249. Note on the Representation of Semi-Groups of Non-Linear Operators

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(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1966)

1. Let X be a Banach space and let $\{T(\xi)\}_{\xi\geqq 0}$ be a family of non-linear operators from X into itself satisfying the following conditions:

(1) $T(0)=I,\ T(\xi)T(\eta)=T(\xi+\eta) \qquad \xi,\eta\geqq 0,$

(2) $\|T(\xi)x-T(\xi)y\|\leqq\|x-y\| \qquad \xi>0,\ x,y\in X,$

(3) There exists a dense subset D in X such that for each $x\in D$, the right derivative

$$D_\xi^+T(\xi)x=\lim_{h\to 0+}h^{-1}(T(\xi+h)x-T(\xi)x)$$

exists and it is continuous for $\xi\geqq 0$. Then we shall call this family $\{T(\xi)\}_{\xi\geqq 0}$ a *non-linear contraction semi-group.*

Definition. *We define the infinitesimal generator A of a non-linear contraction semi-group $\{T(\xi)\}_{\xi\geqq 0}$ by*

$$Ax=\lim_{h\to 0+}A_hx$$

whenever the limit exists, where $A_h=h^{-1}(T(h)-I)$. We denote the domain of A by $D(A)$.

Lately J. W. Neuberger [1] gave the following result: If $\{T(\xi)\}_{\xi\geqq 0}$ is a non-linear contraction semi-group,*) then for each $x\in X$ and each $\xi\geqq 0$

$$\lim_{n\to\infty}\limsup_{\delta\to 0+}\|(I-(\xi/n)A_\delta)^{-n}x-T(\xi)x\|=0.$$

It is well known that if $\{T(\xi)\}_{\xi\geqq 0}$ is a linear contraction semi-group of class (C_0), then for each $x\in X$ and each $\xi\geqq 0$

$$\lim_{n\to\infty}(I-(\xi/n)A)^{-n}x=T(\xi)x$$

(see [2]). In this paper we shall give the representation of this type for non-linear contraction semi-groups.

The main results are the follwing

Theorem. *Let $\{T(\xi)\}_{\xi\geqq 0}$ be a non-linear contraction semi-group and let A be the infinitesimal generator such that $\overline{\mathfrak{R}(I-\xi_0A)}=X$ for some $\xi_0>0$. Then for each $\xi>0$ there exists an inverse operator $(I-\xi A)^{-1}$ and its unique extension $L(\xi)$ onto X, which is a contraction operator, and $T(\xi)$ is represented by*

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*) In his paper the following condition is assumed:

(3)′ There is a dense subset D of X such that if x is in D, then the derivative $T'(\xi)x$ is continuous with domain $[0,\infty)$.

$$\lim_{n\to\infty} L(\xi/n)^n x = T(\xi)x \qquad \xi \geqq 0,\ x \in X,$$

where for each fixed $x \in X$ *the convergence is uniform for any compact set in* $[0, \infty)$ *and for each fixed* $\xi \geqq 0$ *it is the continuous convergence on* X. *Moreover, there exists a unique mapping* $\tilde{A}$, *which is not necessarily one-valued, defined on a region* $\tilde{D} \supset D(A)$ *such that*

(1) *the mapping* $\tilde{D} \ni x \to x - \xi \tilde{A}x$ *is the topological inverse mapping of* $L(\xi)$,

(2) $\tilde{A}x \ni Ax$ *for each* $x \in D(A)$,

(3) *for any* $x \in \tilde{D}$ *there exists a sequence* $\{x_n\} \subset D(A)$ *such that* $\lim x_n = x$ *and* $\lim_{n\to\infty} Ax_n \in \tilde{A}x$.

Corollary 1. *If* $\tilde{A}$ *is one-valued, then in the above Theorem* $L(\xi) = (I - \xi\tilde{A})^{-1}$ *and* $\tilde{A}$ *is the closure of* A *in the sense that the graph* $G(\tilde{A})$ *of* $\tilde{A}$ *is the closure of the graph* $G(A)$ *in* $X \times X$.

