{M}{}^{pq}_{jk}.$$

On the other hand from the definition (22) we have

(29) $$\underset{1}{M}{}^{jk}_{j_1k_1}\underset{1}{M}{}^{pq}_{jk} = 2\,\lambda_1^2\,I^{pq}_{j_1k_1} + 2\,\underset{1}{C}{}^{pq}_{j_1k_1},$$

$$\underset{2}{M}{}^{jk}_{j_1k_1}\underset{2}{M}{}^{pq}_{jk} = 2\,\lambda_2^2\,I^{pq}_{j_1k_1} + 2\,\underset{2}{C}{}^{pq}_{j_1k_1},\ \underset{3}{M}{}^{jk}_{j_1k_1}\underset{3}{M}{}^{pq}_{jk} = 2\,\lambda_1^2\,\lambda_2^2\,I^{pq}_{j_1k_1} + 2\,\underset{3}{C}{}^{pq}_{j_1k_1},$$

and

(30) $$\begin{aligned}
\underset{1}{M}{}^{jk}_{j_1k_1}\underset{2}{M}{}^{pq}_{jk} &= \underset{2}{M}{}^{jk}_{j_1k_1}\underset{1}{M}{}^{pq}_{jk} = -\underset{3}{M}{}^{pq}_{j_1k_1} + \underset{12}{N}{}^{pq}_{j_1k_1},\\
\underset{1}{M}{}^{jk}_{j_1k_1}\underset{3}{M}{}^{pq}_{jk} &= \underset{3}{M}{}^{jk}_{j_1k_1}\underset{1}{M}{}^{pq}_{jk} = -\,\lambda_1^2\,\underset{2}{M}{}^{pq}_{j_1k_1} + \underset{13}{N}{}^{pq}_{j_1k_1},\\
\underset{3}{M}{}^{jk}_{j_1k_1}\underset{2}{M}{}^{pq}_{jk} &= \underset{2}{M}{}^{jk}_{j_1k_1}\underset{3}{M}{}^{pq}_{jk} = -\,\lambda_2^2\,\underset{1}{M}{}^{pq}_{j_1k_1} + \underset{23}{N}{}^{pq}_{j_1k_1}.
\end{aligned}$$

Multiplying (28) by $\underset{1}{M}{}^{jk}_{j_1k_1}$, and sum up with respect to j,k we have

(31) $$2\,\underset{1}{M}{}^{pq}_{j_1k_1} = \frac{1}{\lambda_2}(-\underset{3}{M}{}^{pq}_{j_1k_1} + \underset{12}{N}{}^{pq}_{j_1k_1}) - \frac{1}{\lambda_1}(2\,\lambda_1^2\,I^{pq}_{j_1k_1} + 2\,\underset{1}{C}{}^{pq}_{j_1k_1})$$
$$- \frac{1}{\lambda_1\lambda_2}(-\,\lambda_1^2\,\underset{2}{M}{}^{pq}_{j_1k_1} + \underset{13}{N}{}^{pq}_{j_1k_1}).$$

Similarly

(32) $$2\,\underset{2}{M}{}^{pq}_{j_1k_1} = \frac{1}{\lambda_2}(2\,\lambda_2^2\,I^{pq}_{j_1k_1} + 2\,\underset{2}{C}{}^{pq}_{j_1k_1}) - \frac{1}{\lambda_1}(-\underset{3}{M}{}^{pq}_{j_1k_1} + \underset{12}{N}{}^{pq}_{j_1k_1})$$
$$- \frac{1}{\lambda_1\lambda_2}(-\,\lambda_2^2\,\underset{1}{M}{}^{pq}_{j_1k_1} + \underset{23}{N}{}^{pq}_{j_1k_1}),$$

$$2\underset{3}{M}{}^{pq}_{j_1k_1} = \frac{1}{\lambda_2}\left(-\lambda_2^2\underset{1}{M}{}^{pq}_{j_1k_1} + \underset{23}{N}{}^{pq}_{j_1k_1}\right) - \frac{1}{\lambda_1}\left(-\lambda_1^2\underset{2}{M}{}^{pq}_{j_1k_1} + \underset{13}{N}{}^{pq}_{j_1k_1}\right)$$

$$-\frac{1}{\lambda_1\lambda_2}\left(2\lambda_1^2\lambda_2^2 I^{pq}_{j_1k_1} + 2\underset{3}{C}{}^{pq}_{j_1k_1}\right).$$

