If in (1) the ω functions be such that ω_q^r vanishes when $r > q$, then we have from (2)

$$a_n = \frac{f_a^n}{\omega_n^n} - \sum_{\mu = n+1}^{\mu=\infty} a_\mu \frac{\omega_\mu^n}{\omega_n^n},$$

$$= \frac{f_a^n}{\omega_n^n} - \sum_{\mu=1}^{\mu=\infty} \frac{f^{n+\mu} a}{\prod_{r=1}^{\mu} \omega_{n+r}^{n+r}}. \qquad (5)$$

This is wrong, for the correct value of a_n in this case must be $f^n a/\omega_n^n$. In particular, if $\omega^q = x_q$, then $a_0 = f0$, and

$$a_n = \frac{f^n x}{n!} - \frac{f^{n+1} x}{n!(n+1)!} - \ldots,$$

which should be the coefficient $f^n x/n!$ in Bernouilli's series.

The value (4) under the conditions imposed is correct, and gives that particular case mentioned in the BULLETIN, vol. 2, p. 140. It seems that throughout Wronski's whole work he has aimed at the generalization given in the BULLETIN as quoted above, which I have called a *composite*.

If inferences may be drawn while the investigations are yet incomplete, this composite may prove useful; and if so, it is my sincere hope that it may be the means of lifting in some measure the weight of opprobrium from the memory of the unfortunate Wronski, whose pathetic story appeals so strongly to the sympathy of all of his co-workers.

CHARLOTTESVILLE, *February*, 1893.

NOTE ON THE SUBSTITUTION GROUPS OF SIX, SEVEN, AND EIGHT LETTERS.

BY F. N. COLE, PH.D.

A LIST of the groups of six, seven, and eight letters is given by Mr. Askwith in vol. 24 of the *Quarterly Journal of Mathematics*, and Professor Cayley has revised and tabulated Mr. Askwith's results in vol. 25 of the same journal.* Noticing

* A list of groups as far as ten letters was given by Kirkman in the *Proceedings of the Literary and Philosophical Society of Manchester*, vol. 3 (1864), p. 144.

that several familiar groups were missing in this table, I have re-examined the whole question by an independent method, with the result that I am able to furnish here a supplementary list of over forty omitted groups of these degrees. The precautions which I have taken to insure accuracy give me a considerable degree of confidence that my results are correct and complete.

I. *Six and Seven Letters.*

1. For six letters the intransitive and multiply transitive groups are correctly given in Professor Cayley's enumeration. The three following non-primitive groups are, however, omitted:

Order 36. $36_3 =$

+	+	+	+	−	−
1,	*abc*,	*abc . def*,	*ab . de*,	*ad . be . cf*,	*adbecf*,
	acb,	*abc . dfe*,	*ab . df*,	*ad . bf . ce*,	*adbfce*,
	def,	*acb . def*,	*ab . ef*,	*ae . bd . cf*,	*adcebf*,
	dfe,	*acb . dfe*,	*ac . de*,	*ae . bf . cd*,	*adcfbe*,
			ac . df,	*af . bd . ce*,	*aebdcf*,
			ac . ef,	*af . be . cd*,	*aebfcd*,
			bc . de,		*aecdbf*,
			bc . df,		*aecfbd*,
			bc . ef,		*afbdce*,
					afbecd,
					afcdbe,
					afcebd.

Order 24. $24_5 =$

+	−	+	−	+	−
1,	*ac*,	*ac . bd*,	*ac . bd . ef*,	*abe . cdf*,	*aebcfd*,
	bd,	*ac . ef*,		*abf . cde*,	*abecfd*,
	ef,	*bd . ef*,		*ade . cbf*,	*abfcde*,
				adf . cbe,	*adecbf*,
				aeb . cfd,	*adfcbe*,
				aed . cfb,	*aedcbf*,
				afb . ced,	*afbced*,
				afd . ceb,	*afdcbe*.

Order 12. $12_2 =$

+	+	+	+
1,	*ac . bd*,	*abe . cdf*,	*aeb . cfd*,
	ac . ef,	*abf . cde*,	*afb . ced*,
	bd . ef,	*ade . cbf*,	*aed . cfb*,
		adf . bec,	*afd . bce*.

The first of these is a subgroup of Cayley's $(abcdef)_{72}$; its systems of non-primitivity are a, b, c and d, e, f. It consists of all the even substitutions which permute the letters of each system separately and of the *odd* substitutions which interchange the two systems.

