# Note on the spectra of finite permutation matrices

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§ 1. *Introduction.* The spectrum of the special $n\times n$ permutation matrix

$$(1) \qquad C_n = [\delta_{\nu,\mu-1}] \qquad \delta_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha \equiv \beta (\mathrm{mod}\, n) \\ 0 & \text{for } \alpha \not\equiv \beta (\mathrm{mod}\, n) \end{cases}$$

i.e. the *circulant matrix* with the first row $(0, 1, 0, \ldots, 0)$ is already known from the classical determinant theory (see for example [1]). The matrix $C_n$ has only simple eigenvalues, hence its spectral decomposition is uniquely determined. Let $\varepsilon_n^k$ denote the $n$-th root of unity $\varepsilon_n^k = \exp i\dfrac{2k\pi}{n}$ then the $\lambda_k$ eigenvalue and the eigenvector $u_k$ belonging to $\lambda_k$ are

$$(2) \qquad \left.\begin{aligned} \lambda_k &= \varepsilon_n^k \\ u_k^T &= \frac{1}{\sqrt{n}}(1, \varepsilon_n^k, \varepsilon_n^{2k}, \ldots, \varepsilon_n^{(n-1)k}) \end{aligned}\right\} \qquad k = 0, 1, \ldots, n-1.$$

The raised $T$ is used to denote transposition.

In the general case that is if a permutation matrix $P_n$ has also multiple eigenvalues, one has certain freedom to choose the linearly independent eigenvectors belonging to the same multiple eigenvalue. The present paper is intended to give a formula for a spectral decomposition of any permutation matrix $P_n$, as simple as (2) that is a formula such that *the nonzero components* of an eigenvector belonging to an eigenvalue *contain the powers of this eigenvalue only*.

In discussing the spectra of the permutation matrices it seems very helpful to use some group theoretical relations.

§ 2. The $n\times n$ permutation matrices $P_n$ give an $n$-dimensional real unitary representation of the symmetric group $S_n$. This representation is called the *permutation representation* (see [2] p. 79).

Let $p$ denote a permutation $p\in S_n$ and $\alpha^{|p|}$ the image of the element $\alpha$ $(\alpha = 1, 2, \ldots, n)$ under the permutation $p$, hence

$$(3) \qquad p = \begin{pmatrix} 1 & 2 & \ldots n \\ 1^{|p|} & 2^{|p|} & \ldots n^{|p|} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha^{|p|} \end{pmatrix}$$

We read the product $pq$ from left to right, that is $\alpha^{|pq|}=(\alpha^{|p|})^{|q|}$. Thus, in the above representation the permutation (3) is represented by the matrix

(4) $$P_n(p)=[\delta_{\nu^{|p|}\mu}] \qquad \nu, \mu=1, 2, \dots, n.$$

Here

$$\delta_{\alpha\beta}=\begin{cases}1 & \text{for} \quad \alpha=\beta \\ 0 & \text{for} \quad \alpha\neq\beta\end{cases}$$

is the Kronecker symbol. With this notation the matrix $C_n$ in (1) can be written as

(1a) $$C_n=P_n(c)=[\delta_{\nu^{|c|}\mu}]$$

where the permutation $c$ consists of the single cycle

$$c=(1, 2, \dots n).$$

Postmultiplication by $P_n(p)$ of any $n\times n$ matrix $M$ results in the rearrangement of the columns of the matrix $M$ corresponding to the permutation $p$, while the premultiplication by $P_n(p)$ rearranges the rows of $M$ corresponding to the permutation $p^{-1}$.

Two permutations $p, q\in S_n$ are conjugate if and only if both contain disjunct cycles of the same number and the same length resp. Suppose $p$ and $q$ are conjugate, i.e.

(5) $$q=s^{-1}ps \quad \text{with} \quad s\in S_n$$

then the cyclic form of $q$ is obtained by replacing by $\alpha^{|s|}$ each element $\alpha$ in the cyclic form of $p$.