Substitute (13) and (14) in (24), then make use of the above relations (25), (26), (31) and (32), we get

$$t^m_{jk} = -\frac{1}{16}\left\{3\left(\frac{1}{\lambda_1^2}\underset{1}{M}{}^{pq}_{jk}\underset{1}{f}{}^m_{pq} + \frac{1}{\lambda_2^2}\underset{2}{M}{}^{pq}_{jk}\underset{2}{f}{}^m_{pq} + \frac{1}{\lambda_1^2\lambda_2^2}\underset{3}{M}{}^{pq}_{jk}\underset{3}{f}{}^m_{pq}\right)\right.$$

$$\left.+\frac{1}{\lambda_1^2\lambda_2^2}\left(\underset{12}{N}{}^{pq}_{jk}\underset{3}{f}{}^m_{pq} + \underset{13}{N}{}^{pq}_{jk}\underset{2}{f}{}^m_{pq} + \underset{23}{N}{}^{pq}_{jk}\underset{1}{f}{}^m_{pq}\right)\right\}.$$

From the above preparation, we have the following:

THEOREM 3. *Assume that the manifold is of class* C^ω *and both the real and imaginary parts of each of the tensors* $\underset{1}{F}{}^j_i$, $\underset{2}{F}{}^j_i$, $\underset{3}{F}{}^j_i$ *of the structure* A *are analytic functions of the local coordinates. Then the structure* A *is integrable if and only if all the Nijenhuis tensors* $N^m_{jk}(\underset{1}{F})$, $N^m_{jk}(\underset{2}{F})$, $N^m_{jk}(\underset{3}{F})$ *and the tensor* $M^m_{jk}(\underset{1}{F}, \underset{2}{F}, \underset{3}{F})$ *vanish identically.*

PROOF. From (10) and (13) it follows that both the real and imaginary part of the tensor F^j_i associated to the π-structure corresponding to the considered structure A are also analytic functions of the local coordinates.

If the structure A is integrable, then the corresponding π-structure is integrable, so all of the Nijenhuis tensors $N^m_{jk}(\underset{1}{F})$, $N^m_{jk}(\underset{2}{F})$, $N^m_{jk}(\underset{3}{F})$, and the torsion tensor of the corresponding π-structure vanish identically. Hence from (21) and (23) it follows that $M^m_{jk}(\underset{1}{F}, \underset{2}{F}, \underset{3}{F})$ must also vanish identically Conversely, if all the Nijenhuis tensors $N^m_{jk}(\underset{1}{F})$, $N^m_{jk}(\underset{2}{F})$, $N^m_{jk}(\underset{3}{F})$ and the tensor $M^m_{jk}(\underset{1}{F}, \underset{2}{F}, \underset{3}{F})$ vanish identically, then the torsion tensor of the corresponding π-structure (21) or (33) vanishes identically, so the π-structure and hence also the structure A is integrable.

2. In this section we digress to the $(\underset{1}{F}, \underset{2}{F})$-connection of the manifold having structure A. By definition a $(\underset{1}{F}, \underset{2}{F})$-connection of the manifold with a structure A (or B) is a linear connection which makes all the tensors $\underset{1}{F}{}^j_i$, $\underset{2}{F}{}^j_i$ and $\underset{3}{F}{}^j_i$ covariant constant [1]. A linear connection on a manifold with π-structure is called a π-connection if the connection makes the tensor F^j_i associated

to the π-structure covariant constant [2] [4]. Then from (10), (11), (13), (14) and the fact that $\underset{1}{\mathfrak{F}}, \underset{2}{\mathfrak{F}}, \underset{3}{\mathfrak{F}}$ can also be expressed as the linear combinations with constant coefficients of $\mathfrak{F}$ and $\overset{a}{\mathfrak{F}}$'s ($a = 1, 2, 3$ for 4-π-structure, whereas $a = 1, 2$ for 3-π-structure), it follows that a linear connection on the manifold with the structure A is a $(\underset{1}{F}, \underset{2}{F})$-connection if and only if the connection is the π-connection with respect to the corresponding π-structure. On the other hand it is shown that on the manifold with a π-structure, there exists a connection having the torsion tensor of the π-structure as its torsion tensor [4]. Thus we have

