Note on the space group selection rule for closed strings on orbifolds

Saúl Ramos-Sánchez[a,1] and Patrick K.S. Vaudrevange[b]

[a]*Instituto de Física, Universidad Nacional Autónoma de México,*
POB 20-364, Cd. Mx. 01000, México

[b]*Physik Department T75, Technische Universität München,*
James-Franck-Straße, 85748 Garching, Germany

E-mail: ramos@fisica.unam.mx, patrick.vaudrevange@tum.de

ABSTRACT: It is well-known that the space group selection rule constrains the interactions of closed strings on orbifolds. For some examples, this rule has been described by an effective Abelian symmetry that combines with a permutation symmetry to a non-Abelian flavor symmetry like D_4 or $\Delta(54)$. However, the general case of the effective Abelian symmetries was not yet fully understood. In this work, we formalize the computation of the Abelian symmetry that results from the space group selection rule by imposing two conditions only: (i) well-defined discrete charges and (ii) their conservation. The resulting symmetry, which we call the space group flavor symmetry D_S, is uniquely specified by the Abelianization of the space group. For all Abelian orbifolds with $\mathcal{N} = 1$ supersymmetry we compute D_S and identify new cases, for example, where D_S contains a $\mathbb{Z}_2$ dark matter-parity with charges 0 and 1 for massless and massive strings, respectively.

---

[1]Corresponding author.

Contents

1 Introduction

One of the main goals of string phenomenology is to build a realistic string model that describes and explains the origin of the particles of the Standard Model (SM) and their interactions, as well as of the standard cosmology encoded in the ΛCDM model. In this sense, the compactification of the heterotic string on six-dimensional orbifolds [1, 2] can be seen as a promising string scenario. It has led to very large sets of semi-realistic string models, which, among other features, reproduce the particle content of the SM [3, 4] or of its supersymmetric extensions [5–12].

Having large sets of semi-realistic orbifold models, the next step is to improve the phenomenological constraints on these models. This can be achieved by a detailed study of the low-energy effective field theory in these constructions. First of all, this requires the identification and understanding of all symmetries that govern the interactions of closed strings

on orbifolds. These symmetries include gauge symmetries, discrete R-symmetries [13–15], target-space modular symmetries [16–19] and Abelian discrete symmetries that arise from the so-called space group selection rule [20, 21]. The latter ones are the main focus of this work: the space group selection rule sets the geometric conditions under which closed strings can split and join while propagating on the surface of an orbifold. These geometric conditions depend on the geometrical orbifold under consideration, which is specified by the space group S. Then, for a given space group S one can rephrase the space group selection rule as an effective discrete symmetry which we denote by D_S in the following.

The phenomenological relevance of D_S has been emphasized by showing that they are essential ingredients of the non-Abelian flavor symmetries realized in the effective field theory of orbifold compactifications [12, 22, 23]. Therefore, we call the Abelian symmetry D_S the *space group (SG) flavor symmetry*. Unfortunately, the nature of the SG flavor symmetry D_S has not been fully understood. Furthermore, the computation of D_S has only been restricted to the interactions of massless strings, even though massive strings may also play a crucial role for the phenomenology in orbifold compactifications, for example for CP violation due to the presence of heavy string modes [24].

In this work, we aim at completing the study of the SG flavor symmetries arising from the space group selection rule for closed strings on orbifolds. After an introduction to space groups, orbifolds and the space group selection rule in section 2, we formalize in section 3 the computation of the SG flavor symmetry D_S by imposing two physical conditions only: D_S must be as symmetry with well-defined discrete charges for closed string states and discrete charges must be conserved. This fixes the SG flavor symmetry uniquely via the so-called Abelianization of the space group S. After exemplifying the computation, we present in section 4 the SG flavor symmetries D_S for all space groups S of six-dimensional orbifold geometries with Abelian point groups and $\mathcal{N} = 1$ supersymmetry as classified in refs. [25, 26]. As a consistency check, we verify that all discrete anomalies of the SG flavor symmetry D_S are universal and, hence, can be canceled by a discrete version of the universal Green-Schwarz mechanism [27].

Interestingly, we identify new SG flavor symmetries that act differently on massless and massive strings. In other words, the full string spectrum is subject to a larger symmetry group compared to the massless spectrum. In order to highlight possible phenomenological consequences of this, we present in section 3.6.2 a detailed example for a specific $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. There, the SG flavor symmetry contains a $\mathbb{Z}_4$ factor that acts as a $\mathbb{Z}_2$ symmetry if restricted to massless strings only. As we show, such a symmetry can be used to define a $\mathbb{Z}_2$ dark matter parity, where certain massive strings can only be produced and annihilated in pairs.

2 Space groups and orbifolds

In this section we give a detailed review on space groups and their resulting orbifold geometries. We highlight an unconventional approach to use a so-called presentation of a space group, i.e. a way to define a space group by abstract generators and a set of relations among them without explicitly writing out neither a basis of lattice vectors nor the rotation

matrices. In addition, we briefly discuss closed strings on orbifolds with a focus on those properties that are relevant for the space group selection rule of interacting strings.

2.1 The space group

A general element g of a D-dimensional space group S can be written as

$$g = (\rho, \lambda) \in S . \tag{2.1}$$

We will mostly consider the case $D = 6$ in order to compactify the heterotic string from ten to four dimensions. By definition, the space group element g acts on the D extra-dimensional coordinates $y \in \mathbb{R}^D$ as

$$y \overset{g}{\mapsto} (\rho, \lambda) y = \rho y + \lambda , \tag{2.2}$$

where the so-called twist $\rho \in \mathrm{O}(D)$ is a $D \times D$ rotation matrix (if $\det(\rho) = 1$) or reflection matrix (if $\det(\rho) = -1$) and the vector $\lambda \in \mathbb{R}^D$ yields a translation. Consequently, two space group elements $g_1 = (\rho_1, \lambda_1)$ and $g_2 = (\rho_2, \lambda_2)$ multiply as

$$(\rho_1, \lambda_1)(\rho_2, \lambda_2) = (\rho_1 \rho_2, \rho_1 \lambda_2 + \lambda_1) . \tag{2.3}$$

It follows that $(\mathbb{1}, 0) = \mathbb{1}_S$ is the identity element of S and the inverse element of (ρ, λ) is given by

$$(\rho, \lambda)^{-1} = (\rho^{-1}, -\rho^{-1} \lambda) . \tag{2.4}$$

Now we can define a D-dimensional space group S by specifying finitely many generators: first, one chooses a D-dimensional torus lattice Λ_D, which is generated by D linear independent translations T_i, i.e.

$$T_i = (\mathbb{1}, e_i) \qquad \text{for} \qquad i = 1, \ldots, D , \tag{2.5}$$

where the basis vectors e_i are given by the columns of a vielbein e. In addition, there are generators of the form

$$(\rho, \lambda) \in S \quad \text{with} \quad \rho \neq \mathbb{1} . \tag{2.6}$$

As a remark, if $\lambda \notin \Lambda_D$ in eq. (2.6) such generators are referred to as roto-translations. On the other hand, if $\lambda \in \Lambda_D$ one can choose a pure rotation as an alternative generator to eq. (2.6), i.e.

$$(\rho, 0) \in S . \tag{2.7}$$

As we will be dealing with space groups with at most two rotational generators, we will label them by $g_\theta, g_\omega \in S$, where

$$g_\theta = (\theta, \lambda_\theta) \quad \text{and} \quad g_\omega = (\omega, \lambda_\omega) . \tag{2.8}$$

The space group S must close under multiplication. For example,

$$(\rho, \lambda)(\mathbb{1}, e_i)(\rho, \lambda)^{-1} = (\mathbb{1}, \rho e_i) \overset{!}{\in} S \tag{2.9}$$

and, consequently, $\rho\,e_i$ must be from the D-dimensional lattice Λ_D. In other words, the twist ρ of any space group element has to be an automorphism of the lattice Λ_D. Thus, one can always find a matrix $\hat{\rho}$ from $\mathrm{GL}(D,\mathbb{Z})$ such that

$$\rho\,e \;=\; e\,\hat{\rho}\,. \tag{2.10}$$

$\hat{\rho} = e^{-1}\,\rho\,e$ is called the twist in the lattice basis. Since $\rho \in \mathrm{O}(D)$, we find the condition

$$\hat{\rho}^T G\,\hat{\rho} \;=\; G\,, \tag{2.11}$$

on the D-dimensional torus metric $G = e^T e$.

The twists ρ form a finite group P, called the point group. Its elements act crystallographically: they map the lattice Λ_D to itself. As P is finite, each element $\rho \in P$ has to have finite order, i.e. there exists a smallest integer $N_\rho \in \mathbb{N}$ such that

$$\rho^{N_\rho} \;=\; \mathbb{1}\,. \tag{2.12}$$

A point group can be both, Abelian or non-Abelian. If P is Abelian, it is isomorphic to the cyclic group (or to the direct product of several cyclic groups). In the following we restrict ourselves to Abelian point groups which preserve $\mathcal{N}=1$ supersymmetry, being either $\mathbb{Z}_M$ or $\mathbb{Z}_M \times \mathbb{Z}_N$.