Corollary 2. *If* $\Re(I - \xi_0 A) = X$ *for some* $\xi_0 > 0$, *then* $\tilde{A} = A$ *in the above Corollary* 1.

2. We shall prove the theorems mentioned above by the following successive lemmas:

Lemma 1. $D(A) \supset D$, $D(A) \supset T(\xi)[D]$ *for each* $\xi \geqq 0$. *And the left derivative also exists, and is equal to the right one and*

$$\frac{d}{d\xi} T(\xi)x = AT(\xi)x$$

on $(0, \infty)$ *for each* $x \in D$.

Proof. The first relations of inclusion follow immediately from the condition (3). It follows from

$$\| T(\xi \pm h)x - T(\xi)x \| \leqq \| T(h)x - x \| \qquad (x \in D)$$

and the denseness of D that for any $x \in X$, $T(\xi)x$ is strongly continuous on $[0, \infty)$. Therefore by the same argument as in the linear case we get the above conclusions (see [3]; p. 239). Q.E.D.

Under the conditions (1)–(3) and by virtue of Lemma 1, we can apply the Neuberger's results and get the following

Lemma 2. *For each* $\xi > 0$ *and* $\delta > 0$, $(I - \xi A_\delta)^{-1}$ *exists on* X *and is a contraction operator in the sense that*

$$\|(I - \xi A_\delta)^{-1}x - (I - \xi A_\delta)^{-1}y\| \leqq \| x - y \| \qquad x, y \in X.$$

Lemma 3. *For each* $\xi > 0$, $(I - \xi A)^{-1}$ *exists on* $\Re(I - \xi A)$ *and contraction operator there. And if* $\Re(I - \xi A)$ *is dense in* X, *then the family* $\{(I - \xi A_\delta)^{-1}\}_{\delta > 0}$ *of contraction operators converges to some contraction operator* $L(\xi)$ *defined on* X *onto some region* $\tilde{D}_\xi \supset D(A)$. *This* $L(\xi)$ *is a unique extension of* $(I - \xi A)^{-1}$.

Proof. Let $\tau(x, y)$ be defined by $\lim_{a\to 0+} a^{-1}\{\| x + ay \| - \| x \|\}$. This always exists for each $x, y \in X$ and has the following properties [4]:

(i) $|\tau(x, y)| \leqq \| y \|$,

(ii) $\tau(x, y+z) \leqq \tau(x, y)+\tau(x, z)$,

(iii) $\tau(x, \lambda x+cy)=\Re_e(\lambda)\| x \|+c\tau(x, y) \qquad (c \geqq 0)$.

Using these properties, for any $u, v \in D(A)$ and $\delta>0$ we have

$$\tau(u-v, A_\delta u-A_\delta v)=\tau\left(u-v, \frac{T(\delta)u-T(\delta)v}{\delta}-\frac{u-v}{\delta}\right)$$
$$\leqq \tau(u-v, \delta^{-1}(T(\delta)u-T(\delta)v)-\delta^{-1}\| u-v \|$$
$$\leqq \delta^{-1}\{\| T(\delta)u-T(\delta)v \|-\| u-v \|\} \leqq 0.$$

From the continuity of $\tau(u-v, \cdot)$ we have $\tau(u-v, Au-Av) \leqq 0$ for each $u, v \in D(A)$. Thus we have again from (i), (ii), and (iii) the following estimate for any $u, v \in D(A)$:

$$\|(I-\xi A)u-(I-\xi A)v \| \geqq \tau(u-v, (u-v)-\xi(Au-Av))$$
$$\geqq \| u-v \|-\xi\tau(u-v, Au-Av) \geqq \| u-v \|,$$

which implies the first assertion. For any $x \in \Re(I-\xi A)$ we have, from Lemma 2,

$$\| (I-\xi A_\delta)^{-1}x-(I-\xi A)^{-1}x \|$$
$$\leqq \| (I-\xi A)(I-\xi A)^{-1}x-(I-\xi A_\delta)(I-\xi A)^{-1}x \|$$
$$=\xi \| A_\delta(I-\xi A)^{-1}x-A(I-\xi A)^{-1}x \| \to 0 \text{ as } \delta \to 0.$$