THEOREM 4. *On the manifold with a structure* A, *there exists a* $(\underset{1}{F},\underset{2}{F})$-*connection which is symmetric if the structure is integrable.*

3. Finally we consider a manifold with a structur B. For this case, we have in place of theorem 1 the following:

THEOREM 5. *If the manifold has a structure* B, *then it is of even dimensional* ($n = 2m$) *and there exist two complementary distributions of* m *dimensional subspaces* T', T'' (i.e. $T^C_P = T'_P + T''_P$: *direct sum*) *and a system of isomorphisms* S *of class* C^∞: S_P: $T'_P \to T''_P$. *The converse also holds good.*

PROOF. Using the notations as in theorem 1, $\underset{1}{\mathfrak{F}}$ has the following form with respect to the adapted basis in T^C_P:

$$\underset{1}{\mathfrak{F}} = \begin{pmatrix} \lambda_1 E_{n_1} & 0 \\ 0 & -\lambda_1 E_{n_2} \end{pmatrix}. \tag{34}$$

By B_3 it follows that $\underset{2}{\mathfrak{F}}$ is then represented as follows:

$$\underset{2}{\mathfrak{F}} = \begin{pmatrix} 0 & \underset{2}{F}{}^{\alpha}_{\beta^*} \\ \underset{2}{F}{}^{\alpha^*}_{\beta} & 0 \end{pmatrix}, \tag{35}$$

where $\alpha, \beta = 1, \ldots\ldots, n_1$; $\alpha^*, \beta^* = n_1 + 1, \ldots\ldots, n_1 + n_2 = n$. Since $\underset{2}{\mathfrak{F}}$ is non singular, from (35) we have $n_1 = n_2 \equiv m$.

Now let $v \in T_{1'}$, then $\underset{1}{\mathfrak{F}}v = \lambda_1 v$, hence $\underset{1}{\mathfrak{F}}\underset{2}{\mathfrak{F}}v = -\underset{2}{\mathfrak{F}}\underset{1}{\mathfrak{F}}v = -\underset{2}{\mathfrak{F}}(\lambda_1 v) = -\lambda_1 \underset{2}{\mathfrak{F}}v$, that is $\underset{2}{\mathfrak{F}}v \in T_{1''}$. Since $\underset{2}{\mathfrak{F}}$ is non singular and dim $T_{1'} =$ dim $T_{1''}$, it follows that $\underset{2}{\mathfrak{F}}$ is an isomorphism from $T_{1'}$ onto $T_{1''}$.

Conversely, assume that the manifold is of even dimensional ($n = 2m$) and that there exist two complementary distributions of m dimensional subspaces $T_{1'}$, $T_{1''}$ and a field of differentiable isomorphisms S: S_P: $T_{1'P} \to T_{1''P}$. Denote the projection operations from T_P^C to $T_{1'P}$ and $T_{1''P}$ respectively as P_1 and P_2. Then define

$$\underset{1}{\mathfrak{F}}\, v = \lambda_1(P_1 v - P_2 v), \tag{36}$$
$$\underset{2}{\mathfrak{F}}\, v = \lambda_2 S P_1 v - \lambda_2 S^{-1} P_2 v,$$

where $v \in T_P^C$; λ_1 and λ_2 be any two fixed non zero complex numbers. Then we have

$$\underset{1}{\mathfrak{F}}^2 v = \lambda_1^2 v. \tag{37}$$

Since $SP_1\, v \in T_{1''}$, $S^{-1}P_2 v \in T_{1'}$, it follows from (36) that

$$P_2 \underset{2}{\mathfrak{F}} v = \lambda_2 S P_1 v,\ P_1 \underset{2}{\mathfrak{F}} v = -\lambda_2 S^{-1} P_2 v. \tag{38}$$