The second and third groups are subgroups of Cayley's $(abcdef)_{48}$. Their systems of non-primitivity are a, c ; b, d ; and e, f. The former consists of all the substitutions which permute the letters of the three systems separately, together with the cyclical permutations of the systems. The latter consists of the even substitution of the former, and is a common subgroup of all the three transitive groups of order 24 with six letters.

2. In the case of seven letters one intransitive group and one doubly transitive group are omitted by Professor Cayley:

Order 24. $24_8 =$

$1,$	$1,$	$ab \,.\, cd,$	$ac \,.\, bd,$	$ad \,.\, bc,$
$efg,$	$abc,$	$bdc,$	$adb,$	$acd,$
$egf,$	$acb,$	$bcd,$	$abd,$	$adc,$
$ef,$	$ab,$	$cd,$	$adbc,$	$acbd,$
$gh,$	$bc,$	$ad,$	$acdb,$	$abdc,$
$eg,$	$ac,$	$bd,$	$adcb,$	$abcd,$

where each substitution of e, f, g is to be multiplied by the four substitutions of a, b, c, d which stand opposite it.

Order 168. A doubly transitive group obtained by combining Cayley's $+ (abcdef)_{24} =$

$1,$	$ab \,.\, cd,$	$abe \,.\, cdf,$	$abcd \,.\, ef,$
	$ac \,.\, bd,$	$abf \,.\, cde,$	$adcb \,.\, ef,$
	$ad \,.\, bc,$	$ade \,.\, bfc,$	$aecf \,.\, bd,$
	$ac \,.\, ef,$	$adf \,.\, bec,$	$afce \,.\, bd,$
	$ae \,.\, cf,$	$aeb \,.\, cfd,$	$bedf \,.\, ac,$
	$af \,.\, ce,$	$aed \,.\, bcf,$	$bfde \,.\, ac,$
	$bd \,.\, ef,$	$afb \,.\, ced,$	
	$bf \,.\, de,$	$afd \,.\, bce,$	
	$be \,.\, df,$		

with

$$abcefdg.$$

This is the well-known triad group of seven letters, discussed by Kronecker and others, which occurs in connection with the transformation of the seventh order of the elliptic modular functions. It is certainly remarkable that this group among all others should have been overlooked.

II. *Eight Letters.*

1. The following 20 intransitive groups are not given by Prof. Cayley:

Order 96. $\{(abcd)$ all $(efgh)$ all$\}$sex. $=$

1,	$ab.cd$,	$ac.bd$,	$ad.bc$,	1,	$ef.gh$,	$eg.fh$,	$eh.fg$,
abc,	bdc,	adb,	acd,	efg,	fhg,	ehf,	egh,
acb,	bcd,	abd,	adc,	egf,	fgh,	efh,	ehg,
ab,	cd,	$adbc$,	$acbd$,	ef,	gh,	$egfh$,	$ehfg$,
ac,	bd,	$adcb$,	$abcd$,	eg,	fh,	$ehgf$,	$efgh$,
ad,	bc,	$acbd$,	$abdc$,	eh,	fg,	$eghf$,	$efhg$.

Order 72. $\{(abcdef)_{72}\ (gh)\}\text{dim.}_3$,
$(abcdef)_{36_3}\ (gh)$.

There are three dimidiates of $(abcdef)_{72}$ and (gh), since the former has three subgroups of order 36.

Order 48. $\{(abcdef)_{48}\ (gh)\}\text{dim.}_3$,
$(abcdef)_{24_5}\ (gh)$.

Order 36. $\{(abcdef)_{36_3}\ (gh)\}\text{dim.}_1$,
$(abcdef)_{36_3}\ (gh)\ \text{dim.}_2$,
$\{(abc)$ all (def) cyc.$\}(gh)$.

Order 24. $\{+(abcdef)_{24}\ (gh)\}\text{dim.}$,
$\{\pm(abcdef)_{24}\ (gh)\}\text{dim.}$,
$\{(abcdef)_{24_5}\ (gh)\}\text{dim.}$,
$(abcdef)_{12_2}\ (gh)$.

Order 16. Six groups based on a two-to-two isomorphism between $(abcd)_8$ and $(efgh)_8$. They are

1,	$ac.bd$,	1,	$eg.fh$,
ac,	bd,	ef,	fh,
$ab.cd$,	$ad.bc$,	$ef.gh$,	$eh.fg$,
$abcd$,	$adcb$,	$efgh$,	$ehgf$,

where the last three lines on either side may be permuted in any of the six possible ways.