Thus we have

$$\lim_{\delta \to 0+} (I-\xi A_\delta)^{-1}x=(I-\xi A)^{-1}x \tag{*}$$

for any $x \in \Re(I-\xi A)$. On the other hand, each $(I-\xi A_\delta)^{-1}$ is a contraction operater defined on X from Lemma 2, and so, combining with (*) and the denseness of $\Re(I-\xi A)$, it follows that the family $\{(I-\xi A_\delta)^{-1}\}_{\delta>0}$ converges to some contraction operator $L(\xi)$ defined on X and that this $L(\xi)$ is the unique extension of $(I-\xi A)^{-1}$. Q.E.D.

Lemma 4. *If* $\overline{\Re(I-\xi_0 A)}=X$ *for some* $\xi_0>0$, *then* $\overline{\Re(I-\xi A)}=X$ *for any* $\xi>0$. *And if* $\Re(I-\xi_0 A)=X$ *for some* $\xi_0>0$, *then* $\Re(I-\xi A)=X$ *for any* $\xi>0$.

Proof. Since $\overline{\Re(I-\xi_0 A)}=X$, from Lemma 3 there exists a unique extension $L(\xi_0)$ of $(I-\xi_0 A)^{-1}$, which is also a contraction. Changing $I-\xi A$ to the form

$$I-\xi A=\frac{\xi}{\xi_0}\left[I-\left(1-\frac{\xi_0}{\xi}\right)L(\xi_0)\right](I-\xi_0 A);$$

for any $x \in X$, we put $Ky=x+(1-(\xi_0/\xi))L(\xi_0)y$ for each $y \in X$. Then K becomes a contraction mapping for ξ with $(\xi_0/2)<\xi$, since $\| Ky-Ky' \| \leqq | 1-(\xi_0/\xi) | \cdot \| y-y' \|$. Thus there exists a unique fixed point z of K; $Kz=z$, and so we have

$$x=z-(1-(\xi_0/\xi))L(\xi_0)z=[1-(1-(\xi_0/\xi))L(\xi_0)]z.$$

Since $\overline{\Re(I-\xi_0 A)}=X$, there exists a sequence $\{x_n\} \subset \Re(I-\xi_0 A)$ such that $\lim_{n\to\infty} x_n=z$. Putting $y_n=(I-\xi_0 A)^{-1}x_n$, we have

$$\frac{\xi}{\xi_0}\left[I-\left(1-\frac{\xi_0}{\xi}\right)L(\xi_0)\right]x_n=\frac{\xi}{\xi_0}\left[I-\left(1-\frac{\xi_0}{\xi}\right)L(\xi_0)\right](I-\xi_0 A)y_n=(I-\xi A)y_n,$$

where the left hand side tends to $(\xi/\xi_0)x$ as $n\to\infty$. Therefore it follows that $\overline{\Re(I-\xi A)}=X$ for all $\xi>(\xi_0/2)$. Thus in particular $\overline{\Re(I-(2\xi_0/3)A)}=X$. Again change the $I-\xi A$ to the form

$$I-\xi A=(3\xi/2\xi_0)[I-(1-(2\xi_0/3\xi))L(2\xi_0/3)](I-(2\xi_0/3)A).$$

For any $x\in X$, putting $K_1y=x+(1-(2\xi_0/3\xi))L(2\xi_0/3)y$ for each $y\in X$, K_1 becomes a contraction mapping for ξ with $(\xi_0/3)<\xi$. In the similar way as in the abovementioned we have $\overline{\Re(I-\xi A)}=X$ for $\xi>(\xi_0/3)$. Inductively we can prove $\overline{\Re(I-(\xi_0/k)A)}=X$ $(k=3,4,5,\cdots)$ and thus we have $\overline{\Re(I-\xi A)}=X$ for $\xi>0$. The last assertion is now evident. Q.E.D.