Hence

$$\underset{2}{\mathfrak{F}}(\underset{2}{\mathfrak{F}} v) = \lambda_2 S P_1(\underset{2}{\mathfrak{F}} v) - \lambda_2 S^{-1} P_2(\underset{2}{\mathfrak{F}} v) = -\lambda_2^2 P_2 v - \lambda_2^2 P_1 v,$$

that is

$$\underset{2}{\mathfrak{F}}^2 v = -\lambda_2^2 v \tag{39}$$

Moreover, since

$$\underset{2}{\mathfrak{F}} P_1 v = \lambda_2 S P_1 v = P_2 \underset{2}{\mathfrak{F}} v \in T_{1''},\ \underset{2}{\mathfrak{F}} P_2 v = -\lambda_2 S^{-1} P_2 v = P_1 \underset{2}{\mathfrak{F}} v \in T_{1'},$$

we have

$$\underset{1}{\mathfrak{F}}\underset{2}{\mathfrak{F}} v = \underset{1}{\mathfrak{F}}(\underset{2}{\mathfrak{F}} P_1 v + \underset{2}{\mathfrak{F}} P_2 v) = \lambda_1 \underset{2}{\mathfrak{F}} P_2 v - \lambda_1 \underset{2}{\mathfrak{F}} P_1 v,$$
$$\underset{2}{\mathfrak{F}}\underset{1}{\mathfrak{F}} v = \underset{2}{\mathfrak{F}}(\lambda_1 P_1 v - \lambda_1 P_2 v) = \lambda_1 \underset{2}{\mathfrak{F}} P_1 v - \lambda_1 \underset{2}{\mathfrak{F}} P_2 v,$$

therefore, we get

$$\underset{1}{\mathfrak{F}}\underset{2}{\mathfrak{F}} v = -\underset{2}{\mathfrak{F}}\underset{1}{\mathfrak{F}} v. \tag{40}$$

If we put

$$\underset{1}{\mathfrak{F}}\underset{2}{\mathfrak{F}} = -\underset{2}{\mathfrak{F}}\underset{1}{\mathfrak{F}} \equiv \underset{3}{\mathfrak{F}}, \tag{41}$$

then we have

$$\underset{2}{\mathfrak{F}}\underset{3}{\mathfrak{F}} = -\underset{3}{\mathfrak{F}}\underset{2}{\mathfrak{F}} = \lambda_2^2 \underset{1}{\mathfrak{F}},\ \underset{3}{\mathfrak{F}}\underset{1}{\mathfrak{F}} = -\underset{1}{\mathfrak{F}}\underset{3}{\mathfrak{F}} = -\lambda_1^2 \underset{2}{\mathfrak{F}}. \tag{42}$$

Q. E. D.

Let the proper subspaces corresponding to the proper values $i\lambda_2$ and $-i\lambda_2$ of $\underset{2}{\mathfrak{F}}$ be denoted respectively as $T_{2'}$ and $T_{2''}$, then $\underset{1}{\mathfrak{F}}$ restricted to $T_{2'}$ is an isomorphism between $T_{2'}$ and $T_{2''}$. Because, $\underset{1}{\mathfrak{F}}$ is non singular and if $u \in T_{2'}$, we have $\underset{2}{\mathfrak{F}} u = i\lambda_2 u$ and $\underset{2}{\mathfrak{F}}(\underset{1}{\mathfrak{F}} u) = -\underset{1}{\mathfrak{F}}(\underset{2}{\mathfrak{F}} u) = -\underset{1}{\mathfrak{F}}(i\lambda_2 u) = -i\lambda_2 \underset{1}{\mathfrak{F}} u$, thus $\underset{1}{\mathfrak{F}} u$

$\in T_{2''}$. Moreover, any two of $T_{1'}$, $T_{1''}$, $T_{2'}$, $T_{2''}$ are complementary to each other. For if $v \in T_{1'}$, it follows $P_2 v = 0$, $P_1 v = v$ and $\underset{2}{\mathfrak{F}} v \in T_{1''}$. If $v \in T_{2'}$, also holds, then $\underset{2}{\mathfrak{F}} v = i\lambda_2 v$, hence $v \in T_{1''}$ and consequently $v = 0$.

From the above, it is evident that the results of π-structure can not be applied to the structure B. For this case quite similar reasoning as in the case of the integrability of quaternion structure treated by Obata [1] can be applied and one can get an analoguous theorem to the Theorem 5. 1 of Obata's paper. We do not go in detail in this matter.

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