2.2 Presentation of the space group

The commutator of two space group elements $g, h \in S$ is defined as

$$[g, h] \;=\; g^{-1}h^{-1}g\,h\,. \tag{2.13}$$

Then, since we restrict ourselves to Abelian point groups, the commutator of any pair of elements $g, h \in S$ is always a pure translation $\lambda_{(g,h)} \in \Lambda_D$, i.e.

$$[g, h] \;=\; (\mathbb{1}, \lambda_{(g,h)})\,. \tag{2.14}$$

Following ref. [29], we can specify a space group S uniquely by a presentation that involves all relations between all generators. For example, in the case of a $\mathbb{Z}_M \times \mathbb{Z}_N$ point group we have the following generators of S: the rotations (or roto-translations) $g_\theta = (\theta, \lambda_\theta)$ and $g_\omega = (\omega, \lambda_\omega)$, and the translations $T_i = (\mathbb{1}, e_i)$, $i = 1, \ldots, D$. Then, all relations are given by

$$g_\theta^M \;=\; (T_1)^{a_{(\theta,1)}} \ldots (T_D)^{a_{(\theta,D)}}\,, \tag{2.15a}$$

$$g_\omega^N \;=\; (T_1)^{a_{(\omega,1)}} \ldots (T_D)^{a_{(\omega,D)}}\,, \tag{2.15b}$$

$$[g_\theta, g_\omega] \;=\; (T_1)^{a_1} \ldots (T_D)^{a_D}\,, \tag{2.15c}$$

$$[T_i, T_j] \;=\; \mathbb{1}_S\,, \tag{2.15d}$$

$$[g_\theta, T_i] \;=\; (T_1)^{b_{(i,1)}} \ldots (T_D)^{b_{(i,D)}}\,, \tag{2.15e}$$

$$[g_\omega, T_i] \;=\; (T_1)^{c_{(i,1)}} \ldots (T_D)^{c_{(i,D)}}\,, \tag{2.15f}$$

where M and N denote the order of θ and ω, respectively, and $a_{(\theta,i)}, a_{(\omega,i)}, a_i, b_{(i,j)}, c_{(i,j)} \in \mathbb{Z}$ for $i, j = 1, \ldots, D$. A few remarks are in order: the right-hand sides of eqs. (2.15a), (2.15b) and (2.15c) are non-trivial only in the case where $g_\theta = (\theta, \lambda_\theta)$ or $g_\omega = (\omega, \lambda_\omega)$ are roto-translations, i.e. $\lambda_\theta \notin \Lambda_D$ or $\lambda_\omega \notin \Lambda_D$. Furthermore, two translations $T_i = (\mathbb{1}, e_i)$ and $T_j = (\mathbb{1}, e_j)$ necessarily commute, see eq. (2.15d). Finally, the action of a twist (e.g. $\theta \in P$) on the lattice (e.g. $\hat\theta = e^{-1}\theta\, e$, see eq. (2.10)) is uniquely specified by eqs. (2.15e) and (2.15f). In detail, comparing

$$[g_\theta, T_i] \;=\; (\mathbb{1}, (\mathbb{1} - \theta^{-1})e_i)\,, \tag{2.16}$$

with eq. (2.15e) we obtain $\left(\hat\theta^{-1}\right)_{ij} = \delta_{ij} - b_{(j,i)}$.

Now, in order to write down a presentation of S, we rewrite each relation (2.15) such that one has the identity element $\mathbb{1}_S$ on the right-hand side. For example, we modify eq. (2.15a) to

$$g_\theta^M\; (T_1)^{-a_{(\theta,1)}} \ldots (T_D)^{-a_{(\theta,D)}} \;=\; \mathbb{1}_S\,. \tag{2.17}$$

Then, we suppress the identity element $\mathbb{1}_S$ and do the same for all relations (2.15). Consequently, a presentation of the space group S reads

$$S \;=\; \langle g_\theta,\, g_\omega,\, T_1,\, \ldots,\, T_D \mid g_\theta^M\; (T_1)^{-a_{(\theta,1)}} \ldots (T_D)^{-a_{(\theta,D)}},\, \ldots,\, \text{all relations} \rangle\,. \tag{2.18}$$

One can use the presentation of a space group S to uniquely specify S without writing out the torus vielbein e and the twist matrices explicitly. Indeed, the relations (2.18) contain all information about the space group S. This fact is used in section 3 where we discuss the effective symmetries arising from the space group selection rule.

2.3 Geometrical orbifolds and closed strings

Having defined the space group S, a D-dimensional orbifold is defined geometrically as a quotient space $O = \mathbb{R}^D/S$ using the equivalence relation

$$y_1 \;\sim\; y_2 \qquad \Leftrightarrow \qquad \exists\, g \in S \quad \text{such that} \quad y_1 = g\, y_2\,, \tag{2.19}$$

for $y_1, y_2 \in \mathbb{R}^D$. In words, two points y_1 and y_2 from $\mathbb{R}^D$ are identified on the orbifold O if there exists a space group element $g \in S$ that maps y_2 to y_1.

A closed string on an orbifold is characterized by the so-called constructing element $g = (\rho, \lambda) \in S$ that specifies the boundary condition for the string to close up to the action of g. For example, considering the worldsheet boson $X(\tau, \sigma)$ with worldsheet time and space coordinates τ and $\sigma \in [0, 1]$, respectively, we impose the boundary condition

$$X(\tau, \sigma + 1) \;\overset{!}{=}\; g\, X(\tau, \sigma) \;=\; \rho\, X(\tau, \sigma) + \lambda\,. \tag{2.20}$$

If $\rho \neq \mathbb{1}$ the boundary condition eq. (2.20) describes a so-called twisted string that is localized at the fixed point of g. On the other hand, if $\rho = \mathbb{1}$ the boundary condition eq. (2.20) describes a so-called untwisted string, which in general can be massless only if the winding vanishes, $\lambda = 0$.

Since $h\,X(\tau,\sigma)$ and $X(\tau,\sigma)$ are identified on the orbifold for all $h \in S$, see eq. (2.19), the boundary condition eq. (2.20) with constructing element $g \in S$ and the corresponding one with constructing element $h\,g\,h^{-1} \in S$ describe the same string on the orbifold. Hence, a closed string with constructing element $g \in S$ is associated to the conjugacy class

$$[g] \;=\; \{h\,g\,h^{-1} \mid \text{ for all } h \in S\}\,. \tag{2.21}$$

The resulting closed string state is denoted by $|[g]\rangle$.

Conventionally, one (ambiguously) labels an orbifold by the abstract finite group that is isomorphic to its point group P, for example $\mathbb{Z}_6$ (and sometimes additional labels to distinguish between different representations of the same abstract finite group, for example $\mathbb{Z}_6$-I or $\mathbb{Z}_6$-II). In terms of the modern nomenclature (see e.g. [25, 26]), this corresponds to the so-called $\mathbb{Q}$-class. For a given $\mathbb{Q}$-class, there can be several inequivalent torus lattices Λ_D, called $\mathbb{Z}$-classes. Furthermore, for a given $\mathbb{Z}$-class there can be several inequivalent roto-translations, called affine classes. Then, $\mathbb{Z}$- and affine classes are consecutively enumerated. For example, the space group $\mathbb{Z}_2 \times \mathbb{Z}_2$–2–5 from table 1 belongs to the $\mathbb{Q}$-class $\mathbb{Z}_2 \times \mathbb{Z}_2$, therein to the second $\mathbb{Z}$-class and, finally, therein to the fifth affine class.

2.4 The space group selection rule

Let us denote a coupling among L closed string states $|[g_a]\rangle$, $a = 1, \dots, L$, moving on an orbifold by

$$|[g_1]\rangle\,|[g_2]\rangle\,\cdots\,|[g_L]\rangle\,, \tag{2.22}$$

where the closed string states are characterized by the conjugacy classes $[g_a]$ of their constructing elements $g_a \in S$.

In the underlying CFT of string theory, the amplitude of such couplings of string states is given by the L-point correlation functions [20, 21] $\langle V^{(1)} V^{(2)} \cdots V^{(L)} \rangle$ among their corresponding vertex operators $V^{(a)}$, which contain all quantum numbers of the strings, including their boundary conditions given by g_a. These correlation functions are non-zero only if certain conditions on the quantum numbers encoded in the vertex operators are fulfilled. These constraints are known as *string selection rules*.

In particular, the so-called *space group selection rule* indicates under which circumstances the boundary conditions of certain closed strings are compatible, so that they can combine together and still render a closed string. As an example, consider two closed strings with constructing elements g_1 and g_2. When these incoming strings merge, they yield an outgoing string with constructing element $g_3 = g_1\,g_2$. In terms of three incoming strings this condition reads

$$\prod_{a=1}^{L} g_a \;=\; \mathbb{1}_S\,, \tag{2.23}$$

for $L = 3$. This is schematically represented in figure 1.

As a closed string is not represented by a single constructing element g_a but by the full conjugacy class $[g_a]$, it suffices to choose one element of each of the conjugacy classes
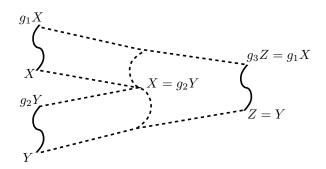
Figure 1. Two incoming strings combine to yield an outgoing string. The equalities are requirements for the coupling to occur. Combining these conditions leads to the space group selection rule on the corresponding constructing elements, $g_3 Y = g_1 g_2 Y$.

of the constructing elements to satisfy the space group selection rule. That is, for L-point couplings one can choose elements $h_a \in S$, such that

$$\prod_{a=1}^{L} h_a \, g_a \, h_a^{-1} \;=\; \mathbb{1}_S \tag{2.24}$$

in order to obtain a non-vanishing coupling among the states $|[g_a]\rangle$. As a remark, if eq. (2.24) is satisfied, then the order of string states in eq. (2.22) does not matter.

Depending on the complexity of the space group S under consideration, the space group selection rule is difficult to apply as one has to check eq. (2.24) for all possible choices $h_a \in S$ for $a = 1, \ldots, L$. Hence, we want to identify the effective SG flavor symmetry D_S that incorporates the space group selection rule without this ambiguity. Such a symmetry has to fulfill certain conditions that we now discuss.

3 Consequences of the space group selection rule

In this section, we derive the SG flavor symmetry D_S that emerges from the geometric restrictions on the interactions of closed strings while moving on the surface of an orbifold.

3.1 Conditions on the effective symmetry

We look for a mapping, denoted by s, from the space group S to a discrete group D_S, i.e. $s : S \to D_S$, with the following two properties

(i) s is a class function:

$$s(h \, g \, h^{-1}) \;=\; s(g) \quad \text{for all} \quad h, g \in S \,. \tag{3.1}$$

(ii) s is a representation of the space group S:

$$s(g_1) \, s(g_2) \;=\; s(g_1 \, g_2) \,, \tag{3.2}$$

Let us briefly discuss some immediate implications of these conditions. A necessary condition for the space group selection rule (2.24) to be satisfied is given by

$$s(g_1) \cdots s(g_L) \; = \; s(\mathbb{1}_S) \,, \tag{3.3}$$

which follows from eq. (2.24) by applying s on both sides and using our conditions (i) and (ii). Thus, we arrive at a much simpler version of the space group selection rule, where the ambiguity of choosing the elements $h_a \in S$ for $a = 1, \ldots, L$ has been eliminated. Moreover, from a physical point of view, condition (ii) can be understood as a discrete charge conservation, where $s(g_a)$ corresponds to the discrete charge of a string state $|[g_a]\rangle$ with constructing element g_a, i.e.

$$|[g_a]\rangle \; \mapsto \; s(g_a) \, |[g_a]\rangle \,, \tag{3.4}$$

for $a = 1, \ldots, L$. Condition (i) ensures that the discrete charge corresponding to $s(g_a)$ is uniquely defined as a string state $|[g_a]\rangle$ with constructing element $g_a \in S$ is characterized by the conjugacy class $[g_a]$, see section 2.3.