Order 12. $A\ \{(abcd)$ pos. $(efgh)$ pos.$\}$ $=$

1,	$ab.ef$,	$abc.efg$,	$abcd.efgh$,
	$ac.eg$,	$acb.egf$,	$adcb.ehgf$,
	$ad.eh$,	$abd.efh$,	$abdc.efhg$,
	$bc.fg$,	$adb.ehf$,	$acdb.eghf$,
	$bd.fh$,	$acd.egh$,	$acbd.egfh$,
	$cd.gh$,	$adc.ehg$,	$adbc.ehfg$.
		$bcd.fgh$,	
		$bdc.fhg$,	

Order 8. A group based on a one-to-two isomorphism between $(abcd)_4$ and $(efgh)_8$:

$1,$	$1,$	$eg \,.\, fh,$
$ab \,.\, cd,$	$eg,$	$fh,$
$ac \,.\, bd,$	$ef \,.\, gh,$	$eh \,.\, fg,$
$ad \,.\, bc,$	$efgh,$	$ehgf.$

2. Of the non-primitive groups there are also 20 missing:

Order 576. $\{(abcd)$ all $(efgh)$ all$\}$ pos. $(aebf \,.\, cg \,.\, dh)$.

This is composed of all the even substitutions which affect the letters of the systems a, b, c, d and e, f, g, h separately, together with the *odd* substitutions which interchange the two systems.

Order 192. I find four groups of this order, while Cayley gives only two. The four are

$$\{(abcd) \text{ all } (efgh) \text{ all}\} \text{sex. } (ae \,.\, bf \,.\, cg \,.\, dh),$$
$$\{(ab)\,(cd)\,(ef)\,(gh)\} \text{all } (ABCD)_{12},$$
$$+[\{(ab)\,(cd)\,(ef)\,(gh)\} \text{pos. } (ABCD) \text{ all}],$$
$$\pm[\{(ab)\,(cd)\,(ef)\,(gh)\} \text{pos. } (ABCD) \text{ all}].$$

Here A, B, C, D represent the systems a, b; c, d; e, f; g, h. In the third group the substitutions of A, B, C, D are even in a, b, c, d, e, f, g, h, while in the fourth group the substitutions of A, B, C, D are even or odd in a, b, c, d, e, f, g, h, according as they are even or odd in A, B, C, D.

Order 128. $\{(abcd)_8\ (efgh)_8\}\ (ae \,.\, bf \,.\, cg \,.\, dh)$,

Order 96. $\{(ab)\,(cd)\,(ef)\,(gh)\}$pos. $(ABCD)$ pos.,

Order 64. $\{(abcd)_8$ cyc. $(efgh)_8$ cyc.$\}$dim. $(aecg \,.\, bf \,.\, dh)$,

$\{(abcd)_8$ pos. $(efgh)_8$ pos.$\}$dim. $(aebfcgdh)$,

$\{(abcd)_8$ com. $(efgh)$ com.$\}$dim. $(ae \,.\, bf \,.\, cg \,.\, dh)$,

$\{(abcd_8$ com. $(efgh)_8$ com.$\}$dim.$(aebfcgdh)$.

Order 48. $(ab \,.\, cd \,.\, ef \,.\, gh)\ (ABCD)$ all.

This is obtained by combining

$$ABC = ade \,.\, bcf, \quad ABCD = acfhbdeg.$$

Order 32. Four groups obtained by combining

$1,$	$ac \,.\, bd,$	$1,$	$eg \,.\, fh,$
$ac,$	$bd,$	$eg,$	$fh,$
$ab \,.\, cd$	$ad \,.\, bc,$	$ef \,.\, gh,$	$eh \,.\, fg,$
$abcd,$	$adcb,$	$efgh,$	$ehgf.$

with one of

$$ae \,.\, bf \,.\, cg \,.\, dh, \quad aebf \,.\, cgdh, \quad aecg \,.\, bfdh, \quad aebfcgdh..$$

Two groups

$$\left\{\begin{array}{cc|cc} 1, & ac\,.\,bd, & 1, & eg\,.\,fh, \\ ac, & bd, & ef\,.\,gh, & eh\,.\,fg, \\ ab\,.\,cd, & ad\,.\,bc, & eg, & fh, \\ abcd, & adcb, & efgh, & ehgf, \end{array}\right\}(ae\,.\,bf\,.\,cg\,.\,dh),$$

$$\left\{\begin{array}{cc|cc} 1, & ac\,.\,bd, & 1, & eg\,.\,fh, \\ ac, & bd, & efgh, & ehgf, \\ ab\,.\,cd, & ad\,.\,bc, & ef\,.\,gh & eh\,.\,fg, \\ abcd, & adcb, & eg, & fh, \end{array}\right\}(ae\,.\,bf.\,cg\,.\,dh).$$