By virtue of this Lemma 4, we assume in the following Lemmas that $\overline{\Re(I-\xi_0A)}=X$ for some $\xi_0>0$, which insures the existence of the limit operator $L(\xi)$ for each $\xi>0$ (by Lemma 3).

Lemma 5. *The relation*

$$L(\xi)\left[\frac{\xi}{\xi'}y+\frac{\xi'-\xi}{\xi'}L(\xi')y\right]=L(\xi')y$$

holds for any $y\in X$ and $\xi,\xi'>0$. And $L(\xi)[X]=L(\xi')[X]$ for any $\xi,\xi'>0$. In particular, $\tilde{D}_\xi$ of Lemma 3 is independent of $\xi>0$.

Proof. For any $\delta>0$, $\xi,\xi'>0$ and $y\in X$, we have

$$(I-\xi'A_\delta)^{-1}y=(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}(I-\xi'A_\delta)^{-1}y+\frac{\xi}{\xi'}y\right]$$

and thus

$$\left\|L(\xi')y-L(\xi)\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]\right\|\leqq\|L(\xi')y-(I-\xi'A_\delta)^{-1}y\|$$
$$+\left\|(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}(I-\xi'A_\delta)^{-1}y+\frac{\xi}{\xi'}y\right]-(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]\right\|$$
$$+\left\|(I-\xi A_\delta)^{-1}\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]-L(\xi)\left[\frac{\xi'-\xi}{\xi'}L(\xi')y+\frac{\xi}{\xi'}y\right]\right\|.$$

Passing to the limit as $\delta\to0$, we have the required relation for each $y\in X$. From this it follows that $L(\xi')[X]\subset L(\xi)[X]$ for any $\xi,\xi'>0$ and thus we have $L(\xi')[X]=L(\xi)[X]$. The last assertion is now evident. Q.E.D.

By virtue of this Lemma, we denote the set $L(\xi)[X]=\tilde{D}_\xi$, independent of $\xi>0$, by $\tilde{D}$.

Lemma 6. *For any $\xi,\xi'>0$ we have the relation of inclusion: $\frac{1}{\xi}(x-L(\xi)^{-1}x)=\frac{1}{\xi'}(x-L(\xi')^{-1}x)\subset X$, $x\in\tilde{D}$, where $L(\xi)^{-1}$ is the topological inverse mapping of $L(\xi)$.*

Proof. It suffices to prove that for any $x\in\tilde{D}$, $\xi,\xi'>0$

$$\xi^{-1}(x-L(\xi)^{-1}x)\supseteqq\xi'^{-1}(x-L(\xi')^{-1}x).$$

From Lemma 5, $L(\xi)^{-1}L(\xi)\left[\frac{\xi}{\xi'}y+\frac{\xi'-\xi}{\xi'}L(\xi')y\right]=L(\xi)^{-1}L(\xi')y$. Thus

$L(\xi)^{-1}L(\xi')y \ni \frac{\xi}{\xi'}y+\frac{\xi'-\xi}{\xi'}L(\xi')y$ for each $y \in X$. Therefore we have $L(\xi')y-y \in (\xi'/\xi)L(\xi')y-(\xi'/\xi)L(\xi)^{-1}L(\xi')y$ for each $y \in X$. And for any $u \in L(\xi')^{-1}x$ we have

$$\xi^{-1}(x-L(\xi)^{-1}x)=\xi^{-1}(L(\xi')u-L(\xi)^{-1}L(\xi')u)$$
$$=\xi'^{-1}((\xi'/\xi)L(\xi')u-(\xi'/\xi)L(\xi)^{-1}L(\xi')u).$$

From this and the abovementioned it follows that the above right hand side contains the element $\xi'^{-1}(L(\xi')u-u)$, which implies the required relation of inclusion. Q.E.D.