Using the conditions (i) and (ii) of eqs. (3.1) and (3.2), it turns out that $s(g)$ (and consequently the SG flavor symmetry D_S) is fully specified for all space groups S. In the remainder of this paper we will compute our main result: the space group flavor symmetries D_S for all six-dimensional orbifold geometries with Abelian point groups and $\mathcal{N} = 1$ supersymmetry, as classified in refs. [25, 26].

As a remark, if a coupling is not invariant under D_S then the space group selection rule is also not fulfilled. However, eq. (3.3) is a necessary condition but not sufficient: there are cases, where a coupling is invariant under D_S, i.e. eq. (3.3) is satisfied, but the space group selection rule (2.24) is not satisfied for any choice $h_a \in S$. This happens for example for the $\mathbb{Z}_6$–II–1–1 orbifold, where the charges under D_S do not depend on n_1 and n_2, i.e. on the localization of a twisted string in the first two-torus (often called the G_2 torus), see table 1. However, the space group selection rule applied to the first two-torus still constrains the allowed interactions [28]. Consequently, in such a case the SG flavor symmetry D_S is not equivalent to the space group selection rule. It remains unknown whether one can modify our assumptions eqs. (3.1) and (3.2) such that the resulting discrete symmetry is fully equivalent to the space group selection rule.

3.2 Consequences of the conditions

Since s is a representation of the space group S we easily see that

$$s(\mathbb{1}_S) \, s(\mathbb{1}_S) \; = \; s(\mathbb{1}_S \, \mathbb{1}_S) \; = \; s(\mathbb{1}_S) \quad \Rightarrow \quad s(\mathbb{1}_S) \; = \; \mathbb{1}_{D_S} \,, \tag{3.5}$$

where $\mathbb{1}_{D_S}$ is the identity element of D_S. Another consequence of s being a representation of S reads

$$\mathbb{1}_{D_S} \; = \; s(\mathbb{1}_S) \; = \; s(g^{-1} \, g) \; = \; s(g^{-1}) \, s(g) \quad \Rightarrow \quad s(g^{-1}) \; = \; s(g)^{-1} \,, \tag{3.6}$$

for all $g \in S$. In addition, eq. (3.1) yields

$$s(h \, g) \; = \; s(g \, h) \quad \Leftrightarrow \quad [s(h), s(g)] \; = \; 0 \,, \tag{3.7}$$

for all $h, g \in S$. Thus, the SG flavor symmetry D_S must be Abelian and the representation s is in general not faithful. Furthermore, eqs. (3.6) and (3.7) yield the important condition

$$s([g, h]) \;=\; \mathbb{1}_{D_S} \;, \tag{3.8}$$

where the commutator is defined as $[g, h] = g^{-1} h^{-1} g \, h$, see section 2.2. Consequently, the representation s of the space group S is equivalent to the Abelianization of the space group S,

$$D_S \;\cong\; S/[S, S] \;, \tag{3.9}$$

see e.g. ref. [29].[1]

Indeed, it turns out that D_S is a *finite* Abelian group for all space groups S under consideration (i.e. in all cases the Abelian group D_S does not contain any factors of $\mathbb{Z}$). Hence, we can represent $s(g)$ for all $g \in S$ by a complex phase,

$$s(g) \;=\; \exp\left(\mathrm{i}\,\alpha(g)\right) \;, \tag{3.10}$$

and set $\mathbb{1}_{D_S} = 1$ in the following. In addition, D_S is a direct product of $\mathbb{Z}_{M_i}$ factors of various orders M_i.

In order to obtain the discrete transformation $s(g)$ for a string state $|[g]\rangle$ with constructing element $g \in S$ explicitly, we first express g as a product of the generators of S, see section 2.1. In detail, in the case of a $\mathbb{Z}_M \times \mathbb{Z}_N$ point group the generators of S read

$$T_i \;=\; (\mathbb{1}, e_i) \;, \; g_\theta \;=\; (\theta, \lambda_\theta) \quad \text{and} \quad g_\omega \;=\; (\omega, \lambda_\omega) \;, \tag{3.11}$$

such that we can decompose any space group element $g \in S$ as

$$g \;=\; (T_1)^{n_1} \ldots (T_D)^{n_D} \, (g_\theta)^k \, (g_\omega)^\ell \;, \tag{3.12}$$

where $k = 0, \ldots, M-1$, $\ell = 0, \ldots, N-1$ and $n_i \in \mathbb{Z}$ for $i = 1, \ldots, D$. Then, using the representation property eq. (3.2), we obtain

$$s(g) \;=\; (s(T_1))^{n_1} \ldots (s(T_D))^{n_D} \, (s(g_\theta))^k \, (s(g_\omega))^\ell \;. \tag{3.13}$$

Thus, we can easily compute the discrete transformation $s(g)$ of a string state $|[g]\rangle$ once we know the discrete transformations of the generators of the space group

$$s(T_i) \;, \; s(g_\theta) \quad \text{and} \quad s(g_\omega) \;. \tag{3.14}$$

Hence, we have reduced our problem to the task of identifying these building-blocks of the general discrete transformation $s(g)$.

---

[1]The Abelianization of space groups was also used in ref. [30] to constrain the gauge embeddings via shifts and Wilson lines, and in relation to Gauged Linear Sigma Models (see also [31]).

3.3 Space group selection rule and the Abelianization of the space group

As noted before, our conditions eqs. (3.1) and (3.2) that define the transformation $s(g)$ of a closed string state $|[g]\rangle$ with constructing element $g \in S$ correspond to the Abelianization of the space group S. In this section we show that the Abelianization of S can be computed using the so-called presentation of the space group S. To do so, each generator g of S is replaced by $s(g)$ and each relation in the presentation of S is replaced by $s(\text{relation})$ using that $s([g, h]) = 1$, i.e.

$$g_\theta = (\theta, \lambda_\theta) \longmapsto s(g_\theta), \tag{3.15a}$$

$$g_\omega = (\omega, \lambda_\omega) \longmapsto s(g_\omega), \tag{3.15b}$$

$$T_i = (\mathbb{1}, e_i) \longmapsto s(T_i), \quad i = 1, \ldots, D, \tag{3.15c}$$

and from the presentation

$$S = \langle g_\theta, g_\omega, T_1, \ldots, T_D \mid g_\theta^M (T_1)^{-a_{(\theta,1)}} \ldots (T_D)^{-a_{(\theta,D)}}, \ldots, \text{all relations} \rangle, \tag{3.16}$$

see section 2.2, we obtain via the map $S \mapsto D_S$ the presentation of D_S as

$$\begin{aligned} D_S = \langle s(g_\theta), s(g_\omega), s(T_1), \ldots, s(T_D) \mid \\ s(g_\theta)^M (s(T_1))^{-a_{(\theta,1)}} \ldots (s(T_D))^{-a_{(\theta,D)}}, \ldots, s(\text{all relations}) \rangle. \end{aligned} \tag{3.17}$$

In most cases, some generators $s(g)$ in the presentation of D_S are no longer independent compared to their preimages $g \in S$. Thus, one has to solve the relations in D_S such that only the independent generators remain.

Furthermore, for all space groups under consideration all remaining relations in D_S can be solved explicitly such that one can identify the SG flavor symmetry D_S as the direct product of cyclic groups, i.e.

$$D_S \cong \mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2} \times \ldots. \tag{3.18}$$

In the following we present two approaches to how this computation can be performed in detail.

3.4 Direct computation

Among the relations in the presentation of D_S in eq. (3.17) there are two sets of relations that are of special interest: eqs. (2.15e) and (2.15f) (or equivalently eq. (2.9)) are related to the charges of translations $s(T_i)$, while eqs. (2.15a) and (2.15b) are related to the charges of rotations $s(g_\theta)$ and $s(g_\omega)$. Using these relations one can easily compute the orders of $s(g_\theta)$, $s(g_\omega)$ and $s(T_i)$, as we do next.

The charges of translations $s(T_i)$. We embed eq. (2.9) into D_S and obtain[2]

$$s(\mathbb{1}, e_i) = s(\mathbb{1}, \rho\, e_i), \tag{3.19}$$

---

[2] For a space group element $g = (\rho, \lambda) \in S$ we write $s(g) = s(\rho, \lambda)$ instead of $s(g) = s((\rho, \lambda))$.

for all $\rho \in P$. In other words, two vectors e_i and $\rho\, e_i$ that are rotated to each other by a point group element ρ give rise to the same element of D_S.

Now, we assume that there is a point group element $\rho \in P$ and a smallest integer M_i such that

$$e_i + \rho\, e_i + \ldots + \rho^{M_i-1}\, e_i \; = \; 0 \; . \tag{3.20}$$

We can translate this equation into space group elements of pure translations and apply s. This yields

$$\underbrace{s(\mathbb{1}, e_i)\, s(\mathbb{1}, \rho\, e_i) \ldots s(\mathbb{1}, \rho^{M_i-1}\, e_i)}_{M_i \text{ factors}} \; = \; 1 \; . \tag{3.21}$$

Then, using eq. (3.19) we get

$$s(\mathbb{1}, e_i)^{M_i} \; = \; 1 \; . \tag{3.22}$$

Consequently, $s(T_i) = s(\mathbb{1}, e_i)$ generates a finite Abelian group of order M_i, being $\mathbb{Z}_{M_i}$, and we can express $s(T_i)$ as a complex phase

$$s(T_i) \; = \; \exp\left(\frac{2\pi \mathrm{i}}{M_i}\beta_i\right) \; , \tag{3.23}$$

for $\beta_i \in \mathbb{Z}$.

The charges of rotations $s(g_\theta)$ and $s(g_\omega)$. For example, consider the generator $\theta \in P$ of order M in the case without roto-translations, i.e. $g_\theta = (\theta, 0)$. Then, we obtain the following identities

$$s(\theta, 0)^M \; = \; s(\theta^M, 0) \; = \; s(\mathbb{1}_S) \; = \; 1 \; . \tag{3.24}$$

Consequently, $s(g_\theta) = s(\theta, 0)$ generates a finite Abelian group of order M, being $\mathbb{Z}_M$, and we can express $s(g_\theta)$ as a complex phase

$$s(g_\theta) \; = \; \exp\left(\frac{2\pi \mathrm{i}}{M}\alpha_\theta\right) \; , \tag{3.25}$$

for $\alpha_\theta \in \mathbb{Z}$. Analogously, using $\omega^N = \mathbb{1}$ we find

$$s(g_\omega) \; = \; \exp\left(\frac{2\pi \mathrm{i}}{N}\alpha_\omega\right) \; , \tag{3.26}$$

for $\alpha_\omega \in \mathbb{Z}$.