From (5) we have the orthogonal similarity relation between the matrices $P_n(p)$ and $P_n(q)$:

(6) $$P_n(q)=P_n(s^{-1})P_n(p)P_n(s)=P_n^T(s)P_n(p)P_n(s).$$

Hence, obviously

**Lemma.** *Suppose $q=s^{-1}ps$ and let $u$ be an eigenvector of $P_n(p)$ corresponding to the eigenvalue $\lambda$,* i.e.

(7) $$P_n(p)u=\lambda u$$

*then the vector*

(8) $$v=P_n^T(s)u$$

*is an eigenvector of $P_n(q)$ corresponding to the eigenvalue $\lambda$. The relations between the components of the two vectors $u^T=(u_1, u_2, \dots, u_n)$ and $v^T=(v_1, v_2, \dots, v_n)$ are given by*

(9) $$v_{\nu^{|s|}}=u_\nu \qquad \nu=1, 2, \dots, n.$$

§ 3. It follows from the above discussion that for determining a spectral decomposition of any permutation matrix $P_n(p)$ ($p\in S_n$) it is enough to choose any single permutation $p$ in each conjugate class $K$ of $S_n$ and determine a spectral decomposition of the matrix $P_n(p)$. Moreover in each conjugate class $K$ we can find a permutation $p_0$ such that the matrix $P_n(p_0)$ decomposes into the direct sum of circulant matrices. For illustration let us consider the permutation $p_0$ which consists of $m$ disjunct cycles of length $l_1, l_2, \ldots, l_m$

$$(10)\quad p_0 = (12 \ldots l_1) \ldots \big(a_k(a_k+1) \ldots (a_k+l_k-1)\big) \ldots \big(a_m(a_m+1) \ldots (a_m+l_m-1)\big)$$

$$a_k = 1+\sum_{\nu=1}^{k-1} l_\nu \qquad (k = 2, \ldots, m); \quad a_1 = 1; \quad a_m+l_m-1 = n.$$

Obviously the permutation matrix $P_n(p_0)$ decomposes into the direct sum of $m$ circulants of order $l_1, l_2, \ldots, l_m$ as

$$(11)\qquad P_n(p_0) = C_{l_1} \dot{+} C_{l_2} \dot{+} \ldots \dot{+} C_{l_m} = \operatorname{diag}(C_{l_1}, C_{l_2}, \ldots, C_{l_m}).$$

Thus a spectral decomposition of $P_n(p_0)$ is known by (2): The cycle $\big(a_k\,(a_k+1) \ldots$ $\ldots (a_k+l_k-1)\big)$ contributes to the spectra of $P_n(p_0)$ the eigenvector $u_{k\nu}$ corresponding to the eigenvalue $\varepsilon_{l_k}^{\nu} = \exp i\dfrac{2\nu\pi}{l_k}$ where

$$(12)\quad u_{k\nu}^T = \frac{1}{\sqrt{l_k}}(\overset{1}{0}, \overset{2}{0}, \ldots, \overset{a_k-1}{0}, \overset{a_k}{1}, \overset{a_k+1}{\varepsilon_{l_k}^{\nu}}, \ldots, \overset{a_k+l_k-1}{\varepsilon_{l_k}^{(l_k-1)\nu}}, 0, \ldots, \overset{n}{0}) \qquad \nu = 0, 1, \ldots, l_k-1;$$
$$k = 1, 2, \ldots, m.$$