Order 24. $A\ \{(abcd)\text{ pos. }(efgh)\text{ pos.}\}(ae\,.\,bf\,.\,cg\,.\,dh)$.

$$(ab\,.\,cd\,.\,ef\,.\,gh)\ (ABCD)\text{ pos.}$$

The latter is obtained by combining

$$ABC = ade\,.\,bcf, \qquad AC\,.\,BD = afbe\,.\,chdg.$$

Order 16. Two groups obtained by combining

$$\begin{array}{llll} 1, & ac\,.\,ef\,.\,gh, & ab\,.\,cd\,.\,ef\,.\,gh, & abcd\,.\,efgh, \\ & bd\,.\,eh\,.\,fg, & & adcb\,.\,ehgf, \\ & ab\,.\,cd\,.\,fh, & & \\ & ad\,.\,bc\,.\,eg, & & \end{array}$$

with one of the two substitutions

$$aebfcgdh, \qquad afbgchde.$$

5. Of the multiply transitive groups Professor Cayley omits two:

Order 56. $\{A\ (abcdefgh), \quad bcedghf\}$.

Order 168. There are two doubly transitive groups of this order:

$$\{A(abcdefgh), \quad (bcedghf)_{21}\},$$
$$\{(abcdefg)_{21}, \quad (ad\,.\,be\,.\,ch\,.\,fg)\}.$$

The latter is given by Mr. Askwith, and is the *simple* group of 8 letters. It is isomorphic with the group of the same order in 7 letters. Professor Cayley's notation (p. 154) is misleading, inasmuch as there is no cyclical substitution of 8 letters in either group, both being composed of even substitutions.

The two groups of order 168 are discussed by Hölder, *Math. Annalen*, vol. 40, p. 75.

One misprint in Professor Cayley's list remains to be noted. The group 4_7 of 8 letters should read:

$$\{(abcd)_4, \quad (ef)(gh)\}\text{ quad.}$$

The corrected total number of groups of degrees 2–8 is as given below:

No. of letters ..	2,	3,	4,	5,	6,	7,	8.
No. of groups..	1,	2,	7,	8,	37,	40,	199.
Transitives	1,	2,	5,	5,	16,	7,	48.

ANN ARBOR, *April*, 1893.

MATHEMATICAL BIBLIOGRAPHY.

Revue semestrielle des publications mathématiques rédigée sous les auspices de la Société mathématique d'Amsterdam par P. H. SCHOUTE (Groningue), D. J. KORTEWEG (Amsterdam), W. KAPTEYN (Utrecht), J. C. KLUYVER (Leyde), P. ZEEMAN (Delft), avec la collaboration de MM. C. VAN ALLER, F. DE BOER, J. CARDINAAL, D. COELINGH, R. J. ESCHER, W. MANTEL, P. MOLENBROEK, P. VAN MOURIK, M. C. PARAIRA, A. E. RAHUSEN, G. SCHOUTEN, J. W. TESCH, J. VERSLUYS, J. DE VRIES et de Madlle. A. G. WIJTHOFF. Tome I. (Première partie). Amsterdam, W. Versluys, 1893. [8vo. 104 pp. 4 florins or 8.50 francs per year.]

AT the international congress for mathematical bibliography held at Paris, in July 1889, under the auspices of the French mathematical society, a detailed classification was adopted to serve as a basis for a general bibliography of the mathematical sciences. It was resolved to prepare a *répertoire bibliographique* of the mathematical literature of the present century (1800–1889) arranged by subjects in logical order. This bibliography was to be continued by supplements issued at intervals of ten years.

The execution of this work was entrusted to a permanent commission composed of five French members (H. Poincaré, D. André, Humbert, d'Ocagne, Charles Henry) and twelve foreign members (Catalan, Bierens de Haan, Glaisher, Teixeira, Holst, Valentin, Em. Weyr, Guccia, Eneström, Gram, Liguine, Stephanos). It is not known to the present writer what progress may have been made with this undertaking which will no doubt require several years for its completion.

The *Revue semestrielle*, whose complete title is given above, is a new bibliographical journal of the current periodical literature of mathematics based on this classification. The number now at hand (part 1 of vol. 1) covers the period from March 1 (why not April 1?) to October 1, 1892, and is supposed to have appeared January 1, 1893; the second part will report the period from October 1, 1892, to March 1, 1893, and is to appear