Combination of translations and rotations. As we have seen in eq. (3.13) the representation $s(g)$ of a general space group element eq. (3.12) is given by

$$s(g) \; = \; (s(T_1))^{n_1} \ldots (s(T_D))^{n_D} \; (s(g_\theta))^k \; (s(g_\omega))^\ell \; , \tag{3.27}$$

where one might be tempted to simply replace the previous results for the representations of the translations, $s(T_i)$, and roto-translations, $s(g_\theta), s(g_\omega)$. However, in general, the presentation of D_S eq. (3.17) establishes non-trivial relations among different elements, that can alter the result.

In fact, only in the simplest case without roto-translations and without non-trivial relations between translations and twists, this naïve expectation holds. In this case, a general space group element simplifies to $g = (\theta^k \omega^\ell, n_i e_i) \in S$ and its representation is given by

$$s \; : \; (\theta^k \omega^\ell, n_i e_i) \;\to\; \exp\left(2\pi\mathrm{i}\left(\frac{k}{M}\alpha_\theta + \frac{\ell}{N}\alpha_\omega + \sum_{i=1}^{D} \frac{n_i}{M_i}\beta_j\right)\right) , \qquad (3.28)$$

where k, ℓ and n_i for $i = 1, \ldots, D$ are discrete charges. They are conserved in an allowed coupling, i.e.

$$\sum_{a=1}^{L} k^{(a)} \;=\; 0 \bmod M , \quad \sum_{a=1}^{L} \ell^{(a)} \;=\; 0 \bmod N , \quad \sum_{a=1}^{L} n_i^{(a)} \;=\; 0 \bmod M_i , \qquad (3.29)$$

where $k^{(a)}$, $\ell^{(a)}$ and $n_i^{(a)}$ define the a-th constructing element $g_a \in S$ in the coupling (2.22). As a remark, those symmetries that constrain the twisted sectors $k^{(a)}$ (and $\ell^{(a)}$) are conventionally referred to as point group selection rule (PG).

In the more general case, α_θ, α_ω, β_i and β_j, with $i \neq j$, turn out to be connected by the relations in the presentation of D_S. These relations as well as their implications shall be discussed in our examples, in section 3.6.

3.5 Space group selection rule and remnant discrete symmetries from spontaneous symmetry breaking

As before, we begin with a space group S generated by $D + 2$ generators

$$T_i \;=\; (\mathbb{1}, e_i) , \; i = 1, \ldots, D , \; g_\theta \;=\; (\theta, \lambda_\theta) \quad \text{and} \quad g_\omega \;=\; (\omega, \lambda_\omega) \qquad (3.30)$$

subject to K relations as listed in the presentation of S. Then, the SG flavor symmetry D_S obtained from the space group selection rule is a subgroup of $\mathrm{U}(1)^{D+2}$. It can be computed as follows: each relation $\alpha = 1, \ldots, K$ of the presentation of S is mapped to a relation in D_S and can be written in the form

$$s(T_1)^{q_1^\alpha} \ldots s(T_D)^{q_D^\alpha} s(g_\theta)^{q_{D+1}^\alpha} s(g_\omega)^{q_{D+2}^\alpha} \;=\; 1 , \qquad (3.31)$$

see eq. (2.15) in section 2.2 using $s([g, h]) = 1$. To each relation (3.31) one can associate an auxiliary field $\phi_\alpha(x)$ with integer $\mathrm{U}(1)^{D+2}$ charges given by the exponents in eq. (3.31), i.e. the charges of $\phi_\alpha(x)$ read

$$(q_1^\alpha, \ldots, q_D^\alpha, q_{D+1}^\alpha, q_{D+2}^\alpha) . \qquad (3.32)$$

These auxiliary fields have no physical interpretation but are only used to describe the symmetry breaking associated to the relations (3.31). Then, under a general $\mathrm{U}(1)^{D+2}$ transformation the auxiliary field $\phi_\alpha(x)$ picks up a phase, i.e.

$$\phi_\alpha(x) \;\mapsto\; \exp\left(\mathrm{i}\sum_{j=1}^{D+2} \xi_j q_j^\alpha\right) \phi_\alpha(x) , \qquad (3.33)$$

where $\xi_j \in \mathbb{R}$, $j = 1, \ldots, D+2$, denote the U(1) transformation parameters. Now, we turn on the vacuum expectation values (VEVs) of the auxiliary fields $\langle \phi_\alpha \rangle \neq 0$ for all $\alpha = 1, \ldots, K$. Consequently, the unbroken remnant symmetry is given by the solutions ξ_j of

$$\exp\left(\mathrm{i} \sum_{j=1}^{D+2} \xi_j \, q_j^\alpha \right) \langle \phi_\alpha \rangle \; = \; \langle \phi_\alpha \rangle \tag{3.34}$$

for all auxiliary fields $\alpha = 1, \ldots, K$. Identifying

$$s(T_i) \; = \; \exp\left(\mathrm{i}\, \xi_i \right) \,, \; s(g_\theta) \; = \; \exp\left(\mathrm{i}\, \xi_{D+1} \right) \quad \text{and} \quad s(g_\omega) \; = \; \exp\left(\mathrm{i}\, \xi_{D+2} \right) \,, \tag{3.35}$$

we realize that this spontaneous U(1) symmetry breaking exactly corresponds to solving the relations (3.31).

This method to compute the SG flavor symmetry as a remnant discrete symmetry via spontaneous symmetry breaking of $\mathrm{U}(1)^{D+2}$ can be automatized easily using for example the mathematica package "DiscreteBreaking" developed in ref. [32].

3.6 Examples

In this section we present three examples how to compute the SG flavor symmetries explicitly: first, we consider a $\mathbb{Z}_3$ toy example in $D = 2$ dimensions and then two $\mathbb{Z}_2 \times \mathbb{Z}_2$ examples in $D = 6$ dimensions, one with roto-translations and the other with a freely-acting shift.

3.6.1 $\mathbb{Z}_3$ space group in $D = 2$ dimensions

We define the two-dimensional $\mathbb{Z}_3$ space group S by the generators

$$T_1 \; = \; (\mathbb{1}, e_1) \,, \quad T_2 \; = \; (\mathbb{1}, e_2) \quad \text{and} \quad g_\theta \; = \; (\theta, 0) \,, \tag{3.36}$$

where the basis vectors e_1 and e_2 enclose an angle of $120°$ and have equal length. Furthermore, θ is a counter-clockwise rotation by $120°$ such that

$$\theta \, e \; = \; e \, \hat{\theta} \quad \text{where} \quad \hat{\theta} \; = \; \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \,. \tag{3.37}$$

This space group can be defined alternatively by the following abstract presentation, based on the three generators g_θ, T_1 and T_2 subject to four relations, i.e.

$$S \; = \; \langle g_\theta, T_1, T_2 \mid g_\theta^3 \,, [T_1, T_2] \,, [g_\theta, T_1] \, T_1^{-2} \, T_2^{-1} \,, [g_\theta, T_2] \, T_1 \, T_2^{-1} \rangle \,, \tag{3.38}$$

see section 2.2.

To compute the SG flavor symmetry D_S we map the three generators of S to $s(g_\theta)$, $s(T_1)$ and $s(T_2)$. Then, the presentation of D_S reads

$$D_S \; = \; \langle s(g_\theta), s(T_1), s(T_2) \mid s(g_\theta)^3, \underbrace{s([T_1, T_2])}_{=1}, \underbrace{s([g_\theta, T_1])}_{=1} \, s(T_1)^{-2} \, s(T_2)^{-1}, $$
$$\underbrace{s([g_\theta, T_2])}_{=1} \, s(T_1) \, s(T_2)^{-1} \rangle \,. \tag{3.39}$$

Next, we omit the trivial relation $s([T_1, T_2]) = 1$ and obtain

$$D_S \;=\; \langle s(g_\theta),\, s(T_1),\, s(T_2) \mid s(g_\theta)^3,\, s(T_1)^{-2}\, s(T_2)^{-1},\, s(T_1)\, s(T_2)^{-1} \rangle \,. \tag{3.40}$$

In the final step, we observe that only two of the three generators of D_S are independent. For example, we can use the last relation $s(T_1)\, s(T_2)^{-1} = 1$ in eq. (3.40) to replace $s(T_2)$ by $s(T_1)$ using $s(T_2) = s(T_1)$. Thus, we get

$$D_S \;=\; \langle s(g_\theta),\, s(T_1) \mid s(g_\theta)^3,\, s(T_1)^3 \rangle \;\cong\; \mathbb{Z}_3 \times \mathbb{Z}_3 \,, \tag{3.41}$$

and a general string state $|[g]\rangle$ transforms as

$$|[g]\rangle \;\longmapsto\; s(T_1)^{n_1}\, s(T_2)^{n_2}\, s(g_\theta)^k\, |[g]\rangle \;=\; s(T_1)^{n_1+n_2}\, s(g_\theta)^k\, |[g]\rangle \tag{3.42}$$

$$=\; \exp\left(\frac{2\pi\mathrm{i}}{3} \alpha_1\, (n_1 + n_2) \right) \exp\left(\frac{2\pi\mathrm{i}}{3} \alpha_\theta\, k \right) |[g]\rangle \,, \tag{3.43}$$

for $\alpha_1,\, \alpha_\theta \in \{0, 1, 2\}$ using $s(g_\theta)^3 = s(T_1)^3 = 1$. Then, a coupling $|[g_1]\rangle \dots |[g_L]\rangle$ of string states $|[g_a]\rangle$ with constructing elements $g_a = (\theta^{k^{(a)}}, n_1^{(a)} e_1 + n_2^{(a)} e_2)$ is allowed by the SG flavor symmetry $D_S \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ if

$$\sum_{a=1}^{L} k^{(a)} \;=\; 0 \mod 3 \quad \text{and} \quad \sum_{a=1}^{L} \left(n_1^{(a)} + n_2^{(a)} \right) \;=\; 0 \mod 3 \,, \tag{3.44}$$

where the first equation in eq. (3.44) is called point group selection rule (PG).

In summary, the space group selection rule of the two-dimensional $\mathbb{Z}_3$ space group yields a $\mathbb{Z}_3 \times \mathbb{Z}_3$ SG flavor symmetry with discrete charges k and $n_1 + n_2$, respectively [20].