Now suppose we want to determine a spectrum of the permutation matrix $P_n(q)$ where the permutation $q$ is conjugate permutation to (10)

$$q = s^{-1} p_0 s \qquad s \in S_n.$$

Then the cycle decomposition of $q$ is

$$(13)\quad \begin{aligned} q &= (1^{|s|}2^{|s|} \ldots l_1^{|s|}) \ldots \big(a_k^{|s|}(a_k+1)^{|s|} \ldots (a_k+l_k-1)^{|s|}\big) \ldots \big(a_m^{|s|}(a_m+1)^{|s|} \ldots n^{|s|}\big) \\ &= (\alpha_1\alpha_2 \ldots \alpha_{l_1}) \ldots (\varkappa_1\varkappa_2 \ldots \varkappa_{l_k}) \ldots (\mu_1\mu_2 \ldots \mu_{l_m}). \end{aligned}$$

By the lemma and the spectrum (12) of the matrix $P_n(p_0)$ in (11) a spectral decomposition of $P_n(q)$ can be written down without calculation to and we have the

**Theorem.** *Suppose $q \in S_n$ is the permutation of the cycle decomposition* (13), *then a spectral decomposition of the permutation matrix $P_n(q)$ can be chosen as follows. Each cycle*

$$(\varkappa_1\varkappa_2 \ldots \varkappa_{l_k})$$

*contributes to the spectrum the eigenvector $v_{k\nu}$ corresponding to the eigenvalue $\varepsilon_{l_k}^{\nu} = \exp i\dfrac{2\nu\pi}{l_k}$ $(\nu = 0, 1, \ldots, l_k-1)$ where the vector $v_{k\nu}$ has only $l_k$ nonzero components namely those, corresponding to the indices $\varkappa_j$, denoted by $v_{k\nu}(\varkappa_j)$ $(j = 1, 2, \ldots, l_k)$, when the values of these components are*

$$v_{k\nu}(\varkappa_j) = \frac{1}{\sqrt{l_k}}\,\varepsilon_{l_k}^{(j-1)\nu} \begin{cases} j = 1, 2, \ldots, l_k \\ \nu = 0, 1, \ldots, l_k-1 \\ k = 1, 2, \ldots, m. \end{cases}$$

It is worthy of note that any symmetric permutation matrix represents a permutation which contains at most transpositions i.e. the lengths of its disjunct cycles are one or two. Thus a symmetric permutation matrix has only two differen eigenvalues $\pm 1$ and the corresponding eigenvectors can be chosen such that they contain at most two nonzero components.

§ 4. Let us see now an example. We want to determine a spectral decomposition of the permutation matrix

$$P_7(q) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The cyclic form of $q$ is

$$q = (143)(26)(57).$$

The cycle (143) contribute to the spectrum the eigenvalues $\varepsilon_3^\nu = \exp i \frac{2\nu\pi}{3}$ $(\nu = 0, 1, 2)$ with the eigenvectors

$$\lambda = 1 = \varepsilon_3^0; \quad \frac{1}{\sqrt{3}}(1, 0, 1, 1, 0, 0, 0)$$

$$\lambda = \varepsilon_3; \quad \frac{1}{\sqrt{3}}(1, 0, \varepsilon_3^2, \varepsilon_3, 0, 0, 0) = \frac{1}{\sqrt{3}}(1, 0, \bar{\varepsilon}_3, \varepsilon_3, 0, 0, 0)$$

$$\lambda = \varepsilon_3^2 = \bar{\varepsilon}_3; \quad \frac{1}{\sqrt{3}}(1, 0, \varepsilon_3^4, \varepsilon_3^2, 0, 0, 0) = \frac{1}{\sqrt{3}}(1, 0, \varepsilon_3, \bar{\varepsilon}_3, 0, 0, 0).$$

The contribution of the cycles (26) and (57) to the spectrum are the eigenvalues $\pm 1$ with the eigenvectors

$$\lambda = 1; \qquad \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, 1, 0) \quad \text{and} \quad \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, 0, 1)$$

$$\lambda = -1; \quad \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, -1, 0) \quad \text{and} \quad \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, 0, -1)$$

respectively.

## References

[1] E. Pascal, I determinanti. *Milano*, 1897.
[2] H. Wielandt, Finite permutation groups. *New York*, 1964.

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