Lemma 7. *The not necessarily one valued mapping $\tilde{A}$ is defined on $\tilde{D}$ by*

$$\tilde{A}x=\xi^{-1}(x-L(\xi)^{-1}x)\subset X \qquad x\in\tilde{D},$$

which has the properties (1)-(3) *mentioned in the main theorem.*

Proof. Such an operator $\tilde{A}$ is well defined by Lemma 6. For each $x \in \tilde{D}$ we have

$$\xi\tilde{A}x=x-L(\xi)^{-1}x\subset X \text{ and so, } x-\xi\tilde{A}x=L(\xi)^{-1}x\subset X.$$

But since $L(\xi)[x-\xi\tilde{A}x]=L(\xi)[L(\xi)^{-1}x]=x$, the mapping $x\to x-\xi\tilde{A}x$ is the topological inverse mapping of $L(\xi)$, which implies (1). Since $L(\xi)$ is the unique extension of $(I-\xi A)^{-1}$ by Lemma 3, $L(\xi)(I-\xi A)x=x$ for each $x \in D(A)$ and thus $L(\xi)^{-1}x=x-\xi\tilde{A}x \ni (I-\xi A)x$, from which $\tilde{A}x \ni Ax$. Thus (2) is proved. Finally we shall prove (3). For any $x \in \tilde{D}$ there exists $x' \in X$ such that $x=L(1)x'$. Since $\Re(I-A)$ is dense in X, there exists a sequence $\{x_n\}\subset D(A)$ such that $(I-A)x_n\to x'$ as $n\to\infty$. Thus $x_n=L(1)(I-A)x_n\to L(1)x'=x$ and so, $Ax_n=x_n-(I-A)x_n\to x-x' \in x-L(1)^{-1}x=\tilde{A}x$. Q.E.D.

Lemma 8. *For each $\xi\geqq 0$, $\{L(\xi/n)^n\}$ converges continuously to $T(\xi)$ on X and for each $x \in X$, $\{L(\xi/n)^n x\}$ converges to $T(\xi)x$ uniformly in ξ for any compact subset in $[0, \infty)$.*

Proof. Since $T'(\xi)x=AT(\xi)x$ for each $x \in D$ from Lemma 1, we have the following estimate:

$$\begin{aligned}
&\| L(\xi/n)^n x-T(\xi)x \| \\
&\quad =\| L(\xi/n)^n x-T(\xi/n)^n x \| \\
&\quad \leqq \sum_{i=1}^{n} \| L(\xi/n)^{n-i+1}T(\xi(i-1)/n)x-L(\xi/n)^{n-i}T(\xi i/n)x \| \\
&\quad \leqq \sum_{i=1}^{n} \| L(\xi/n)T(\xi(i-1)/n)x-L(\xi/n)(I-(\xi/n)A)T(\xi i/n)x \| \\
&\quad \leqq \sum_{i=1}^{n} (\xi/n) \| A_{\frac{\xi}{n}}T(\xi(i-1)/n)x-AT(\xi i/n)x \| \\
&\quad = \sum_{i=1}^{n} (\xi/n) \| (\xi/n)^{-1}(T(\xi i/n))-T(\xi(i-1)/n)x-AT(\xi i/n)x \| \\
&\quad \leqq (\xi/n) \sum_{i=1}^{n} (\xi/n)^{-1} \int_{\frac{\xi(i-1)}{n}}^{\frac{\xi i}{n}} \| T'(\sigma)x-T'(\xi i/n)x \| d\sigma \\
&\quad \leqq \xi \max_{1\leqq i\leqq n} \max_{\sigma\in\left[\frac{\xi(i-1)}{n},\frac{\xi i}{n}\right]} \| T'(\sigma)x-T'(\xi i/n)x \|.
\end{aligned}$$

The above right hand tends to 0 as $n \to \infty$, since $T'(\sigma)x$ is uniformly continuous on $[0, \xi]$. Thus $\lim_{n\to\infty} L(\xi/n)^n x = T(\xi)x$ for each $x \in D$. On the other hand, $L(\xi/n)^n$ is a contraction operator for each n. And so, $\{L(\xi/n)^n\}$ converges continuously to $T(\xi)$ on X [5]. Moreover the uniform convergence in ξ for any compact subset of $[0, \infty)$ is evident from the abovementioned estimate. Q.E.D.

Finally the author expresses his sincere thanks to Professor I. Miyadera for many useful advices.

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