3.6.2 Space group with freely-acting shift: $\mathbb{Z}_2 \times \mathbb{Z}_2 - 5 - 1$

Consider the so-called Blaszczyk-geometry $\mathbb{Z}_2 \times \mathbb{Z}_2$–5–1 [25, 26] (in ref. [33] it is labeled as 1–1, see also [34], and [35] for an MSSM-like orbifold model based on this geometry using a different convention). The twists in the lattice basis (see eq. (2.10)) read

$$\hat\theta \;=\; \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \hat\omega \;=\; \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \,. \tag{3.45}$$

The space group S is generated by six translations $T_i = (\mathbb{1}, e_i)$ and two rotations

$$g_\theta \;=\; (\theta, 0) \quad \text{and} \quad g_\omega \;=\; (\omega, 0) \,. \tag{3.46}$$

By writing down the presentation of this space group S we identify the relations

$$s(T_1) = s(T_2) \;=\; s(T_3) \quad \text{where} \quad s(T_1)^2\, s(T_2)\, s(T_3) \;=\; 1 \,, \tag{3.47a}$$

$$s(T_4)^2 = s(T_5)^2 \;=\; s(T_6)^2 \;=\; s(g_\theta)^2 \;=\; s(g_\omega)^2 \;=\; 1 \,, \tag{3.47b}$$

see eq. (2.15). Consequently, the most general representation $s(g)$ is given by

$$s(g) = s(T_1)^{n_1} s(T_2)^{n_2} s(T_3)^{n_3} s(T_4)^{n_4} s(T_5)^{n_5} s(T_6)^{n_6} s(g_\theta)^k s(g_\omega)^\ell \qquad (3.48a)$$

$$= s(T_1)^{n_1+n_2+n_3} s(T_4)^{n_4} s(T_5)^{n_5} s(T_6)^{n_6} s(g_\theta)^k s(g_\omega)^\ell \,, \qquad (3.48b)$$

where we have used $s(T_1) = s(T_2) = s(T_3)$. The order of the basic transformations $s(g_\theta)$, $s(g_\omega)$, $s(T_4)$, $s(T_5)$ and $s(T_6)$ is 2. Furthermore, from eq. (3.47a) we obtain

$$s(T_1)^2 \, s(T_2) \, s(T_3) \;=\; s(T_1)^4 \;=\; 1 \,, \qquad (3.49)$$

and $s(T_1)$ generates a $\mathbb{Z}_4$ factor. Consequently, all transformations $s(g_\theta)$, $s(g_\omega)$, $s(T_1)$, $s(T_4)$, $s(T_5)$ and $s(T_6)$ are independent. Hence, the SG flavor symmetry D_S obtained from the space group selection rule is given by

$$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\mathrm{PG}} \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \,, \qquad (3.50)$$

with discrete charges

$$k \;,\;\; \ell \;,\;\; n_1 + n_2 + n_3 \;,\;\; n_4 \;,\;\; n_5 \;\;\text{and}\;\; n_6 \,, \qquad (3.51)$$

respectively, and we have verified this result additionally using the VEV-method of section 3.5.

$\mathbb{Z}_2$ dark matter-parity. By considering the fixed points of this $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, one can check that all massless strings carry even $\mathbb{Z}_4$ charges, i.e.

$$n_1 + n_2 + n_3 \;\in\; \{0, 2\} \,. \qquad (3.52)$$

Hence, the allowed interactions of massless strings are not constrained by a $\mathbb{Z}_4$ factor in eq. (3.50), but only by $\mathbb{Z}_2$. However, there are massive strings with odd discrete charges $n_1 + n_2 + n_3 \in \{1, 3\}$ under $\mathbb{Z}_4$, for example a winded string with constructing element $(\mathbb{1}, e_1)$. Consequently, massive strings with odd $\mathbb{Z}_4$ charges can only be produced and annihilated in pairs. Thus, the lightest massive string with odd $\mathbb{Z}_4$ charge could serve as a dark matter candidate, which is stable because there is a $\mathbb{Z}_2$ dark matter-parity with

$$|\text{matter}\rangle \;\longmapsto\; +|\text{matter}\rangle \,, \qquad (3.53a)$$

$$|\text{dark matter}\rangle \;\longmapsto\; -|\text{dark matter}\rangle \,, \qquad (3.53b)$$

and the mass of the dark matter particle depends on the compactification radii. This fact is common to many SG flavor symmetries in table 1 and might be relevant for the observed dark matter content of the universe and also for its cosmological evolution.

3.6.3 Space group with roto-translation: $\mathbb{Z}_2 \times \mathbb{Z}_2 - 2 - 5$

Consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$–2–5 orbifold from the classification in refs. [25, 26] (it corresponds to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold labeled 1–9 in ref. [33]). The twists in the lattice basis (see eq. (2.10))

are given by

$$\hat{\theta} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \hat{\omega} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \qquad (3.54)$$

The space group S is generated by six translations $T_i = (\mathbb{1}, e_i)$ and two roto-translations

$$g_\theta \;=\; \left(\theta, \tfrac{1}{2} e_3\right) \quad \text{and} \quad g_\omega \;=\; \left(\omega, \tfrac{1}{2} e_5\right) . \qquad (3.55)$$

By writing down the presentation of S we identify the relations (see eq. (2.15))

$$s(T_1) = s(T_2) \;,\;\; s(T_3) \;=\; s(T_5) \;=\; s(g_\theta)^2 \;=\; s(g_\omega)^2 \quad \text{and} \qquad (3.56a)$$
$$s(T_i)^2 = s(g_\theta)^4 \;=\; s(g_\omega)^4 \;=\; 1 \quad \text{for} \quad i = 1, \dots, 6 . \qquad (3.56b)$$

Consequently, the most general charge $s(g)$ is given by

$$s(g) = s(T_1)^{n_1} s(T_2)^{n_2} s(T_3)^{n_3} s(T_4)^{n_4} s(T_5)^{n_5} s(T_6)^{n_6} s(g_\theta)^k s(g_\omega)^\ell \qquad (3.57a)$$
$$= s(T_1)^{n_1+n_2} s(T_4)^{n_4} s(T_6)^{n_6} s(g_\theta)^{k+2(n_3+n_5)} s(g_\omega)^\ell , \qquad (3.57b)$$

where we have used $s(T_1) = s(T_2)$ and $s(T_3) = s(T_5) = s(g_\theta)^2$. Next, we analyze the consequences of $s(g_\theta)^2 = s(g_\omega)^2$ from eq. (3.56a), i.e. we make the ansatz

$$s(g_\theta) \;=\; \exp\left(\frac{2\pi i}{4} \alpha_\theta\right) \quad \text{and} \quad s(g_\omega) \;=\; \exp\left(\frac{2\pi i}{4} \alpha_\omega\right) . \qquad (3.58)$$

Then, $s(g_\theta)^2 = s(g_\omega)^2$ yields

$$\exp\left(\frac{2\pi i}{2} \alpha_\theta\right) \;=\; \exp\left(\frac{2\pi i}{2} \alpha_\omega\right) \quad \Leftrightarrow \quad \alpha_\omega \;=\; \alpha_\theta + 2\, x , \qquad (3.59)$$

for some $x \in \mathbb{Z}$. Thus,

$$s(g_\omega) \;=\; \exp\left(\frac{2\pi i}{4} \alpha_\omega\right) \;=\; \exp\left(\frac{2\pi i}{4} \alpha_\theta\right) \exp\left(\frac{2\pi i}{2} x\right) \;=\; s(g_\theta)\, s_x . \qquad (3.60)$$

where $s(g_\theta)$ and s_x are now independent and of order 4 and 2, respectively. Using eq. (3.60) in eq. (3.57b) we obtain

$$s(g) \;=\; s(g_\theta)^{k+\ell+2(n_3+n_5)} s_x^\ell\, s(T_1)^{n_1+n_2} s(T_4)^{n_4} s(T_6)^{n_6} , \qquad (3.61)$$

where the orders of $s(g_\theta)$, s_x, $s(T_1)$, $s(T_4)$ and $s(T_6)$ are 4, 2, 2, 2 and 2, respectively. Now, we have solved all relations (3.56) and, consequently, all transformations $s(g_\theta)$, s_x, $s(T_1)$,

$s(T_4)$ and $s(T_6)$ are independent. Hence, the space group selection rule results in a SG flavor symmetry

$$(\mathbb{Z}_4 \times \mathbb{Z}_2)^{\mathrm{PG}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \,, \tag{3.62}$$

with discrete charges

$$k + \ell + 2(n_3 + n_5) \,\,,\,\, \ell \,\,,\,\, n_1 + n_2 \,\,,\,\, n_4 \quad \text{and} \quad n_6 \,, \tag{3.63}$$

respectively. This result has also been verified using the VEV-method of section 3.5.

Naively, one would expect a $\mathbb{Z}_2 \times \mathbb{Z}_2$ point group selection rule with charges k and ℓ if the point group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, we have seen that the point group selection rule yields $\mathbb{Z}_4 \times \mathbb{Z}_2$ with charges $k + \ell + 2(n_3 + n_5)$ and ℓ. If one considers the $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_4$ factor, one identifies the corresponding $\mathbb{Z}_2$ charges as $k + \ell$. Thus, the naive $\mathbb{Z}_2 \times \mathbb{Z}_2$ point group selection rule is a subgroup of the full $\mathbb{Z}_4 \times \mathbb{Z}_2$ point group selection rule.

4 Results

In this section we present the main result of this work: we compute the Abelian SG flavor symmetries D_S for all 138 space groups S with Abelian point group and $\mathcal{N}=1$ supersymmetry. To do so, we adopt the convention of refs. [25, 26] as specified in the geometry files for the `orbifolder` [36] (see ancillary files in arXiv.org). The results are listed in table 1.

$\mathbb{Q}$-class twist vector	$\mathbb{Z}$- and affine class	SG flavor symmetry D_S	discrete charge
$\mathbb{Z}_3$ $\left(0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$	1–1	$\mathbb{Z}_3^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^3$	k $(n_1 + n_2, n_3 + n_4, n_5 + n_6)$
$\mathbb{Z}_4$ $\left(0, \frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right)$	1–1	$\mathbb{Z}_4^{(\mathrm{PG})}$ $(\mathbb{Z}_2)^4$	k $(n_1 + n_2, n_3 + n_4, n_5, n_6)$
	2–1	$\mathbb{Z}_4^{(\mathrm{PG})}$ $\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	k $(n_3 + n_4 + n_5, n_1 + n_2, n_6)$
	3–1	$\mathbb{Z}_4^{(\mathrm{PG})}$ $(\mathbb{Z}_4)^2$	k $(n_1 + n_2 + n_3, n_4 + n_5 + n_6)$
$\mathbb{Z}_6$–I $\left(0, \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$	1–1	$\mathbb{Z}_6^{(\mathrm{PG})}$ $\mathbb{Z}_3$	k $n_5 + n_6$
	2–1	$\mathbb{Z}_6^{(\mathrm{PG})}$ $\mathbb{Z}_3$	k $n_3 + n_4 + n_5 + n_6$
$\mathbb{Z}_6$–II $\left(0, \frac{1}{6}, \frac{1}{3}, -\frac{1}{2}\right)$	1–1	$\mathbb{Z}_6^{(\mathrm{PG})}$ $\mathbb{Z}_3 \times (\mathbb{Z}_2)^2$	k $(n_3 + n_4, n_5, n_6)$
	2–1	$\mathbb{Z}_6^{(\mathrm{PG})}$ $\mathbb{Z}_3 \times (\mathbb{Z}_2)^2$	k $(n_1 + n_2 + n_3 + n_4, n_5, n_6)$
	3–1	$\mathbb{Z}_6^{(\mathrm{PG})}$ $\mathbb{Z}_3 \times (\mathbb{Z}_2)^2$	k $(n_4 + n_5, n_1 + n_2 + n_3, n_6)$

ℚ-class twist vector	ℤ- and affine class	SG flavor symmetry D_S	discrete charge
	4–1	$\mathbb{Z}_6^{(\mathrm{PG})}$	k
		$\mathbb{Z}_6 \times \mathbb{Z}_2$	$(n_1 + n_2 + n_3 + n_4 + n_5, n_6)$
$\mathbb{Z}_7$ $\left(0, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7}\right)$	1–1	$\mathbb{Z}_7^{(\mathrm{PG})}$	k
		$\mathbb{Z}_7$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
$\mathbb{Z}_8$–I $\left(0, \frac{1}{8}, \frac{1}{4}, -\frac{3}{8}\right)$	1–1	$\mathbb{Z}_8^{(\mathrm{PG})}$	k
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_5 + n_6)$
	2–1	$\mathbb{Z}_8^{(\mathrm{PG})}$	k
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_5 + n_6)$
	3–1	$\mathbb{Z}_8^{(\mathrm{PG})}$	k
		$\mathbb{Z}_4$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
$\mathbb{Z}_8$–II $\left(0, \frac{1}{8}, \frac{3}{8}, -\frac{1}{2}\right)$	1–1	$\mathbb{Z}_8^{(\mathrm{PG})}$	k
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_3 + n_4, n_5, n_6)$
	2–1	$\mathbb{Z}_8^{(\mathrm{PG})}$	k
		$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(n_1 + n_2 + n_3 + n_4 + n_5, n_6)$
$\mathbb{Z}_{12}$–I $\left(0, \frac{1}{12}, \frac{1}{3}, -\frac{5}{12}\right)$	1–1	$\mathbb{Z}_{12}^{(\mathrm{PG})}$	k
		$\mathbb{Z}_3$	$n_5 + n_6$
	2–1	$\mathbb{Z}_{12}^{(\mathrm{PG})}$	k
		$\mathbb{Z}_3$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
$\mathbb{Z}_{12}$–II $\left(0, \frac{1}{12}, \frac{5}{12}, -\frac{1}{2}\right)$	1–1	$\mathbb{Z}_{12}^{(\mathrm{PG})}$	k
		$(\mathbb{Z}_2)^2$	(n_5, n_6)
$\mathbb{Z}_2 \times \mathbb{Z}_2$ $\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right)$ $\left(0, \frac{1}{2}, 0, -\frac{1}{2}\right)$	1–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^6$	$(n_1, n_2, n_3, n_4, n_5, n_6)$
	1–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^5$	$(n_1, n_3, n_4, n_5, n_6)$
	1–3	$(\mathbb{Z}_4 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	$(k + 2(n_2 + n_6), \ell)$
		$(\mathbb{Z}_2)^4$	(n_1, n_3, n_4, n_5)
	1–4	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	$(k + 2(n_2 + n_6), \ell + 2(n_4 + n_6))$
		$(\mathbb{Z}_2)^3$	(n_1, n_3, n_5)
	2–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^5$	$(n_1 + n_2, n_3, n_4, n_5, n_6)$
	2–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^4$	$(n_1 + n_2, n_4, n_5, n_6)$
	2–3	$(\mathbb{Z}_4 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	$(k + 2(n_3 + n_6), \ell)$
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_4, n_5)$
	2–4	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^4$	$(n_1 + n_2, n_3, n_4, n_6)$

Q-class twist vector	ℤ- and affine class	SG flavor symmetry D_S	discrete charge
	2–5	$(\mathbb{Z}_4 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^3$	$(k + \ell + 2(n_3 + n_5), \ell)$ $(n_1 + n_2, n_4, n_6)$
	2–6	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{\text{(PG)}}$ $(\mathbb{Z}_2)^2$	$(k + 2(n_3 + n_6), \ell + 2(n_5 + n_6))$ $(n_1 + n_2, n_4)$
	3–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^5$	(k, ℓ) $(n_1, n_2 + n_3, n_4, n_5, n_6)$
	3–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^4$	(k, ℓ) $(n_1, n_2 + n_3, n_4, n_5)$
	3–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{\text{(PG)}}$ $(\mathbb{Z}_2)^3$	$(k, \ell + 2(n_5 + n_6))$ $(n_1, n_2 + n_3, n_4)$
	3–4	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{\text{(PG)}}$ $(\mathbb{Z}_2)^2$	$(k + 2(n_4 + n_6), \ell + 2(n_5 + n_6))$ $(n_1, n_2 + n_3)$
	4–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^4$	(k, ℓ) $(n_1 + n_2, n_3, n_4, n_5 + n_6)$
	4–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^3$	(k, ℓ) $(n_1 + n_2, n_3, n_5 + n_6)$
	5–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $\mathbb{Z}_4 \times (\mathbb{Z}_2)^3$	(k, ℓ) $(n_1 + n_2 + n_3, n_4, n_5, n_6)$
	5–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	(k, ℓ) $(n_1 + n_2 + n_3, n_5, n_6)$
	5–3	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^4$	(k, ℓ) $(n_1 + n_2 + n_3, n_4, n_5, n_6)$
	5–4	$(\mathbb{Z}_4 \times \mathbb{Z}_2)^{\text{(PG)}}$ $\mathbb{Z}_4 \times \mathbb{Z}_2$	$(k + \ell + 2(n_4 + n_5), \ell)$ $(n_1 + n_2 + n_3, n_6)$
	5–5	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{\text{(PG)}}$ $\mathbb{Z}_4$	$(k + 2(n_4 + n_6), \ell + 2(n_5 + n_6))$ $n_1 + n_2 + n_3$
	6–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^4$	(k, ℓ) $(n_1 + n_2, n_3 + n_4, n_5, n_6)$
	6–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^3$	(k, ℓ) $(n_1 + n_2, n_3 + n_4, n_6)$
	6–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{\text{(PG)}}$ $(\mathbb{Z}_2)^2$	$(k, \ell + 2(n_5 + n_6))$ $(n_1 + n_2, n_3 + n_4)$
	7–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^4$	(k, ℓ) $(n_1 + n_2, n_3, n_4 + n_5, n_6)$
	7–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\text{(PG)}}$ $(\mathbb{Z}_2)^3$	(k, ℓ) $(n_1 + n_2, n_3, n_4 + n_5)$

Q-class twist vector	Z- and affine class	SG flavor symmetry D_S	discrete charge
	8–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^4$	$(n_1, n_2 + n_3, n_4, n_5 + n_6)$
	9–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3, n_4 + n_5, n_6)$
	9–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(n_1 + n_2 + n_3, n_4 + n_5)$
	9–3	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_3, n_4 + n_5, n_6)$
	10–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3, n_4, n_5 + n_6)$
	10–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_3, n_4, n_5 + n_6)$
	11–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_3 + n_4, n_5 + n_6)$
	12–1	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_4)^2$	$(n_1 + n_2 + n_3, n_4 + n_5 + n_6)$
	12–2	$(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(PG)}$	(k, ℓ)
		$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(n_1 + n_2 + n_3, n_4 + n_5 + n_6)$
$\mathbb{Z}_2 \times \mathbb{Z}_4$ $\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right)$ $\left(0, \frac{1}{4}, 0, -\frac{1}{4}\right)$	1–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^4$	$(n_1 + n_2, n_3, n_4, n_5 + n_6)$
	1–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_3, n_4, n_5 + n_6)$
	1–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_5 + n_6, n_3, n_4)$
	1–4	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_3, n_5 + n_6)$
	1–5	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_4, n_3, n_5 + n_6)$
	1–6	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_4, n_3, n_4 + n_5 + n_6)$
	2–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^4$	$(n_1 + n_2, n_3 + n_4, n_5, n_6)$
	2–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(PG)}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_3 + n_4, n_5)$
	2–3	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(PG)}$	$(k + 2(n_1 + n_2 + n_3 + n_4), \ell)$
		$(\mathbb{Z}_2)^2$	(n_5, n_6)

Q-class twist vector	ℤ- and affine class	SG flavor symmetry D_S	discrete charge
	2–4	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	$(k + 2(n_3 + n_4 + n_6), \ell)$
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_6, n_5)$
	2–5	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_5, n_6)$
	2–6	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_3 + n_4 + n_6, n_5)$
	3–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_3, n_4, n_5 + n_6)$
	3–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3, n_4)$
	3–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3, n_5 + n_6)$
	3–4	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3, n_4 + n_5 + n_6)$
	3–5	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_4, n_5 + n_6)$
	3–6	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_5 + n_6, n_4)$
	4–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_3 + n_4 + n_5, n_6)$
	4–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2, n_3 + n_4 + n_5)$
	4–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4 + n_5, n_6)$
	4–4	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_6, n_3 + n_4 + n_5 + n_6)$
	4–5	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2, n_6)$
	5–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_3 + n_4, n_5, n_6)$
	5–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_5)$
	6–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_4 + n_5, n_3, n_6)$
	6–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	(n_3, n_6)

Q-class twist vector	Z- and affine class	SG flavor symmetry D_S	discrete charge
	6–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_4 + n_5, n_3)$
	6–4	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_4 + n_5 + n_6, n_3)$
	6–5	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	$(k + 2(n_1 + n_2 + n_3 + n_4 + n_5), \ell)$
		$\mathbb{Z}_2$	n_6
	7–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2 + n_3 + n_4, n_5, n_6)$
	7–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_5)$
	7–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_6)$
	8–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3, n_4 + n_5 + n_6)$
	8–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_4 + n_5 + n_6$
	8–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
	9–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(n_1 + n_2 + n_3 + n_4 + n_5, n_6)$
	9–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_4$	$n_1 + n_2 + n_3 + n_4 + n_5$
	9–3	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4 + n_5, n_6)$
	10–1	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4 + n_5, n_6)$
	10–2	$(\mathbb{Z}_2 \times \mathbb{Z}_4)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2 + n_3 + n_4 + n_5$
$\mathbb{Z}_2 \times \mathbb{Z}_6$–I $\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right)$ $\left(0, \frac{1}{6}, 0, -\frac{1}{6}\right)$	1–1	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	(n_3, n_4)
	1–2	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	n_3
	2–1	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	(n_5, n_6)
	2–2	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	n_5

ℚ-class twist vector	ℤ- and affine class	SG flavor symmetry D_S	discrete charge
$\mathbb{Z}_2 \times \mathbb{Z}_6$–II $\left(0,0,\frac{1}{2},-\frac{1}{2}\right)$ $\left(0,\frac{1}{6},\frac{1}{6},-\frac{1}{3}\right)$	1–1	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k,ℓ)
	2–1	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k,ℓ)
	3–1	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k,ℓ)
	4–1	$(\mathbb{Z}_2 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k,ℓ)
$\mathbb{Z}_3 \times \mathbb{Z}_3$ $\left(0,0,\frac{1}{3},-\frac{1}{3}\right)$ $\left(0,\frac{1}{3},0,-\frac{1}{3}\right)$	1–1	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^3$	(k,ℓ) $(n_1+n_2, n_3+n_4, n_5+n_6)$
	1–2	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^2$	(k,ℓ) (n_1+n_2, n_3+n_4)
	1–3	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^2$	(k,ℓ) $(2(n_1+n_2)+n_5+n_6, n_3+n_4)$
	1–4	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^2$	(k,ℓ) $(n_1+n_2+2(n_3+n_4), 2(n_3+n_4)+n_5+n_6)$
	2–1	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^2$	(k,ℓ) $(n_1+n_2+n_3+n_4, n_5+n_6)$
	2–2	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) $n_1+n_2+n_3+n_4$
	2–3	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) n_5+n_6
	2–4	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) $n_1+n_2+n_3+n_4+2(n_5+n_6)$
	3–1	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^2$	(k,ℓ) $(n_1+n_2+n_3+n_4, n_5+n_6)$
	3–2	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) n_5+n_6
	3–3	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) $n_1+n_2+n_3+n_4+n_5+n_6$
	4–1	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $(\mathbb{Z}_3)^2$	(k,ℓ) $(n_1+n_4, n_2+n_3+n_5+n_6)$
	4–2	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) n_1+n_4
	4–3	$(\mathbb{Z}_9 \times \mathbb{Z}_3)^{(\mathrm{PG})}$	$(k+2\ell+6(n_1+n_2+n_3+n_4+n_5+n_6), \ell)$
	5–1	$(\mathbb{Z}_3 \times \mathbb{Z}_3)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) $n_1+n_2+n_3+n_4+n_5+n_6$
$\mathbb{Z}_3 \times \mathbb{Z}_6$ $\left(0,0,\frac{1}{3},-\frac{1}{3}\right)$ $\left(0,\frac{1}{6},0,-\frac{1}{6}\right)$	1–1	$(\mathbb{Z}_3 \times \mathbb{Z}_6)^{(\mathrm{PG})}$ $\mathbb{Z}_3$	(k,ℓ) n_3+n_4
	1–2	$(\mathbb{Z}_3 \times \mathbb{Z}_6)^{(\mathrm{PG})}$	(k,ℓ)

Q-class twist vector	Z- and affine class	SG flavor symmetry D_S	discrete charge
	2–1	$(\mathbb{Z}_3 \times \mathbb{Z}_6)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_3$	$n_5 + n_6$
	2–2	$(\mathbb{Z}_3 \times \mathbb{Z}_6)^{(\text{PG})}$	(k, ℓ)
$\mathbb{Z}_4 \times \mathbb{Z}_4$ $\left(0, 0, \frac{1}{4}, -\frac{1}{4}\right)$ $\left(0, \frac{1}{4}, 0, -\frac{1}{4}\right)$	1–1	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^3$	$(n_1 + n_2, n_3 + n_4, n_5 + n_6)$
	1–2	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2, n_3 + n_4)$
	1–3	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_5 + n_6, n_3 + n_4)$
	1–4	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_3 + n_4 + n_5 + n_6)$
	2–1	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2 + n_3 + n_4, n_5 + n_6)$
	2–2	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_5 + n_6$
	2–3	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2 + n_3 + n_4$
	2–4	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
	3–1	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2, n_3 + n_4 + n_5 + n_6)$
	3–2	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
	4–1	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$(\mathbb{Z}_2)^2$	$(n_1 + n_2, n_3 + n_4 + n_5 + n_6)$
	4–2	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_3 + n_4 + n_5 + n_6$
	4–3	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2$
	5–1	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
		$\mathbb{Z}_2$	$n_1 + n_2 + n_3 + n_4 + n_5 + n_6$
	5–2	$(\mathbb{Z}_4 \times \mathbb{Z}_4)^{(\text{PG})}$	(k, ℓ)
$\mathbb{Z}_6 \times \mathbb{Z}_6$ $\left(0, 0, \frac{1}{6}, -\frac{1}{6}\right)$ $\left(0, \frac{1}{6}, 0, -\frac{1}{6}\right)$	1–1	$(\mathbb{Z}_6 \times \mathbb{Z}_6)^{(\text{PG})}$	(k, ℓ)

Table 1. Space group flavor symmetries obtained from the space group selection rule for all space groups with Abelian point group and $\mathcal{N}$=1 supersymmetry [25, 26].

4.1 Discrete anomalies of the SG flavor symmetry D_S

Discrete groups can be anomalous and their anomaly coefficients can be computed directly using Fujikawa's method for the Jacobian of the path integral measure [37, 38]. Following refs. [27, 39], the mixed $\mathbb{Z}_N - G_i - G_i$ anomaly coefficient of a discrete group $\mathbb{Z}_N$ and a non-Abelian gauge group factor G_i is given by

$$A_{\mathbb{Z}_N - G_i - G_i} = \sum_{\boldsymbol{r}^{(f)}} q^{(f)} \, \ell\left(\boldsymbol{r}^{(f)}\right) \, . \tag{4.1}$$

Here, the summation runs over all fermions that transform in the representation $\boldsymbol{r}^{(f)}$ of the gauge group factor G_i and $q^{(f)}$ denotes their discrete $\mathbb{Z}_N$ charge as given in table 1 for a string with constructing element[3]

$$g = (T_1)^{n_1} \dots (T_6)^{n_6} \, (g_\theta)^k \, (g_\omega)^\ell \, \in \, S \, , \tag{4.2}$$

with $n_1, \dots, n_6, k, \ell \in \mathbb{Z}$. In addition, the Dynkin index $\ell\left(\boldsymbol{r}^{(f)}\right)$ is normalized such that $\ell(\boldsymbol{N}) = 1$ for the fundamental representation $\boldsymbol{N}$ of SU(N) and $\ell(\boldsymbol{2N}) = 2$ for the vector representation $\boldsymbol{2N}$ of SO($2N$). Furthermore, the $\mathbb{Z}_N - \text{grav.} - \text{grav.}$ anomaly coefficient reads

$$A_{\mathbb{Z}_N - \text{grav.} - \text{grav.}} = \sum_m q^{(m)} \dim(\boldsymbol{R}^{(m)}) \, , \tag{4.3}$$

where the summation runs over all fermions that transform in the representation $\boldsymbol{R}^{(m)}$ of $G_1 \times G_2 \times \dots$, being the full non-Abelian gauge group of the theory.

For all space groups from table 1 we have constructed more than 1,000 random orbifold models using the `orbifolder` [36]. For every orbifold model we have checked that for each $\mathbb{Z}_N$ factor of D_S there is a discrete Green-Schwarz constant Δ_{GS} [27, 40] such that for all non-Abelian gauge group factors G_i the anomalies are universal, i.e.

$$A_{\mathbb{Z}_N - G_i - G_i} = \Delta_{\text{GS}} \mod N \, , \tag{4.4a}$$

$$A_{\mathbb{Z}_N - \text{grav.} - \text{grav.}} = 12 \, A_{\mathbb{Z}_N - G_i - G_i} \mod N \, . \tag{4.4b}$$

Consequently, if $\Delta_{\text{GS}} \neq 0$ the universal discrete anomalies can be canceled by a discrete Green-Schwarz mechanism involving a single, universally coupled axion. Thus, we have performed a non-trivial test of all SG flavor symmetries.

5 Conclusions

There are large sets of semi-realistic string orbifold models, whose phenomenology may reveal interesting features of the string landscape, restricting thereby string constructions in the search of an ultraviolet completion of low-energy physics. Studying the phenomenology of orbifold compactifications requires, as one of the first steps, the identification and understanding of all symmetries of a string model.

---

[3]Note that in the presence of roto-translations this constructing element g is equal to $g = (\theta^k \, \omega^\ell, \lambda_{(k,\ell)} + n_i e_i)$ for some non-trivial translation $\lambda_{(k,\ell)}$. The discrete charges $q^{(f)}$, however, do not depend on $\lambda_{(k,\ell)}$ but only on the integers $n_i, k, \ell \in \mathbb{Z}$.

In this work, we have studied the Abelian SG flavor symmetry D_S that arises from the constraints on closed strings to split and join while propagating on a six-dimensional orbifold, defined by a space group S. By demanding that D_S be a (non-faithful) representation of S and that the discrete charges of all closed strings be well-defined and conserved, we find that the SG flavor symmetry D_S corresponds to the Abelianization of the space group S. We have discussed how this observation can be applied to space groups with different features, rendering the precise structure of D_S and their charges for all closed strings.

In section 4, we computed D_S for all 138 space groups with Abelian point group that yield $\mathcal{N} = 1$ effective field theories in four dimensions. It is known that the identified symmetries, displayed in table 1, play a key role as part of the flavor symmetries at low energies and are thus essential for phenomenology. As a cross check of the validity of these symmetries, we have also explicitly verified in thousands of $E_8 \times E_8$ heterotic orbifold models that all identified D_S lead to universal anomalies, which allows them to be canceled by a universal Green-Schwarz mechanism.

The SG flavor symmetries D_S are respected in interactions of both massless and massive strings. Interestingly, there are cases where the charges of massless strings are restricted such that the symmetry of the massless sector is only a subgroup of D_S. An intriguing consequence is that the lightest massive string (which can be a winding mode) can only be produced and annihilated in pairs. Thus, the lightest massive string is stable and, hence, contributes to a dark sector of the effective model, see section 3.6.2. Whether this feature might be seen as an explanation of (some of) the dark matter of the universe and its evolution, or whether it rules out some of the orbifold geometries shall be studied elsewhere.

A natural extension of our work is the study of non-Abelian orbifolds, i.e. orbifolds whose point groups P are non-Abelian [41–43]. One should explore how the constraints we imposed on the SG flavor symmetries D_S and their charges apply to orbifolds with non-Abelian point groups, to answer whether this contribution to flavor symmetries is also Abelian in those scenarios.

Acknowledgments

References

[1] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678 [inSPIRE].

[2] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds. 2*, *Nucl. Phys.* **B 274** (1986) 285 [ɪɴSPIRE].

[3] M. Blaszczyk, S. Groot Nibbelink, O. Loukas and S. Ramos-Sánchez, *Non-supersymmetric heterotic model building*, *JHEP* **10** (2014) 119 [arXiv:1407.6362] [ɪɴSPIRE].

[4] M. Blaszczyk, S. Groot Nibbelink, O. Loukas and F. Ruehle, *Calabi-Yau compactifications of non-supersymmetric heterotic string theory*, *JHEP* **10** (2015) 166 [arXiv:1507.06147] [ɪɴSPIRE].

[5] O. Lebedev et al., *A mini-landscape of exact MSSM spectra in heterotic orbifolds*, *Phys. Lett.* **B 645** (2007) 88 [hep-th/0611095] [ɪɴSPIRE].

[6] O. Lebedev, H.P. Nilles, S. Ramos-Sánchez, M. Ratz and P.K.S. Vaudrevange, *Heterotic mini-landscape. (II). Completing the search for MSSM vacua in a Z_6 orbifold*, *Phys. Lett.* **B 668** (2008) 331 [arXiv:0807.4384] [ɪɴSPIRE].

[7] D.K. Mayorga Peña, H.P. Nilles and P.-K. Oehlmann, *A zip-code for quarks, leptons and Higgs bosons*, *JHEP* **12** (2012) 024 [arXiv:1209.6041] [ɪɴSPIRE].

[8] S. Groot Nibbelink and O. Loukas, *MSSM-like models on Z_8 toroidal orbifolds*, *JHEP* **12** (2013) 044 [arXiv:1308.5145] [ɪɴSPIRE].

[9] H.P. Nilles and P.K.S. Vaudrevange, *Geography of fields in extra dimensions: string theory lessons for particle physics*, *Mod. Phys. Lett.* **A 30** (2015) 1530008 [arXiv:1403.1597] [ɪɴSPIRE].

[10] B. Carballo-Pérez, E. Peinado and S. Ramos-Sánchez, $\Delta(54)$ *flavor phenomenology and strings*, *JHEP* **12** (2016) 131 [arXiv:1607.06812] [ɪɴSPIRE].

[11] S. Ramos-Sánchez, *On flavor symmetries of phenomenologically viable string compactifications*, *J. Phys. Conf. Ser.* **912** (2017) 012011 [arXiv:1708.01595] [ɪɴSPIRE].

[12] Y. Olguín-Trejo, R. Pérez-Martínez and S. Ramos-Sánchez, *Charting the flavor landscape of MSSM-like Abelian heterotic orbifolds*, *Phys. Rev.* **D 98** (2018) 106020 [arXiv:1808.06622] [ɪɴSPIRE].

[13] H.P. Nilles, S. Ramos-Sánchez, M. Ratz and P.K.S. Vaudrevange, *A note on discrete R symmetries in Z_6-II orbifolds with Wilson lines*, *Phys. Lett.* **B 726** (2013) 876 [arXiv:1308.3435] [ɪɴSPIRE].

[14] N.G. Cabo Bizet, T. Kobayashi, D.K. Mayorga Peña, S.L. Parameswaran, M. Schmitz and I. Zavala, *Discrete R-symmetries and anomaly universality in heterotic orbifolds*, *JHEP* **02** (2014) 098 [arXiv:1308.5669] [ɪɴSPIRE].

[15] H.P. Nilles, *Stringy origin of discrete R-symmetries*, PoS(CORFU2016)017 (2017) [arXiv:1705.01798] [ɪɴSPIRE].

[16] J. Lauer, J. Mas and H.P. Nilles, *Duality and the role of nonperturbative effects on the world sheet*, *Phys. Lett.* **B 226** (1989) 251 [ɪɴSPIRE].

[17] J. Lauer, J. Mas and H.P. Nilles, *Twisted sector representations of discrete background symmetries for two-dimensional orbifolds*, *Nucl. Phys.* **B 351** (1991) 353 [ɪɴSPIRE].

[18] L.E. Ibáñez and D. Lüst, *Duality anomaly cancellation, minimal string unification and the effective low-energy Lagrangian of 4D strings*, *Nucl. Phys.* **B 382** (1992) 305 [hep-th/9202046] [ɪɴSPIRE].

[19] D. Bailin, A. Love, W.A. Sabra and S. Thomas, *Modular symmetries in Z_N orbifold compactified string theories with Wilson lines*, *Mod. Phys. Lett.* **A 9** (1994) 1229 [hep-th/9312122] [INSPIRE].

[20] S. Hamidi and C. Vafa, *Interactions on orbifolds*, *Nucl. Phys.* **B 279** (1987) 465 [INSPIRE].

[21] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13 [INSPIRE].

[22] T. Kobayashi, H.P. Nilles, F. Plöger, S. Raby and M. Ratz, *Stringy origin of non-Abelian discrete flavor symmetries*, *Nucl. Phys.* **B 768** (2007) 135 [hep-ph/0611020] [INSPIRE].

[23] H.P. Nilles, M. Ratz and P.K.S. Vaudrevange, *Origin of family symmetries*, *Fortsch. Phys.* **61** (2013) 493 [arXiv:1204.2206] [INSPIRE].

[24] H.P. Nilles, M. Ratz, A. Trautner and P.K.S. Vaudrevange, *CP violation from string theory*, *Phys. Lett.* **B 786** (2018) 283 [arXiv:1808.07060] [INSPIRE].

[25] M. Fischer, M. Ratz, J. Torrado and P.K.S. Vaudrevange, *Classification of symmetric toroidal orbifolds*, *JHEP* **01** (2013) 084 [arXiv:1209.3906] [INSPIRE].

[26] M. Fischer, M. Ratz, J. Torrado and P.K.S. Vaudrevange, *Classification of symmetric toroidal orbifolds webpage*, http://users.ph.tum.de/ga57raj/Orbifolds/ClassificationOrbifolds/index.html.

[27] T. Araki, *Anomaly of discrete symmetries and gauge coupling unification*, *Prog. Theor. Phys.* **117** (2007) 1119 [hep-ph/0612306] [INSPIRE].

[28] W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, *Supersymmetric Standard Model from the heterotic string (II)*, *Nucl. Phys.* **B 785** (2007) 149 [hep-th/0606187] [INSPIRE].

[29] J.G. Ratcliffe and S.T. Tschantz, *Abelianization of space groups*, *Acta Cryst.* **A 65** (2008) 18.

[30] M. Blaszczyk, *Heterotic particle models from various perspectives*, Ph.D. thesis, University of Bonn, Bonn, Germany (2012).

[31] M. Blaszczyk and P.-K. Oehlmann, *Tracing symmetries and their breakdown through phases of heterotic $(2,2)$ compactifications*, *JHEP* **04** (2016) 068 [arXiv:1512.03055] [INSPIRE].

[32] B. Petersen, M. Ratz and R. Schieren, *Patterns of remnant discrete symmetries*, *JHEP* **08** (2009) 111 [arXiv:0907.4049] [INSPIRE].

[33] R. Donagi and K. Wendland, *On orbifolds and free fermion constructions*, *J. Geom. Phys.* **59** (2009) 942 [arXiv:0809.0330] [INSPIRE].

[34] S. Förste, T. Kobayashi, H. Ohki and K.-J. Takahashi, *Non-factorisable $Z_2 \times Z_2$ heterotic orbifold models and Yukawa couplings*, *JHEP* **03** (2007) 011 [hep-th/0612044] [INSPIRE].

[35] M. Blaszczyk, S. Groot Nibbelink, M. Ratz, F. Ruehle, M. Trapletti and P.K.S. Vaudrevange, *A $Z_2 \times Z_2$ Standard Model*, *Phys. Lett.* **B 683** (2010) 340 [arXiv:0911.4905] [INSPIRE].

[36] H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange and A. Wingerter, *The orbifolder: a tool to study the low energy effective theory of heterotic orbifolds*, *Comput. Phys. Commun.* **183** (2012) 1363 [arXiv:1110.5229] [INSPIRE].

[37] K. Fujikawa, *Path integral measure for gauge invariant fermion theories*, *Phys. Rev. Lett.* **42** (1979) 1195 [INSPIRE].

[38] K. Fujikawa, *Path integral for gauge theories with fermions*, *Phys. Rev.* **D 21** (1980) 2848 [*Erratum ibid.* **D 22** (1980) 1499] [INSPIRE].

[39] T. Araki, T. Kobayashi, J. Kubo, S. Ramos-Sánchez, M. Ratz and P.K.S. Vaudrevange, *(Non-)Abelian discrete anomalies*, *Nucl. Phys.* **B 805** (2008) 124 [arXiv:0805.0207] [INSPIRE].

[40] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric D = 10 gauge theory and superstring theory*, *Phys. Lett.* **B 149** (1984) 117 [INSPIRE].

[41] Z. Kakushadze, G. Shiu and S.-H. Henry Tye, *Asymmetric non-Abelian orbifolds and model building*, *Phys. Rev.* **D 54** (1996) 7545 [hep-th/9607137] [INSPIRE].

[42] S.J.H. Konopka, *Non Abelian orbifold compactifications of the heterotic string*, *JHEP* **07** (2013) 023 [arXiv:1210.5040] [INSPIRE].

[43] M. Fischer, S. Ramos-Sánchez and P.K.S. Vaudrevange, *Heterotic non-Abelian orbifolds*, *JHEP* **07** (2013) 080 [arXiv:1304.7742] [INSPIRE].